

On a Pair of Operator Series Expansions Implying a Variety of Summation Formulas

Leetsch C. Hsu*

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

Received 20 May 2015; Accepted (in revised version) 20 June 2015

Abstract. With the aid of Mullin-Rota's substitution rule, we show that the Sheffer-type differential operators together with the delta operators Δ and D could be used to construct a pair of expansion formulas that imply a wide variety of summation formulas in the discrete analysis and combinatorics. A convergence theorem is established for a fruitful source formula that implies more than 20 noted classical formulas and identities as consequences. Numerous new formulas are also presented as illustrative examples. Finally, it is shown that a kind of lifting process can be used to produce certain chains of (∞^m) degree formulas for $m \geq 3$ with $m \equiv 1 \pmod{2}$ and $m \equiv 1 \pmod{3}$, respectively.

Key Words: Delta operator, Sheffer-type operator, (∞^m) degree formula, triplet, lifting process.

AMS Subject Classifications: 12E10, 13F25, 16S32, 65B10

1 Introduction and preliminaries

Throughout this paper the theory of formal power series and of differential operators will be utilized. \mathbf{R} , \mathbf{C} , \mathbf{N} and \mathbf{Z} denote, respectively, the sets of real numbers, complex numbers, natural numbers (including 0) and rational integers. Generally, we will use $A(t)$, $g(t)$, $f(t)$, $\varphi(t)$, etc. to denote either the formal power series (fps) or the infinitely differentiable functions (members of C^∞) defined in \mathbf{R} or \mathbf{C} .

We will make use of the ordinary operators Δ (difference), D (differentiation) and E (shift operator) which are defined by the relations respectively

$$\Delta f(t) = f(t+1) - f(t), \quad Df(t) = \frac{d}{dt}f(t), \quad Ef(t) = f(t+1).$$

Consequently they satisfy some simple symbolic relations such as

$$E = 1 + \Delta, \quad E = e^D, \quad \Delta = e^D - 1, \quad D = \log(1 + \Delta),$$

*Corresponding author. Email address: xulizhi63@hotmail.com (L. C. Hsu)

where 1 serves as an identity operator such that $1f(x) = f(x)$. Also, we define $E^\alpha f(t) = f(t+\alpha)$ for $\alpha \in \mathbf{R}$ or \mathbf{C} with $E^0 = D^0 = \Delta^0 = 1$. Hereafter, we always assume that all the operators are acting at the variable t of a function or a fps.

Generally, an operator T is called a shift-invariant operator (see [8]) if $TE^\alpha = E^\alpha T$ for every $\alpha \in \mathbf{R}$. Moreover, if in addition, $Tt \neq 0$ (a non-zero constant), then T is called a delta operator. Obviously, both Δ and D are delta operators.

We shall frequently utilize a general proposition due to Mullin and Rota as stated in what follows (cf. [8]).

Let Q be a delta operator, and let Γ be the ring of formal power series in t , over the same number field (\mathbf{R} or \mathbf{C}). Then there exists an isomorphism from Γ into the ring Σ of shift-invariant operators, which carries

$$f(t) = \sum_{k \geq 0} \frac{a_k}{k!} t^k \quad \text{into} \quad f(Q) \equiv f(t, Q) = \sum_{k \geq 0} \frac{a_k}{k!} Q^k.$$

The above assertion is called Mullin-Rota's substitution rule.

For the fps $f(t)$ given above, note that $D^k f(t) \equiv f^{(k)}(t)$ means a k th order formal derivative of $f(t)$, so that $D^k f(0) \equiv f^{(k)}(0) = a_k$, and that $f(t)$ can also be written as a formal Taylor series.

Also, we shall need three basic concepts as given by the following definitions.

Definition 1.1. For any given fps $A(t)$ and $g(t)$ such that $A(0) = 1$, $g(0) = 0$ and $g'(0) = Dg(0) \neq 0$, the polynomials $p_k(z)$ ($k \in \mathbf{N}$), defined by the generating function (GF)

$$A(t)e^{zg(t)} = \sum_{k \geq 0} p_k(z)t^k \tag{1.1}$$

are called the Sheffer-type polynomials, where $p_0(z) = 1$. More explicitly, we may denote

$$p_k(z) \equiv p_k(z, A(t), g(t)) = [t^k] A(t)e^{zg(t)}, \tag{1.2}$$

where $[t^k]$ is the so-called extracting coefficient operator.

Accordingly, $p_k(D)$ with $D \equiv d/dt$ is called the Sheffer-type differential operator of degree k . In particular, $p_0(D) \equiv 1$ is the identity operator.

Definition 1.2. Any expansion formula or a summation formula in the theory of formal power series as well as in the computational analysis is called an (∞^m) degree formula if it consists of m arbitrary functions that could be chosen in infinitely many ways, where m is called the freedom degree of the formula.

For example, the expansion (1.1) is an (∞^2) degree formula.

Definition 1.3. For $A(t)$ and $g(t)$ as given in Definition 1.1, the numbers d_{kj} as defined by (cf. [6, 7, 12] etc).

$$d_{kj} = [t^k] A(t)(g(t))^j, \quad 0 \leq j \leq k \in \mathbf{N}, \tag{1.3}$$

are said to form a Riordan array (d_{kj}) which may be denoted by $(A(t), g(t))$.

In substance, this paper consists of two main parts plus a few remarks. The first part is concerned with a pair of (∞^4) degree expansion formulas that could lead to various specializations and examples. The object of the second part is to investigate a source formula in some details. It will be shown that the source formula is really a particular consequence of an (∞^4) degree formula, and it becomes an exact formula under certain general convergence conditions. Finally, as one of concluding remarks, we will explain why there could exist an infinitely many (∞^m) formulas via a kind of lifting process.

