Some Generalization for an Operator Which Preserving Inequalities Between Polynomials

Ahmad Zireh1,* , Susheel Kumar2 and Kum Kum Dewan2

1 Department of Mathematics, University of Shahrood, Shahrood, Iran
2 Department of Mathematics, Jamia Millia Islamia, New Delhi, India

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Abstract. For a polynomial $p(z)$ of degree $n$ which has no zeros in $|z| < 1$, Dewan et al., (K. K. Dewan and Sunil Hans, Generalization of certain well known polynomial inequalities, J. Math. Anal. Appl., 363 (2010), 38–41) established

$$\left| z p'(z) + \frac{n \beta}{2} p(z) \right| \leq \frac{n}{2} \left( \left( \left| \frac{\beta}{2} \right| + \left| 1 - \frac{\beta}{2} \right| \right) \max_{|z|=1} |p(z)| - \left( \left| 1 + \frac{\beta}{2} \right| - \left| \frac{\beta}{2} \right| \right) \min_{|z|=1} |p(z)| \right),$$

for any complex number $\beta$ with $|\beta| \leq 1$ and $|z| = 1$. In this paper we consider the operator $B$, which carries a polynomial $p(z)$ into

$$B[p(z)] := \lambda_0 p(z) + \lambda_1 \left( \frac{nz}{2} \right) p'(z) + \lambda_2 \left( \frac{nz}{2} \right)^2 p''(z),$$

where $\lambda_0, \lambda_1$, and $\lambda_2$ are such that all the zeros of $u(z) = \lambda_0 + c(n,1) \lambda_1 z + c(n,2) \lambda_2 z^2$ lie in the half plane $|z| \leq |z - n/2|$. By using the operator $B$, we present a generalization of result of Dewan. Our result generalizes certain well-known polynomial inequalities.

Key Words: B-operator, inequality, polynomial, maximum modulus, restricted zeros.

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1 Introduction and statement of results

Let $p(z)$ be a polynomial of degree $n$ and $p'(z)$ its derivative. Then it is well known that

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \quad (1.1)$$

*Corresponding author. Email addresses: azireh@gmail.com or azireh@shahroodut.ac.ir (A. Zireh), ahlavat_skumar@yahoo.co.in (S. Kumar), kkdewan123@yahoo.co.in (K. K. Dewan)
and

\[
\max_{|z| = R > 1} |p(z)| \leq R^n \max_{|z| = 1} |p(z)|. \tag{1.2}
\]

Inequality (1.1) is a famous result due to Bernstein [7], whereas inequality (1.2) is a simple consequence of maximum modulus principle (see [16]). Both the above inequalities are sharp and equality in each holds for the polynomials having all its zeros at the origin.

For the class of polynomials having no zeros in \(|z| < 1\), inequalities (1.1) and (1.2) have respectively been replaced by

\[
\max_{|z| = 1} |p'(z)| \leq \frac{n}{2} \max_{|z| = 1} |p(z)|, \tag{1.3}
\]

and

\[
\max_{|z| = R > 1} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z| = 1} |p(z)|. \tag{1.4}
\]

Inequality (1.3) was conjectured by Erdős and later proved by Lax [13], whereas inequality (1.4) was proved by Ankeny and Rivlin [1], for which they made use of (1.3). Both these inequalities are also sharp and equality in each holds for polynomials having all its zeros on \(|z| = 1\).

Aziz and Dawood [4] used \(\min_{|z| = 1} |p(z)|\) to obtain a refinement of inequalities (1.3) and (1.4) by demonstrating if \(p(z)\) is a polynomial of degree \(n\) which does not vanish in \(|z| < 1\), then

\[
\max_{|z| = 1} |p'(z)| \leq \frac{n}{2} \left\{ \max_{|z| = 1} |p(z)| - \min_{|z| = 1} |p(z)| \right\}, \tag{1.5}
\]

and

\[
\max_{|z| = R > 1} |p(z)| \leq \left( \frac{R^n + 1}{2} \right) \max_{|z| = 1} |p(z)| - \left( \frac{R^n - 1}{2} \right) \min_{|z| = 1} |p(z)|. \tag{1.6}
\]

Both these inequalities are also sharp and equality in each holds for polynomials having all its zeros on \(|z| = 1\).