2 A pair of (∞^4) degree formulas

A main result to be presented in this section is a basic theorem that involves a pair of (∞^4) degree expansion formulas.

Theorem 2.1. *Let $A(t)$, $g(t)$ and $f(t)$ be fps such that $A(0)=1$, $g(0)=0$ and $g'(0)=Dg(0)\neq 0$. Suppose that $p_k(D)$ ($k \in \mathbf{N}$) are the Sheffer-type differential operators associated with $A(t)$ and $g(t)$. Then for any given $\phi(t) \in C^\infty$ there hold formally a pair of (∞^4) degree expansion formulas as follows*

$$A(D)f(g(D))\phi(t) = \sum_{k \geq 0} (p_k(D)f(0))D^k\phi(t), \quad (2.1a)$$

$$A(\Delta)f(g(\Delta))\phi(t) = \sum_{k \geq 0} (p_k(D)f(0))\Delta^k\phi(t), \quad (2.1b)$$

where $p_k(D)$ have the explicit expression

$$p_k(D) = \sum_{j=0}^k \left(\frac{1}{j!} d_{kj} \right) D^j, \quad (2.2)$$

with $(d_{kj}) = (A(t), g(t))$ being the Riordan array. Moreover, if it is assumed that

$$\theta = \overline{\lim}_{k \rightarrow \infty} |p_k(D)f(0)|^{1/k} > 0, \quad (2.3)$$

then the expansions (2.1a) and (2.1b) are absolutely convergent at $t=0$, under the following conditions, respectively

$$\overline{\lim}_{k \rightarrow \infty} |D^k\phi(0)|^{1/k} < 1/\theta, \quad (2.4a)$$

$$\overline{\lim}_{k \rightarrow \infty} |\Delta^k\phi(0)|^{1/k} < 1/\theta. \quad (2.4b)$$

Proof. First, notice that the generating function for the Sheffer-type polynomial sequence $\{p_k(z)\}$ given by (1.1) with replacement $t \mapsto x$ is a fps in x as well as in z :

$$A(x) \sum_{k \geq 0} \frac{(g(x))^k z^k}{k!} = \sum_{k \geq 0} p_k(z) x^k. \quad (2.5)$$

Actually this is a formal identity involving arguments z and x . Thus one may apply Mullin-Rota's substitution rule to the formal series (2.5) with replacement $Z \mapsto D$. Accordingly, by letting the resultant operator series act on the fps $f(t)$ at $t = 0$, it gives formally

$$A(x) \sum_{k \geq 0} \frac{(g(x))^k}{k!} (D^k f(t))_{t=0} = \sum_{k \geq 0} x^k (p_k(D) f(t))_{t=0}. \quad (2.6)$$

Observe that the left-hand side (LHS) of (2.6) apart from $A(x)$ is just a formal Taylor series expansion of $f(g(x))$ in powers of $g(x)$. Consequently (2.6) can be rewritten in the form

$$A(x) f(g(x)) = \sum_{k \geq 0} (p_k(D) f(0)) x^k. \quad (2.7)$$

This is actually a known formula (cf. e.g., Roman's [11, Theorems 2.3.2-2.3.3] and He-Hsu-Shiue's [6, Expression (2.3)]). Using the conditions for $A(t)$, $g(t)$ and $f(t)$ as stated in the theorem, one may see that Mullin-Rota's substitution rule can also be applied to the formal identity (2.7). Thus the expansion formulas (2.1a) and (2.1b) can be obtained with the substitutions $x \mapsto D$ and $x \mapsto \Delta$, respectively.

Moreover, it is clear that the condition $\theta|x| < 1$ for the absolute convergence of the expansion (2.7) just follows from Cauchy's root-test. Certainly the similar argument also applies to the conditions (2.4a) and (2.4b).

It remains to verify the equality (2.2). Clearly the condition $Dg(0) = g'(0) \neq 0$ implies that the formal power series expansion in t of the following formal series

$$\sum_{j > k} z^j (g(t))^j / j!$$

involves powers of t greater than k . Thus one may deduce from (1.1) that

$$p_k(z) = [t^k] A(t) \sum_{j=0}^k z^j (g(t))^j / j! = \sum_{j=0}^k [t^k] (A(t) (g(t))^j / j!) z^j = \sum_{j=0}^k (d_{kj} / j!) z^j,$$

where $d_{kj} = [t^k] A(t) (g(t))^j$ with $0 \leq j \leq k \in \mathbf{N}$ just form the Riordan array $(d_{kj}) = (A(t), g(t))$. This completes the proof of the theorem. \square

It may be worth noticing that the formula (2.7) involved in the proof is deducible from either of (2.1a) and (2.1b) as a particular consequence. More Precisely we have the following

Corollary 2.1. Either of the choices $\phi(t) = e^{xt}$ in (2.1a) and $\phi(t) = (1+x)^t$ in (2.1b) yields the (∞^2) degree formula (2.7) that is convergent absolutely under the condition $|x| < 1/\theta$ with θ being defined by (2.3).

Obviously, one may find by easy computations

$$[D^k e^{xt}]_{t=0} = x^k, \quad [\Delta^k (1+x)^t]_{t=0} = x^k, \quad x \in \mathbf{R}, \quad k \in \mathbf{N}.$$

Note that $A(D)f(g(D))$ and $A(\Delta)f(g(\Delta))$ have formal power series expansions in D and Δ , respectively. Thus it follows that

$$\begin{aligned} [A(D)f(g(D))e^{xt}]_{t=0} &= A(x)f(g(x)), \\ [A(\Delta)f(g(\Delta))(1+x)^t]_{t=0} &= A(x)f(g(x)). \end{aligned}$$

Hence (2.7) is implied by either of (2.1a) and (2.1b), with Cauchy-type convergence condition $\theta|x| < 1$.