As refinement of inequalities (1.5) and (1.6), Dewan et al. [8, 9] proved that under the same hypothesis, for every \(|\beta| \leq 1, R > 1\) and \(|z| = 1\) we have

\[
|zp'(z) + \frac{n\beta}{2} p(z)| \leq \frac{n}{2} \left\{ \left| \left| 1 + \frac{\beta}{2} \right| + \left| -\frac{\beta}{2} \right| \right| \max_{|z| = 1} |p(z)| - \left| \left| 1 + \frac{\beta}{2} \right| - \left| -\frac{\beta}{2} \right| \right| \min_{|z| = 1} |p(z)| \right\}, \tag{1.7}
\]

and

\[
|p(Rz) + \beta \left( \frac{R + 1}{2} \right)^n p(z)| \leq \frac{1}{2} \left\{ \left( \left| R^n + \beta \left( \frac{R + 1}{2} \right)^n \right| + \left| 1 + \beta \left( \frac{R + 1}{2} \right)^n \right| \right) \max_{|z| = 1} |p(z)| - \left( \left| R^n + \beta \left( \frac{R + 1}{2} \right)^n \right| - \left| 1 + \beta \left( \frac{R + 1}{2} \right)^n \right| \min_{|z| = 1} |p(z)| \right\}. \tag{1.8}
\]
Also they [8] proved if \( p(z) \) has all its zeros in \( |z| \leq 1 \), then for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), we have

\[
\min_{|z|=1} \left| z p'(z) + \frac{n \beta}{2} p(z) \right| \geq n \left| 1 + \frac{\beta}{2} \right| \min_{|z|=1} |p(z)|. \tag{1.9}
\]

**Definition 1.1** (see [16]). Let \( \mathcal{P}_n \) be the class of all polynomials of degree at most \( n \), and let \( T \) be a linear operator carrying polynomials in \( \mathcal{P}_n \) into polynomials in \( \mathcal{P}_n \). We say \( T \) is a \( B_n \)-operator if, for every polynomial \( p \) of degree \( n \) which having all its zeros in the closed unit disc (\( |z| \leq 1 \)), \( T(p) \) also has all its zeros in the closed unit disc.

It is an interesting problem, as pointed out by Professor Q. I. Rahman to characterize all such operators. As an attempt to this characterization, it proved [16] that the operator \( B \) which carries a polynomial \( p(z) \) into polynomial

\[
B[p(z)] := \lambda_0 p(z) + \lambda_1 \left( \frac{nz}{2} \right) \frac{p'(z)}{1!} + \lambda_2 \left( \frac{nz}{2} \right)^2 \frac{p''(z)}{2!}
\]

is a \( B_n \)-operator if all the zeros of

\[
u(z) = \lambda_0 + c(n,1) \lambda_1 z + c(n,2) \lambda_2 z^2
\]

lie in the half plane

\[
|z| \leq \left| z - \frac{n}{2} \right|.
\]

As an extension of Bernstein’s inequality, it was observed by Rahman [15], that if \( |p(z)| \leq M \) for \( |z| = 1 \), then

\[
|B[p(z)]| \leq M |B[z^n]|, \quad |z| \geq 1.
\]

In this paper, we first prove the following theorem and obtain a compact generalization of inequality (1.9).

**Theorem 1.1.** If \( p(z) \) is a polynomial of degree \( n \) having all zeros in \( |z| \leq 1 \), then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1 \) and \( |z| \geq 1 \), we have

\[
\left| R^n - \alpha r^n + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right) r^n \| \min_{|z|=1} |p(z)||B[z^n]| \right|
\leq \left| B[p(Rz)] - \alpha B[p(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right) B[p(rz)] \right|. \tag{1.14}
\]

If we take \( \alpha = 1 \) in Theorem 1.1, then we have
Corollary 1.1. If $p(z)$ is a polynomial of degree $n$ having all zeros in $|z| \leq 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$, we have

$$R^n - r^n + \beta \left( \frac{R+1}{r+1} \right)^n - 1 \right) r^n \left| \min_{|z|=1} |p(z)| \right| B[z^n] \right| \leq \left| B[p(Rz)] - B[p(rz)] + \beta \left( \frac{R+1}{r+1} \right)^n - 1 \right) B[p(rz)] \right|. \quad (1.15)$$

Dividing the two sides of the inequality (1.15) by $(R-r)$ and then making $R \to r$. Since the operator $B$ is linear, we get the following interesting result.