As suggested by two basic theorems of He-Hsu-Yin's paper [8], we could further give a next corollary of Theorem 2.1. Let

$$G(x, y, z) = F_1(x, y, z) / F_2(x, y, z)$$

denote a rational function, namely, both F_1 and F_2 are polynomials involving the variables x , y and z . Then, as a consequence of Theorem 2.1, we have the following:

Corollary 2.2. Let $A(t)$, $g(t)$ and $f(t)$ be given as in Theorem 2.1. Then we have two statements.

(i) Suppose that $A(t)f(g(t))$ is expressible as a rational function in t , e^t and $e^{\alpha t}$ of the form

$$A(t)f(g(t)) = G(t, e^t, e^{\alpha t}), \quad \alpha \in \mathbf{R} \text{ or } \mathbf{C}.$$

Then there holds a formal expansion formula for any $\phi(t) \in C^\infty$:

$$G(D, E, E^\alpha)\phi(t) = \sum_{k \geq 0} (p_k(D)f(0))D^k\phi(t). \quad (2.8)$$

(ii) For the case $A(t)f(g(t)) = G(t, \log(1+\alpha t), (1+\alpha t)^\beta)$ with $\alpha \neq 0$ and $\beta \in \mathbf{R}$, there holds a formal expansion for any $\phi(t) \in C^\infty$:

$$G\left(\frac{1}{\alpha}\Delta, D, E^\beta\right)\phi(t) = \sum_{k \geq 0} (p_k(D)f(0))\Delta^k\phi(t)/\alpha^k. \quad (2.9)$$

Note that (2.8)-(2.9) are parallel to the related formulas obtained in [8], but entirely different in formula structures. Certainly $A(t)$, $g(t)$ and $f(t)$ mentioned in the statements of the Corollary are not really arbitrary; as they have to satisfy the conditions imposed by (i) or (ii).

In the follow-up Sections 3-5, the reader will see that a great variety of particular consequences (involving new and old formulas) could be deduced from the expansion formulas displayed in this section.

3 Specializations and examples

Several specializations of (2.1a)-(2.1b) may be displayed in what follows.

There are three pairs of (∞^3) degree formulas of the following forms (applicable to $\phi(t) \in C^\infty$):

$$\begin{cases} f(g(D))\phi(t) = \sum_{k \geq 0} (p_k(D)f(0))D^k\phi(t), \\ f(g(\Delta))\phi(t) = \sum_{k \geq 0} (p_k(D)f(0))\Delta^k\phi(t), \end{cases} \quad (3.1a)$$

$$\begin{cases} A(D)f(D)\phi(t) = \sum_{k \geq 0} (p_k(D)f(0))D^k\phi(t), \\ A(\Delta)f(\Delta)\phi(t) = \sum_{k \geq 0} (p_k(D)f(0))\Delta^k\phi(t), \end{cases} \quad (3.1b)$$

and

$$\begin{cases} A(D)\exp(zg(D))\phi(t) = \sum_{k \geq 0} p_k(z)D^k\phi(t), \\ A(\Delta)\exp(zg(\Delta))\phi(t) = \sum_{k \geq 0} p_k(z)\Delta^k\phi(t), \end{cases} \quad (3.2)$$

where $p_k(D)$ contained in (3.1a) has the expression

$$p_k(D) \equiv p_k(D, 1, g(t)) = \sum_{j=0}^k \left(\frac{1}{j!} d_{kj} \right) D^j, \quad k \in \mathbf{N}, \quad (3.3)$$

with $d_{kj} = [t^k](g(t))^j$; the operator $p_k(D)$ in (3.1b) takes the form

$$p_k(D) \equiv p_k(D, A(t), t) = \sum_{j=0}^k \left(\frac{1}{j!} d_{kj} \right) D^j, \quad k \in \mathbf{N}, \quad (3.4)$$

with $d_{kj} = [t^k](A(t), t^j) = [t^{k-j}]A(t)$, ($0 \leq j \leq k$); and $p_k(z)$ involved in (3.2) is a Sheffer-type polynomial, viz

$$p_k(z) = p_k(z, A(t), g(t)) = \sum_{j=0}^k \left(\frac{1}{j!} d_{kj} \right) z^j, \quad k \in \mathbf{N}, \quad (3.5)$$

with d_{kj} being given by the Riordan array $(d_{kj}) = ([t^k]A(t)(g(t))^j)_{0 \leq j \leq k}$.

Certainly one may get further specializations therefrom.

For instance, as special cases of (3.2) with $g(t) = t$, there are two (∞^2) degree formulas of the forms

$$\begin{cases} A(D)\exp(zD)\phi(t) = \sum_{k \geq 0} p_k(z)D^k\phi(t), \\ A(\Delta)\exp(z\Delta)\phi(t) = \sum_{k \geq 0} p_k(z)\Delta^k\phi(t), \end{cases} \quad (3.6)$$

where

$$p_k(z) \equiv p_k(z, A(t), t) = [t^k](A(t)e^{zt})$$

may be written as

$$p_k(z) = \sum_{j=0}^k \left(\frac{1}{j!} d_{kj}\right) z^j, \quad k \in \mathbf{N}, \tag{3.7}$$

the numbers d_{kj} being given by $d_{kj} = [t^{k-j}]A(t)$ ($0 \leq j \leq k$).