Corollary 1.2. If $p(z)$ is a polynomial of degree $n$ having all zeros in $|z| \leq 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $r \geq 1$ and $|z| \geq 1$, we have

$$n \left| \frac{r^n-1}{r+1} \beta \right| B[z^n] \left| \min_{|z|=1} |p(z)| \right| \leq \left| B[zp'(rz)] + \frac{n\beta}{r+1} B[p(rz)] \right|. \quad (1.16)$$

Taking $\lambda_1 = \lambda_2 = 0$ in Corollary 1.2, we obtain

Corollary 1.3. If $p(z)$ is a polynomial of degree $n$ having all zeros in $|z| \leq 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $r \geq 1$ and $|z| \geq 1$, we have

$$n \left| \frac{r^n-1}{r+1} \beta \right| B[z^n] \left| \min_{|z|=1} |p(z)| \right| \leq |zp'(rz) + \frac{n\beta}{r+1} p(rz)|. \quad (1.17)$$

Inequality (1.17) reduce to inequality (1.9) if we take $r = 1$ and $|z| = 1$. By using Theorem 1.1, we prove the following theorem, which provides a generalization of inequalities (1.7), (1.8).

Theorem 1.2. If $p(z)$ is a polynomial of degree $n$, having no zeros in $|z| < 1$, then for all $a, \beta \in \mathbb{C}$ with $|a| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, $|z| \geq 1$,

$$|B[p(Rz)] - \lambda_0 B[p(rz)] + \beta \left( \frac{R+1}{r+1} \right)^n - |a| B[z^n]| \leq \left( \left| R^n - r^n + \beta \left( \frac{R+1}{r+1} \right)^n - |a| B[z^n] \right| + 1 + \beta \left( \frac{R+1}{r+1} \right)^n - |a| \right) \left| \lambda_0 \right| \max_{|z|=1} |p(z)| \right|.

Equality holds for the polynomial having all zeros on the unit disk.

If we take $\alpha = \lambda_1 = \lambda_2 = 0$ in Theorem 1.1, then we have

Corollary 1.4. If $p(z)$ is a polynomial of degree $n$ such that having no zeros in $|z| < 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$|p(Rz) + \beta \left( \frac{R+1}{r+1} \right)^n p(z)| \leq \left\{ \left| R^n + \beta \left( \frac{R+1}{r+1} \right)^n \right| + \left| 1 + \beta \left( \frac{R+1}{r+1} \right)^n \right| \right\} \max_{|z|=1} |p(z)| \right|.$$

Equality holds for the polynomial having all zeros on the unit disk.
If we take $r = 1$ in Corollary (1.4) then inequality (1.19) reduce to inequality (1.8). Theorem 1.2 reduces to the following result by taking $\alpha = 1$.

**Corollary 1.5.** If $p(z)$ is a polynomial of degree $n$ such that having no zeros in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$, we have

$$
|B[p(Rz)] - B[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - 1 \right\} B[p(rz)] | \\
\leq \frac{1}{2} \left\{ \left| R^n - r^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - 1 \right\} r^n \right| B|z^n| + \left| \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - 1 \right\} |\lambda_0| \right| \right\} \max_{|z|=1} |p(z)| \\
- \left| \left| R^n - r^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - 1 \right\} r^n \right| B|z^n| - \left| \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - 1 \right\} |\lambda_0| \right| \min_{|z|=1} |p(z)| \}. \quad (1.20)
$$

Dividing the two sides of the inequality (1.20) by $(R-r)$ and then making $R \to r$. Since the operator $B$ is linear, we get the following interesting result.

**Corollary 1.6.** If $p(z)$ is a polynomial of degree $n$ such that having no zeros in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $r \geq 1$ and $|z| \geq 1$, we have

$$
|B[p(Rz)] - B[p(rz)] + \frac{n\beta}{r+1} B[p(rz)] | \leq \frac{n}{2} \left\{ \left| r^n + \frac{\beta}{r+1} r^n \right| B|z^n| + \left| \frac{\beta}{r+1} |\lambda_0| \right| \max_{|z|=1} |p(z)| \\
- \left| \left| r^n + \frac{\beta}{r+1} r^n \right| B|z^n| - \left| \frac{\beta}{r+1} |\lambda_0| \right| \min_{|z|=1} |p(z)| \right\}. \quad (1.21)
$$

Taking $\lambda_1 = \lambda_2 = 0$ in Corollary 1.6, we obtain

**Corollary 1.7.** If $p(z)$ is a polynomial of degree $n$ such that having no zeros in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $r \geq 1$ and $|z| \geq 1$, we have

$$
|zp'(rz) + \frac{n\beta}{r+1} p(rz) | \leq \frac{n}{2} \left\{ \left| r^{n-1} + \frac{\beta}{r+1} r^n \right| n |z^n| + \left| \frac{\beta}{r+1} \right| \max_{|z|=1} |p(z)| \\
- \left| \left| r^{n-1} + \frac{\beta}{r+1} r^n \right| n |z^n| - \left| \frac{\beta}{r+1} \right| \min_{|z|=1} |p(z)| \right\}. \quad (1.22)
$$

Inequality (1.22) reduce to inequality (1.7) if we take $r = 1$ and $|z| = 1$. If we take $\beta = 0$ in Theorem 1.2, then we have

**Corollary 1.8.** If $p(z)$ be a polynomial of degree $n$ such that having no zeros in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$, we have

$$
|B[p(Rz)] - \alpha B[p(rz)] | \leq \frac{1}{2} \left\{ \left| R^n - \alpha r^n \right| n B|z^n| + |1-\alpha||\lambda_0| \right\} \max_{|z|=1} |p(z)| \\
- \left\{ \left| R^n - \alpha r^n \right| n B|z^n| - |1-\alpha||\lambda_0| \right\} \min_{|z|=1} |p(z)| \}. \quad (1.23)
$$
If we take $\alpha = \beta = 0$ in Theorem 1.2, then the inequality (1.18) reduces to a result that recently proved by Shah and Liman [17].

Finally in this paper, we establish the following result for self inverse polynomial. Recall that $p(z)$ is a self inverse polynomial, if $p(z) = uq(z)$, where $u \in \mathbb{C}$ with $|u| = 1$ and $q(z) = z^n p(1/z)$.

**Theorem 1.3.** If $p(z)$ is a self inverse polynomial of degree $n$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1, R > r \geq 1, |z| \geq 1$,

\[
\left| B[p(Rz)] - \alpha B[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)] \right|
\leq \frac{1}{2} \left\{ \left| R^n - \alpha r^n + \beta \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right| r^n |B[z^n]|
+ |1 - \alpha + \beta \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right| |\lambda_0| \right\} \max_{|z|=1} |p(z)|. \tag{1.24}
\]

Equality holds for $p(z) = z^n + 1$.

Similar the above we have the following results by suitable choosing $\alpha, \beta, \lambda_0, \cdots$.

**Corollary 1.9.** If $p(z)$ is a self inverse polynomial of degree $n$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1, R > r \geq 1, |z| \geq 1$,

\[
|p(Rz) + \beta \left( \frac{R+1}{r+1} \right)^n p(z)| \leq \frac{1}{2} \left\{ \left| R^n + \beta \left( \frac{R+1}{r+1} \right)^n \right| + \left| 1 + \beta \left( \frac{R+1}{r+1} \right)^n \right| \right\} \max_{|z|=1} |p(z)|. \tag{1.25}
\]

Inequality (1.25) in particular case gives the following result for self inverse polynomial $p(z)$

\[
\max_{|z|=R>1} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|.
\]

**Corollary 1.10.** If $p(z)$ is a self inverse polynomial of degree $n$, then for $\beta \in \mathbb{C}$ with $|\beta| \leq 1, r \geq 1, |z| \geq 1$,

\[
\left| z p'(rz) + \frac{n \beta}{r+1} p(rz) \right| \leq \frac{n}{2} \left\{ \left| r^{n-1} + \frac{\beta}{r+1} \right| |z^n| + \left| \frac{\beta}{r+1} \right| \right\} \max_{|z|=1} |p(z)|. \tag{1.26}
\]

Inequality (1.26) in particular case gives the following result for self inverse polynomial $p(z)$

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.
\]
2 Lemmas

For proof of the theorems, we need the following lemmas. The first lemma follow from Corollary 18.3 of [14, pp. 86].

Lemma 2.1. If all the zeros of a polynomial $p(z)$ of degree $n$ lie in a circle $|z| \leq 1$, then all the zeros of the polynomial $B[p(z)]$ also lie in the circle $|z| \leq 1$.

The following lemma is due to Aziz and Zargar [6].