By means of practice, one may find that various special formulas and identities could be deduced as consequences from (2.1a)-(2.1b) and some of the specializations mentioned above. In what follows we will give several selected examples.

The first three examples are related to the Touchard polynomials $\tau_k(z)$, the Touchard polynomials $(\text{Tos})_k^{(\lambda)}(z)$ with $\lambda \neq 0$, and the generalized Laguerre polynomials $L_k^{(p-1)}(z)$ with $p > 0$ ($k \in \mathbf{N}$). These polynomials are known as special Sheffer-type polynomials (cf. Boas-Buck [1]). In accordance with the denotation (1.2) we may denote them as follows

$$\begin{aligned} \tau_k(z) &= \tau_k(z, 1, e^t - 1), \\ (\text{Tos})_k^{(\lambda)}(z) &= (\text{Tos})_k^{(\lambda)}(z, e^{\lambda t}, 1 - e^t), \\ L_k^{(p-1)}(z) &= L_k^{(p-1)}(z, (1-t)^{-p}, t/(t-1)). \end{aligned}$$

Accordingly, we could get 3 expansion formulas involving special Sheffer-type operators $\tau_k(D)$, $(\text{Tos})_k^{(\lambda)}(D)$ and $L_k^{(p-1)}(D)$, respectively.

Example 3.1. For the case $A(t) \equiv 1$ and $g(t) = e^t - 1$ we have $A(D) = 1$ (identity operator) and $g(D) = e^D - 1 = E - 1 = \Delta$, so that in accordance with (2.1a) or the first equation of (3.1a), we may get an (∞^2) degree formula of the form (for $\phi(t) \in C^\infty$):

$$f(\Delta)\phi(t) = \sum_{k \geq 0} (\tau_k(D)f(0))\phi^{(k)}(t). \tag{3.8}$$

Herein the Touchard operator $\tau_k(D)$ is given by $\tau_0(D) = 1$ and

$$\tau_k(D) = \sum_{j=0}^k \left(\frac{1}{j!} [t^k](e^t - 1)^j\right) D^j = \frac{1}{k!} \sum_{j=1}^k \left\{ \begin{matrix} k \\ j \end{matrix} \right\} D^j, \quad (k \geq 1), \tag{3.9}$$

with $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$ denoting the Stirling numbers of the second kind in Knuth's notation, viz.

$$\left\{ \begin{matrix} k \\ j \end{matrix} \right\} = \frac{1}{j!} (\Delta^j t^k)_{t=0}.$$

Example 3.2. For the case $A(t) = e^{\lambda t}$ and $g(t) = 1 - e^t$ we have

$$A(D) = e^{\lambda D} = E^\lambda, \quad g(D) = 1 - e^D = 1 - E = -\Delta,$$

and

$$A(D)f(g(D))\phi(t) = E^\lambda f(-\Delta)\phi(t) = f(-\Delta)\phi(\lambda + t).$$

Accordingly, as a consequence of (2.1a) we obtain an (∞^2) degree formula for $\phi(t) \in C^\infty$:

$$f(-\Delta)\phi(\lambda + t) = \sum_{k \geq 0} ((\text{Tos})_k^{(\lambda)}(D)f(0))\phi^{(k)}(t), \tag{3.10}$$

with the Toscano operator being given by the expression

$$(\text{Tos})_k^{(\lambda)}(D) = \sum_{j=0}^k \frac{(-1)^j}{k!} S_2^{(\lambda)}(k, j) D^j. \tag{3.11}$$

Herein $S_2^{(\lambda)}(k, j)$ are known as the second kind of weighted Stirling numbers due to Carlitz [2], which are defined by

$$\frac{1}{j!} e^{\lambda t} (e^t - 1)^j = \sum_{m=j}^{\infty} \frac{t^m}{m!} S_2^{(\lambda)}(m, j), \tag{3.12}$$

so that (3.11) follows from (2.2) and (3.12), namely

$$(\text{Tos})_k^{(\lambda)}(D) = \sum_{j=0}^k \left(\frac{1}{j!} [t^k] e^{\lambda t} (1 - e^t)^j \right) D^j = \text{RHS of (3.11)}. \tag{3.13}$$

Note that the simplest case of (3.10) with $f(t) \equiv 1$ just gives the Taylor expansion of $\phi(\lambda + t)$ in powers of λ , since $S_2^{(\lambda)}(m, 0) = \lambda^m$.

Example 3.3. Given $A(t) = (1 - t)^{-p}$, ($p > 0$) and $g(t) = t/(t - 1)$. We have $A(-\Delta) = (1 + \Delta)^{-p} = E^{-p}$, $g(-\Delta) = (-\Delta)/(-\Delta - 1) = \Delta/E$ and $A(-\Delta)f(g(-\Delta)) = E^{-p}f(\Delta/E)$. Consequently, an application of (2.1b) gives an (∞^2) degree formula of the form

$$f\left(\frac{\Delta}{E}\right)\phi(t - p) = \sum_{k \geq 0} (L_k^{(p-1)}(D)f(0))(-1)^k \Delta^k \phi(t), \tag{3.14}$$

where the Laguerre operator has the expression

$$L_k^{(p-1)}(D) = \sum_{j=0}^k \frac{(-1)^j}{j!} \binom{k+p-1}{k-j} D^j. \tag{3.15}$$

Actually the numbers $(-1)^j \binom{n+p-1}{k-j} = d_{kj}$ involved in (3.15) from the special Riordan array $(d_{kj} = ((1 - t)^p, t/(t - 1)))$.

As is easily seen, the simplest case of (3.14) with $f(t) \equiv 1$ and $t=0$ gives the Newton interpolation series for $\phi(-p)$.