Lemma 2.2. If $q(z)$ is a polynomial of degree $n$, having all zeros in the closed disk $|z| \leq k$, where $k \geq 0$, then for every $R \geq r$ and $rR \geq k^2$,

$$|p(Rz)| \geq \left( \frac{R+k}{r+k} \right)^n |p(rz)|, \quad |z| = 1. \quad (2.1)$$

If we take $k=1$ in Lemma 2.2, we have

Lemma 2.3. If $p(z)$ is a polynomial of degree $n$, having all zeros in the closed disk $|z| \leq 1$, then for every $R \geq r \geq 1$,

$$|p(Rz)| \geq \left( \frac{R+1}{r+1} \right)^n |p(rz)|, \quad |z| = 1. \quad (2.2)$$

Lemma 2.4. Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n$ having no zeros in $|z| < 1$, and $q(z) = z^n p(1/z) = \sum_{i=0}^{n} a_n-i z^i$. Then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$, we have

$$|B[p(Rz)] - \alpha B[p(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right) B[p(rz)]| \leq |B[q(Rz)] - \alpha B[q(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right) B[q(rz)]|. \quad (2.3)$$

Equality holds if $q(z) = e^{i\theta} p(z)$ for some $\theta \in \mathbb{R}$.

Proof. Let $q(z) \neq e^{i\theta} p(z)$ for all $\theta \in \mathbb{R}$. Since $|p(z)| = |q(z)|$ for $|z| = 1$, therefore the function $\phi(z) = q(z)/p(z)$ is analytic in the disc $|z| < 1+\epsilon$ for some $\epsilon > 0$ and $|\phi(z)| = 1$ on $|z| = 1$. Hence by the maximum modulus principle $|\phi(z)| < 1$ for $|z| < 1$, or equivalently $|p(z)| < |q(z)|$ for $|z| > 1$ and $|a_n|/|a_0| = |\phi(0)| < 1$. It follows that for $\delta$ with $|\delta| < 1$ the polynomial $q(z) + \delta p(z)$ is of degree $n$ and by using Rouche’s Theorem, it is obvious that all the zeros of $H(z) := q(z) + \delta p(z)$ lie in $|z| \leq 1$. On applying Lemma 2.3, for $H(z)$, we have

$$|H(Rz)| \geq \left( \frac{R+1}{r+1} \right)^n |H(rz)| > |H(rz)|, \quad |z| = 1. \quad (2.4)$$

where $R > r \geq 1$. 

It implies that for any \( \alpha \in \mathbb{C} \) with \(|\alpha| \leq 1\), we get

\[
|H(Rz) - \alpha H(rz)| \geq |H(Rz)| - |\alpha||H(rz)| \geq \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\}|H(rz)|, \quad |z| = 1,
\]

i.e.,

\[
|H(Rz) - \alpha H(rz)| \geq \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\}|H(rz)|, \quad |z| = 1. \tag{2.5}
\]

Since \( H(Rz) \) has all its zeros in \(|z| \leq 1/R < 1\), and \( |H(rz)| < |H(Rz)| \), a direct application of Rouche's Theorem shows that the polynomial \( H(Rz) - \alpha H(rz) \) has all its zeros in \(|z| < 1\). By using again Rouche's Theorem, it follows that for every \( \beta \in \mathbb{C} \) with \(|\beta| < 1\) and \( R > r \geq 1\), all the zeros of the polynomial

\[
T(z) = H(Rz) - \alpha H(rz) + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} H(rz)
\]

lie in \(|z| < 1\).

On applying Lemma 2.1 to the polynomial \( T(z) \), it follows the polynomial \( B[T(z)] \) has all its zeros in \(|z| < 1\). Replacing \( H(z) \) by \( q(z) + \delta p(z) \), since \( B \) is linear, we conclude that all the zeros of

\[
B[T(z)] = B[q(Rz)] - \alpha B[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz)]
\]

\[
+ \delta \left\{ B[p(Rz)] - \alpha B[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)] \right\} \tag{2.6}
\]

lie in \(|z| < 1\), for every \( R > r \geq 1 \), \(|\alpha| \leq 1\), \(|\beta| < 1\) and \(|\delta| < 1\). This implies

\[
\left| B[p(Rz)] - \alpha B[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)] \right|
\]

\[
\leq \left| B[q(Rz)] - \alpha B[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz)] \right|, \tag{2.7}
\]

where \(|z| \geq 1\).