Surely it is possible to get some special formulas or identities from (3.8), (3.10) and (3.14) via suitable choices of $f(t)$ and $\phi(t)$. Here we just mention an application of (3.8) to the derivation of some formulas in Enumerative Combinatorics.

Note that (3.9) leads us to consider the Bell number $\omega(k)$ as defined by

$$\omega(0) = 1, \quad \omega(k) = \sum_{j=1}^k \left\{ \begin{matrix} k \\ j \end{matrix} \right\}, \quad (k \geq 1), \tag{3.16}$$

where $\omega(k)$ is known as the number of ways to partition a set of k things into nonempty subsets. Naturally we have to take $f(t) = e^t$ in order to make (3.8) to yield a useful formula of the form

$$\sum_{k \geq 0} \frac{\Delta^k \phi(t)}{k!} = \sum_{k \geq 0} \frac{\omega(k) \phi^{(k)}(t)}{k!}. \tag{3.17}$$

This formula has some interesting special consequences.

(1) Choosing $\phi(t) = e^{xt}$ with x being a parameter, we have $\Delta^k \phi(0) = (\Delta^k e^{xt})_{t=0} = (e^x - 1)^k$ and $D^k \phi(0) = (D^k e^{xt})_{t=0} = x^k$. Thus we see that (3.17) yields the following equality at $t=0$:

$$e^{(e^x - 1)} = \sum_{k \geq 0} \omega(k) \frac{x^k}{k!}. \tag{3.18}$$

This gives the well-known exponential generating function for the sequence of Bell numbers.

(2) Taking $\phi(t) = m! \binom{t}{m} = (t)_m$ with $(t)_0 = 1$, we have

$$\Delta^k \phi(0) = m! \binom{t}{m-k}_{t=0} = m! \binom{0}{m-k} = m! \delta_{mk} \quad (\text{Kronecker symbol}),$$

and

$$D^k \phi(0) = \left(D^k \sum_{j=0}^m (-1)^{m-j} \begin{bmatrix} m \\ j \end{bmatrix} t^j \right)_{t=0} = (-1)^{m-k} \begin{bmatrix} m \\ k \end{bmatrix} \cdot k!.$$

Consequently (3.17) with $t=0$ yields the following well-known identity

$$\sum_{k=0}^m (-1)^{m-k} \begin{bmatrix} m \\ k \end{bmatrix} \omega(k) = 1, \tag{3.19}$$

where $\begin{bmatrix} m \\ k \end{bmatrix}$ is known as the signless Stirling number of the first kind, which counts the number of ways to arrange m objects into k cycles instead of subsets. We guess (3.19) may have a combinatorial interpretation.

(3) Letting $\phi(t) = B_n(t)$ (Bernoulli polynomial of degree n) with $B_0(t) = 1$ and denoting $B_n = B_n(0)$ (Bernoulli n -th number) with $B_0 = 1$, we have (cf. [10])

$$D^k B_n(t) = (n)_k B_{n-k}(t), \quad D^k B_n(0) = k! \binom{n}{k} B_{n-k},$$

$$\Delta^k B_n(0) = (\Delta^k B_n(t))_{t=0} = \Delta^{k-1}(nt^{n-1})_{t=0} = (k-1)! n \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}, \quad (k \geq 1).$$

Consequently, by using (3.17) we obtain

$$\sum_{k=1}^n \frac{n}{k} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} = \sum_{k=1}^n \binom{n}{k} B_{n-k} \omega(k). \tag{3.20}$$

This is a strange identity showing that a combinatorial convolution of Bell numbers and Bernoulli numbers can be expressed in terms of the second kind of Stirling numbers.

(4) As may be observed, by taking $f(t) = e^{-t}$ one may find that (3.10)-(3.11) lead to an extension of (3.17), viz.

$$\sum_{k \geq 0} \frac{\Delta^k \phi(t + \lambda)}{k!} = \sum_{k \geq 0} \frac{\omega_\lambda(k) D^k \phi(t)}{k!}, \tag{3.21}$$

where $\omega_\lambda(k)$ ($\lambda \in \mathbf{R}, k \in \mathbf{N}$) denotes a kind of generalized Bell numbers, viz.

$$\omega_\lambda(k) = \sum_{j=0}^k S_2^{(\lambda)}(k, j). \tag{3.22}$$

Also, it may be found that (3.21) yields an extension of (3.18) with $\phi(t) = e^{xt}$:

$$\exp(e^x + \lambda x - 1) = \sum_{k \geq 0} \omega_\lambda(k) \frac{x^k}{k!}. \tag{3.23}$$

This gives a generating function for the generalized Bell numbers.

Finding further examples of (3.10)-(3.11) and possible applications of (3.14)-(3.15) may be left to the interested reader.

Example 3.4. Let $\psi(t)$ be a polynomial of degree r , and let $\phi(t) \in C^\infty$. Then the following summations formula (cf. [8])

$$\sum_{k=0}^{\infty} \frac{\psi(k) \phi^{(k)}(0)}{k!} = \sum_{k=0}^r \frac{\Delta^k \psi(0) \phi^{(k)}(1)}{k!} \tag{3.24}$$

is a particular consequence of (2.1a) with $A(t) = e^t, g(t) = t$ and

$$f(t) = \sum_{k=0}^r t^k \Delta^k \psi(0) / k!. \tag{3.25}$$

For proof, it suffices to show that the RHS of (3.24) and the LHS of (3.24) are given respectively by the LHS and the RHS of (2.1a) for the given $A(t)$, $g(t)$ and $f(t)$. Indeed we have

$$A(D)f(g(D))\phi(t)|_{t=0} = \left(e^D \sum_{k=0}^r D^k \Delta^k \psi(0) / k! \right) \phi(t) \Big|_{t=0} = \sum_{k=0}^r \frac{\Delta^k \psi(0)}{k!} \phi^{(k)}(1).$$