If the inequality (3.2) is not true, then there is a point \( z = z_0 \) with \(|z_0| \geq 1\) such that

\[
\left| B[p(Rz_0)] - \alpha B[p(rz_0)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz_0)] \right|
\]

\[
> \left| B[q(Rz_0)] - \alpha B[q(rz_0)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz_0)] \right|.
\]

Take

\[
\delta = \frac{- B[q(Rz_0)] - \alpha B[q(rz_0)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz_0)]}{B[p(Rz_0)] - \alpha B[p(rz_0)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz_0)]},
\]

then \(|\delta| < 1\) and with this choice of \( \delta \), we have, \( B[T(z_0)] = 0 \) for \(|z_0| \geq 1\) from (3.1). But this contradicts the fact that all the zeros of \( B[T(z)] \) lie in \(|z| < 1\). For \( \beta \), with \(|\beta| = 1\), (3.2) follows by continuity. This completes the proof. \(\square\)
The following lemma is due to Gardner, Govil and Musukula [10].

**Lemma 2.5.** If \( p(z) = \sum_{i=0}^{n} a_i z^i \) is a polynomial of degree \( n \), \( p(z) \neq 0 \) in \(|z| < k\), \((k > 0)\), then \( m < |p(z)| \) for \(|z| < k\), and in particular \( \min_{|z|=k} |p(z)| < |a_0| \).

In the lines of Lemma 2.5, one can easily prove the following.

**Lemma 2.6.** If \( p(z) = \sum_{i=0}^{n} a_i z^i \) is a polynomial of degree \( n \) having all zeros in \(|z| \leq 1\), then \( \min_{|z|=1} |p(z)| < |a_n| \).

**Proof.** Since the polynomial \( p(z) \) has all zeros in \(|z| \leq 1\), then the polynomial \( q(z) = z^n p(1/\bar{z}) = \bar{a}_n + \bar{a}_{n-1}z + \cdots + \bar{a}_1 z^{n-1} + \bar{a}_0 z^n \), has no zero in \(|z| < 1\). Thus by applying Lemma 2.5 for the polynomial \( q(z) \), we get

\[
\min_{|z|=1} |q(z)| < |a_n|.
\]  
(2.8)

Since \( \min_{|z|=1} |q(z)| = \min_{|z|=1} |p(z)| \) then (2.8) implies that \( \min_{|z|=1} |p(z)| < |a_n| \). \( \square \)

**Lemma 2.7.** Let \( p(z) \) be a polynomial of degree \( n \), then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1, \) and \( |z| \geq 1 \), we have

\[
\begin{align*}
|B[p(Rz)] - \alpha B[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)]| & \\
+ |B[q(Rz)] - \alpha B[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz)]| & \\
\leq & \left\{ |R^n - \alpha r^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} r^n \right\} \max_{|z|=1} |B[z^n]| \\
& \cdot \left\{ 1 - |\alpha| + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} |\lambda_0| \right\} |\lambda_0|^{-|\alpha|}.
\end{align*}
\]  
(2.9)

where \( q(z) = z^n p(1/\bar{z}) \).

**Proof.** Let \( M = \max_{|z|=1} |p(z)| \). For any \( \delta \) with \(|\delta| > 1\), it follows by Rouche’s Theorem that the polynomial \( G(z) = p(z) - \delta M \) has no zeros in \(|z| < 1\). If we take \( H(z) = z^n G(1/\bar{z}) \), then \( |G(z)| = |H(z)| \) for \(|z| = 1\). On applying Lemma 2.4, we have for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1 \) and \( |z| \geq 1 \),

\[
\begin{align*}
|B[G(Rz)] - \alpha B[G(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[G(rz)]| & \\
\leq & B[H(Rz)] - \alpha B[H(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[H(rz)].
\end{align*}
\]  
(2.10)

Therefore by the equalities

\[
H(z) = z^n G \left( \frac{1}{\bar{z}} \right) = z^n p \left( \frac{1}{\bar{z}} \right) - \delta M z^n = q(z) - \delta M z^n,
\]
or
\[ H(z) = q(z) - \bar{\delta}Mz^n, \]
and using the fact that \( B \) is linear and \( B[1] = \lambda_0 \), we get

\[
\begin{align*}
&\left| \left\{ B[p(Rz)] - aB[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} B[p(rz)] \right\} \\
&\quad - \delta \left\{ 1 - a + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} \right\} M\lambda_0 \right| \\
&\leq \left| \left\{ B[q(Rz)] - aB[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} B[q(rz)] \right\} \\
&\quad - \delta \left\{ R^n - ar^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} r^n \right\} MB[z^n] \right| \\
&\quad + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} B[q(rz)].
\end{align*}
\]