Moreover, the RHS of (2.1a) gives

$$\sum_{k \geq 0} (p_k(D, e^t, t) f(0)) \phi^{(k)}(0).$$

Notice that

$$p_k(z) = p_k(z, e^t, t) = [t^k] e^t \cdot e^{zt} = (1+z)^k / k!$$

and that (3.25) may be rewritten as

$$f(t) = \sum_0^\infty t^k \Delta^k \psi(0) / k!$$

with $\Delta^k \psi(0) = 0$ for all $k > r$, so that $\Delta^k \psi(0)$ has meaning for all $k \geq 0$. Consequently we see that the RHS of (2.1a) can be evaluated as follows

$$\begin{aligned} & \sum_{k \geq 0} (p_k(D) f(0)) \phi^{(k)}(0) \\ &= \sum_{k \geq 0} \left(\frac{1}{k!} (1+D)^k f(0) \right) \phi^{(k)}(0) = \sum_{k \geq 0} \left(\frac{1}{k!} \sum_{j=0}^k \binom{k}{j} D^j f(0) \right) \phi^{(k)}(0) \\ &= \sum_{k \geq 0} \left(\frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \Delta^j \psi(0) \right) \phi^{(k)}(0) = \sum_{k \geq 0} \frac{\psi^{(k)} \cdot \phi^{(k)}(0)}{k!}. \end{aligned}$$

Note that (3.24) has a different proof appeared in [8], and that its particular case with $\psi(t) = t^r$ implies several interesting special identities. For instances, by means of the following special choices (1) $\phi(t) = e^t$, (2) $\phi(t) = 1 + t + \dots + t^m$ ($m \geq 1$) and (3) $\phi(t) = (1 - tx)^{-1}$ with $|tx| < 1$, one may get, respectively, (1) the well-known formula of Dobinski for the Bell number $\omega(r)$, (2) the useful formula of Stirling for the arithmetic progression of higher order, and (3) the formula of Euler for the arithmetic-geometric series

$$\sum_{k=0}^\infty k^r x^k.$$

Example 3.5. Taking $\psi(t) = \binom{t+r}{r}$, it follows from (3.24) that

$$\sum_{k=0}^\infty \binom{k+r}{r} \frac{\phi^{(k)}(0)}{k!} = \sum_{k=0}^r \binom{r}{k} \frac{\phi^{(k)}(1)}{k!}. \tag{3.26}$$

Recall that

$$D^k \cos t = \cos\left(t + \frac{k\pi}{2}\right) \quad \text{and} \quad D^k \sin t = \sin\left(t + \frac{k\pi}{2}\right),$$

so that the special choices $\phi(t) = \cos t$ and $\phi(t) = \sin t$ in (3.26) lead to a pair of combinatorial series sums:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \binom{2k+r}{r} = \sum_{k=0}^r \frac{1}{k!} \binom{r}{k} \cos\left(1 + \frac{1}{2}k\pi\right), \tag{3.27a}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \binom{2k+r+1}{r} = \sum_{k=0}^r \frac{1}{k!} \binom{r}{k} \sin\left(1 + \frac{1}{2}k\pi\right). \tag{3.27b}$$

Example 3.6. Recall that the generalized m -th order Bernoulli polynomials and Euler polynomials are defined by (cf. [10])

$$e^{zt} \left(\frac{t}{e^t - 1}\right)^m = \sum_{k=0}^{\infty} B_k^{(m)}(z) \frac{t^k}{k!}, \quad (|t| < 2\pi), \tag{3.28a}$$

$$e^{zt} \left(\frac{2}{e^t + 1}\right)^m = \sum_{k=0}^{\infty} E_k^{(m)}(z) \frac{t^k}{k!}, \quad (|t| < \pi), \tag{3.28b}$$

where $z \in \mathbf{C}$, $m \in \mathbf{N}$, $m \geq 1$. Evidently both $B_k^{(m)}(z)/k!$ and $E_k^{(m)}(z)/k!$ are Sheffer-type polynomials associated with the (GF) pairs $\{A(t) = (t/(e^t - 1))^m, g(t) = t\}$ and $\{A(t) = (2/(e^t + 1))^m, g(t) = t\}$ respectively (cf. [3, 10]). Also, both the LHS of (3.28a) and (3.28b) may be written as

$$A(t)f(g(t)) = A(t)f(t) = A(t)\exp(zt)$$

with $f(t) = e^{zt}$, which are rational functions in t , e^t and e^{zt} . Thus using the first equation of (3.2) and referring to Corollary 2.2, we find that (3.28a)-(3.28b) with $t \mapsto D$ lead to the following expression (for given $\phi(t) \in C^\infty$)

$$(D/(E-1))^m E^z \phi(t) = \Delta^{-m} \phi^{(m)}(z+t) = \sum_{k=0}^{\infty} \frac{1}{k!} B_k^{(m)}(z) \phi^{(k)}(t), \tag{3.29a}$$

$$(2/(E+1))^m E^z \phi(t) = \nabla^{-m} \phi^{(m)}(z+t) = \sum_{k=0}^{\infty} \frac{1}{k!} E_k^{(m)}(z) \phi^{(k)}(t), \tag{3.29b}$$

where ∇ is the mean-value operator (in Nörlund's notation) defined by

$$\nabla = \frac{1}{2}(1+E) = 1 + \frac{1}{2}\Delta.$$

Now, letting the operators Δ^m and ∇^m be applied to both sides of (3.29a) and (3.29b) (with evaluation at $t=0$) respectively, we obtain

$$\phi^{(m)}(z) = \sum_{k=0}^{\infty} \frac{\Delta^m \phi^{(k)}(0)}{k!} B_k^{(m)}(z), \tag{3.30a}$$

$$\phi^{(m)}(z) = \sum_{k=0}^{\infty} \frac{\nabla^m \phi^{(k)}(0)}{k!} E_k^{(m)}(z). \tag{3.30b}$$

These two expansions are absolutely convergent under the following conditions respectively

$$\sup_k |\Delta^m \phi^{(k)}(0)|^{1/k} < 2\pi, \quad \sup_k |\nabla^m \phi^{(k)}(0)|^{1/k} < \pi. \quad (3.31)$$

Obviously (3.31) follows from Cauchy's root-test and the conditions contained in (3.28a)-(3.28b).

Note that each of (3.28a)-(3.28b) reduces to the Taylor expansion for $m = 0$, and the case for $m = 1$ provides the classical expansions of an analytic function $\phi'(z)$ in terms of the ordinary Bernoulli/Euler polynomials.

4 A further investigation of a source formula

The so-called source formula is a basic result that has been successfully employed in our previous paper [9] to yield a good many special formulas, including a variety of remarkable formulas and identities in the Discrete Mathematics and Combinatorics. The object of this section is to prove two related results that are given by the following theorems.

Theorem 4.1. *Let $\psi(t)$ and $\phi(t)$ be real-valued functions defined on Z and R respectively, and let $\alpha \in R$. Then the source formula of (∞^2) degree of the form*

$$\sum_{k \geq 0} \binom{\alpha}{k} \psi(k) \Delta^k \phi(0) = \sum_{k \geq 0} \binom{\alpha}{k} \Delta^k \psi(0) \cdot \Delta^k \phi(\alpha - k) \quad (4.1)$$

is a consequence form (2.1b) of Theorem 2.1 with $A(t) = (1+t)^\alpha$, $g(t) = t/(t+1)$ and

$$f(t) = \sum_{k \geq 0} \binom{\alpha}{k} t^k \Delta^k \psi(0). \quad (4.2)$$

Theorem 4.2. *Let $\psi(t)$ and $\phi(t)$ be defined as in Theorem 4.1. Denote*

$$\theta_1 = \overline{\lim}_{k \rightarrow \infty} |\Delta^k \phi(0)|^{1/k}, \quad \theta_2 = \overline{\lim}_{k \rightarrow \infty} |\Delta^k \psi(0)|^{1/k}. \quad (4.3)$$

Then (4.1) becomes an exact equality for $\alpha \in R$, provided that

$$\theta_1(1+\theta_2) < 1. \quad (4.4)$$

As will be observed from many examples displayed in the next section that (4.1) is really a useful tool for finding various summation formulas and identities (including classical ones and new ones), and (4.3)-(4.4) provide a kind of general convergence condition which is more available than those intractable ones given in [9].

Proof of Theorem 4.1. It requires to show that (2.1b) of Theorem 2.1 yields (4.1) with the given $A(t)$, $g(t)$ and $f(t)$. Clearly, the LHS of (2.1b) (evaluated at $t=0$) gives

$$\begin{aligned} A(\Delta)f(g(\Delta))\phi(0) &= (1+\Delta)^\alpha \sum_{k \geq 0} \binom{\alpha}{k} \Delta^k \psi(0) \cdot (g(\Delta))^k \phi(t)|_{t=0} \\ &= E^\alpha \sum_{k \geq 0} \binom{\alpha}{k} \Delta^k \psi(0) \left(\frac{\Delta}{E}\right)^k \phi(t)|_{t=0} \\ &= \sum_{k \geq 0} \binom{\alpha}{k} \Delta^k \psi(0) \cdot \Delta^k \phi(\alpha - k) = RHS \quad \text{of (4.1)}. \end{aligned}$$

On the other hand, the RHS of (2.1b) (evaluated at $t=0$) may be written as

$$\sum_{k \geq 0} (p_k(D)f(0))\Delta^k \phi(0) = \sum_{k \geq 0} \left(\sum_{j=0}^k \frac{1}{j!} d_{kj} D^j f(0) \right) \Delta^k \phi(0).$$

Herein d_{kj} are given by

$$d_{kj} = [t^k](1+t)^\alpha (t/(t+1))^j = [t^{k-j}](1+t)^{\alpha-j} = \binom{\alpha-j}{k-j}.$$

Then, according to (4.2), we have

$$\begin{aligned} p_k(D)f(0) &= \sum_{j=0}^k \frac{1}{j!} \binom{\alpha-j}{k-j} D^j f(0) = \sum_{j=0}^k \binom{\alpha}{j} \binom{\alpha-j}{k-j} \Delta^j \psi(0) \\ &= \binom{\alpha}{k} \sum_{j=0}^k \binom{k}{j} \Delta^j \psi(0) = \binom{\alpha}{k} \psi(k). \end{aligned}$$

Finally, the RHS of (2.1b) gives

$$\sum_{k \geq 0} (p_k(D)f(0))\Delta^k \phi(0) = \sum_{k \geq 0} \binom{\alpha}{k} \psi(k) \Delta^k \phi(0) = LHS \quad \text{of (4.1)}.$$