(2.11)

Now by suitable choice of argument of \( \delta \), we get

\[
\begin{align*}
&\left| \left\{ B[q(Rz)] - aB[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} B[q(rz)] \right\} \\
&\quad - \delta \left\{ R^n - ar^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} r^n \right\} MB[z^n] \right| \\
&= \delta \left| R^n - ar^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} r^n \right| M|B[z^n]| - |B[q(Rz)] - aB[q(rz)]| \\
&\quad + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} B[q(rz)].
\end{align*}
\]

(2.12)

By combining right hand sides of (2.11) and (2.12) we can obtain

\[
\begin{align*}
&\left| B[p(Rz)] - aB[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} B[p(rz)] \right| \\
&\quad - |\delta| \left| 1 - a + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} \right| M|\lambda_0| \\
&\leq |\delta| \left| R^n - ar^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} r^n \right| M|B[z^n]| \\
&\quad - |B[q(Rz)] - aB[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} B[q(rz)]|,
\end{align*}
\]

which implies

\[
\begin{align*}
&\left| B[p(Rz)] - aB[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} B[p(rz)] \right| \\
&\quad + |B[q(Rz)] - aB[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} B[q(rz)]| \\
&\leq |\delta| \left| |R^n - ar^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} r^n \right| |B[z^n]| \\
&\quad + \left| 1 - a + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |a| \right\} \right| |\lambda_0| \right| M.
\end{align*}
\]

Making \(|\delta| \to 1\), we have the result. \(\square\)
Lemma 2.8. Let $p(z)$ be a polynomial of degree $n$ having no zeros in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$, we have
\[
|B[p(Rz)] - aB[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)]| \\
\leq \frac{1}{2} \left\{ \left| R^n - ar^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} r^n \right| |B[z^n]| \\
+ |1 - \alpha + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} ||\lambda_0|| \right\} \max |p(z)|. \tag{2.13}
\]

Proof. Since $p(z)$ does not vanish in $|z| < 1$, Lemma 2.4, yields
\[
|B[p(Rz)] - aB[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)]| \\
\leq |B[q(Rz)] - aB[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz)]|. \tag{2.14}
\]
Now by combining the inequality (2.9) and inequality (2.14), we have
\[
2|B[p(Rz)] - aB[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)]| \\
\leq |B[p(Rz)] - aB[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)]| \\
+ |B[q(Rz)] - aB[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz)]| \\
\leq \left\{ R^n - ar^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} r^n \right| |B[z^n]| \\
+ |1 - \alpha + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} ||\lambda_0|| \right\} \max |p(z)|. \tag{2.15}
\]
This gives the result. \hfill \Box

3 Proof of the theorems

Proof of the Theorem 1.1. If $p(z)$ has a zero on $|z| = 1$, then inequality is trivial. Therefore we assume that $p(z)$ has all its zeros in $|z| < 1$. If $m = \min_{|z|=1} |p(z)|$, then $m > 0$ and $|p(z)| \geq m$ for $|z| = 1$. Therefore, if $|\lambda| < 1$ then it follows by Rouche’s Theorem that the polynomial $p(z) - \lambda mz^n$, has all its zeros in $|z| < 1$. Also by using Lemma 2.6 the polynomial $G(z) = p(z) - \lambda mz^n$ is of degree $n$, for $|\lambda| < 1$. On applying Lemma 2.1 it follows the polynomial $B[G(z)]$ has all its zeros in $|z| < 1$. Replacing $G(z)$ by $p(z) - \lambda mz^n$, since $B$ is linear, we conclude that all the zeros of
\[
B[G(z)] = B[p(Rz)] - aB[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)] + \lambda m \{ B[R^nz^n] \\
- aB[r^nz^n] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[r^nz^n] \} \tag{3.1}
\]
Since the polynomial \( G \) or \( m \), no zeros in \( |z| > 1 \). Using Rouche’s Theorem we conclude that the polynomial \( G(rz) \) has all zeros in \( |z| < 1 \). Consider the polynomial \( H(z) = z^n G(1/z) \), then \( |G(z)| = |H(z)| \) for \( |z| = 1 \). Therefore, by applying Lemma 2.4 for the polynomials \( G(z) \) and \( H(z) \), we obtain

\[
|B[G(rz)] - \alpha B[G(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[G(rz)]| \\
\leq |B[H(Rz)] - \alpha B[H(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[H(rz)]|.
\] (3.3)

Using the fact that

\[
H(z) = z^n G(\frac{1}{z}) = z^n p(\frac{1}{z}) - \tilde{\gamma} mz^n = q(z) - \tilde{\gamma} mz^n,
\]
or

\[
H(z) = q(z) - \tilde{\gamma} mz^n,
\]
and substitute \( G(z) \) and \( H(z) \) in (3.3) we get

\[
\left\{ \left[ B[p(Rz)] - \alpha B[p(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)] \right] \\
- \gamma \left\{ 1 - \alpha + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} \right\} B[m] \right\} \\
\leq \left\{ \left[ B[q(Rz)] - \alpha B[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz)] \right] \\
- \tilde{\gamma} \left\{ R^n - \alpha r^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} r^n \right\} m B[z^n] \right\}.
\] (3.4)

Since the polynomial \( q(z) = z^n p(1/z) \) has all zeros in \(|z| \leq 1 \) and \( m = \min_{|z|=1} |p(z)| = \min_{|z|=1} |q(z)| \), by applying Theorem 1.1 for the polynomial \( q(z) \), we obtain

\[
|\gamma| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} r^n |m B[z^n]| \\
\leq |B[q(Rz)] - \alpha B[q(rz)] + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz)]|, \quad |\gamma| < 1.
\]
Therefore by suitable choice of argument of $\gamma$, we get
\[
\begin{align*}
\left\{ |B[q(Rz)] - aB[q(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[q(rz)] \right\} \\
- \gamma \left\{ R^n - ar^n + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) \right\} m |B[z^n]| \\
= |B[q(Rz)] - aB[q(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[q(rz)]| \\
- |\gamma| R^n - ar^n + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) r^n |B[z^n]|.
\end{align*}
\]
(3.5)

Also combining (3.4) and (3.5) we get
\[
\begin{align*}
\left\{ |B[p(Rz)] - aB[p(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[p(rz)] \right\} \\
- |\gamma| |1 - a + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[z^n]| \\
\leq |B[q(Rz)] - aB[q(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[q(rz)]| \\
- |\gamma| R^n - ar^n + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) r^n |B[z^n]|.
\end{align*}
\]
This implies
\[
\begin{align*}
\left\{ |B[p(Rz)] - aB[p(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[p(rz)] \right\} \\
\leq |B[q(Rz)] - aB[q(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[q(rz)]| \\
- |\gamma| \left\{ R^n - ar^n + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) r^n \right\} m |B[z^n]| \\
- |1 - a + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[z^n]|.
\end{align*}
\]
Making $|\gamma| \to 1$, we have
\[
\begin{align*}
\left\{ |B[p(Rz)] - aB[p(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[p(rz)] \right\} \\
\leq |B[q(Rz)] - aB[q(rz)] + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[q(rz)]| \\
- \left\{ R^n - ar^n + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) r^n \right\} m |B[z^n]| \\
- |1 - a + \beta \left( \left( \frac{R+1}{r+1} \right)^n - |z| \right) |B[z^n]|.
\end{align*}
\]
(3.6)
On the other hand, by Lemma 2.7, we have
\[
\left| B[p(Rz)] - \alpha B[p(rz)] \right| + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[p(rz)] \\
+ \left| B[q(Rz)] - \alpha B[q(rz)] \right| + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[q(rz)] \\
\leq \left\{ R^n - \alpha r^n + \beta \left\{ \left( \frac{R+1}{r+1} \right)^n - |\alpha| \right\} r^n \right\} \max |p(z)|.
\] (3.7)

Considering the inequalities (3.6) and (3.7) together gives the result.

Proof of the Theorem 1.3. Since \( p(z) \) is a self inverse, we have for some \( u \), where \( |u| = 1 \), \( p(z) = uq(z) \). This gives \( B[p(z)] = B[q(z)] \) for all \( z \in \mathbb{C} \). In particular
\[
B[p(Rz)] = B[q(Rz)] \quad \text{and} \quad B[p(rz)] = B[q(rz)] \quad \text{for} \quad R > r \geq 1, \quad |z| = 1.
\] (3.8)

Lemma 2.7 in conjunction with (3.8) gives the result.

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