Thus, we complete the proof. □

Proof of Theorem 4.2. Starting with the familiar equalities

$$\psi(k) = \sum_{j=0}^k \binom{R}{j} \Delta^j \psi(0) \quad \text{and} \quad \binom{\alpha}{k} \binom{k}{j} = \binom{\alpha}{j} \binom{\alpha-j}{k-j}, \tag{4.5}$$

which have been employed in the preceding proof, one may see that the LHS of (4.1) can

be computed formally as follows

$$\begin{aligned} \text{LHS of (4.1)} &= \sum_{k=0}^{\infty} \binom{\alpha}{k} \Delta^k \phi(0) \sum_{j=0}^k \binom{k}{j} \Delta^j \psi(0) = \sum_{j=0}^{\infty} \Delta^j \psi(0) \sum_{k=j}^{\infty} \binom{\alpha}{k} \binom{k}{j} \Delta^k \phi(0) \\ &= \sum_{j=0}^{\infty} \binom{\alpha}{j} \Delta^j \psi(0) \sum_{k=j}^{\infty} \binom{\alpha-j}{k-j} \Delta^k \phi(0) = \sum_{j=0}^{\infty} \binom{\alpha}{j} \Delta^j \psi(0) (1+\Delta)^{\alpha-j} \Delta^j \phi(0) \\ &= \sum_{j=0}^{\infty} \binom{\alpha}{j} \Delta^j \psi(0) \Delta^j \phi(\alpha-j) = \text{RHS of (4.1)}. \end{aligned}$$

Apparently, the key step (without justification) in the formal derivation is the exchange of orders of repeated summation. Thus in order to prove the theorem, it suffices to show that the condition (4.4) ensures the absolute convergence of the repeated summations involved. This can be done in what follows.

First, according to (4.4), one may choose $\delta_1 > 0, \delta_2 > 0$ so small that lead to

$$\bar{\theta}_1(1+\bar{\theta}_2) < 1 \quad \text{with} \quad \bar{\theta}_1 = \theta_1 + \delta_1, \quad \bar{\theta}_2 = \theta_2 + \delta_2.$$

The condition (4.3) implies that there exist integers $m > 0$ and $n > 0$, such that

$$|\Delta^k \phi(0)|^{1/k} < \bar{\theta}_1, \quad |\Delta^j \psi(0)|^{1/j} < \bar{\theta}_2,$$

for $k \geq m$ and $j \geq n$, where it is no real restriction to assume that $m > n$. Consequently, it can be shown that the following series consisting of repeated summations

$$S_{m,n} \equiv \sum_{k \geq m} \binom{\alpha}{k} \Delta^k \phi(0) \sum_{j \geq n} \binom{k}{j} \Delta^j \psi(0)$$

is absolutely convergent. Indeed, we have

$$\begin{aligned} |S_{m,n}| &\leq \sum_{k \geq m} \left| \binom{\alpha}{k} \Delta^k \phi(0) \right| \sum_{j \geq n} \binom{k}{j} |\Delta^j \psi(0)| \\ &\geq \sum_{k \geq m} \left| \binom{\alpha}{k} \bar{\theta}_1^k \right| \sum_{j \geq n} \binom{k}{j} \bar{\theta}_2^j \\ &< \sum_{k \geq m} \left| \binom{\alpha}{k} \bar{\theta}_1^k (1+\bar{\theta}_2)^k \right| < \infty. \end{aligned}$$

It remains to show that the absolute convergence of $S_{m,n}$ implies that of $S_{m,0}$ and so of $S_{0,0}$. Notice that

$$S_{m,0} = S_{m,n} + \sum_{k \geq m} \binom{\alpha}{k} \Delta^k \phi(0) \sum_{j=0}^{n-1} \binom{k}{j} \Delta^j \psi(0). \tag{4.6}$$

Clearly, the right-most summation on the RHS of (4.6) has an order estimate (in Landau's denotation) $\mathcal{O}(nk^{n-1})$ ($k \rightarrow \infty$). Consequently, it is seen that the absolute convergence of the series (on the RHS of (4.6))

$$\sum_{k \geq m} \binom{\alpha}{k} \Delta^k \phi(0) \cdot \mathcal{O}(nk^{n-1})$$

follows from Cauchy's root test with

$$\overline{\lim}_{k \rightarrow \infty} \left| \binom{\alpha}{k} \mathcal{O}(nk^{n-1}) \right|^{1/k} = 1, \quad \overline{\lim}_{k \rightarrow \infty} |\Delta^k \phi(0)|^{1/k} = \theta_1 < 1.$$

Moreover, the absolute convergence of $S_{0,0}$ follows from that of $S_{m,0}$. Hence the theorem is proved. □

Corollary 4.1. The formula (4.1) is an exact equality whenever $\theta_1 + \theta_2 < 1$. In particular, the condition may be replaced by $\theta_1 < 1$ if $\psi(t)$ is a polynomial in t .

In fact, we have $\theta_2 = 0$ if $\psi(t)$ is a polynomial.

5 Various consequences of the source formula

In this section we will present 24 selected instances to illustrate the real capacity of (4.1) for finding or deriving special formulas and novel identities. It has been found that some particular choices of the triplet $\{\alpha, \psi(t), \phi(t)\}$ of (4.1) could lead to numerous interesting formulas and combinatorial identities. Some noticeable examples given in [9] may be briefly recalled for references.

First, let us mention that the following 10 special triplets

$$\begin{aligned} T_1 &= \{\alpha = -1, \psi(t), \phi(t) = 2^t\}, \\ T_2 &= \left\{ \alpha = x, \psi(t) \equiv 1, \phi(t) = \binom{y+t}{n} \right\}, \\ T_3 &= \{\alpha = -1, \psi(t) = t^r, \phi(t) = (1-x)^t\}, \\ T_4 &= \left\{ \alpha = -1, \psi(t) = t^r, \phi(t) = \binom{m-t}{m} \right\}, \\ T_5 &= \left\{ \alpha = -s-1, \psi(t) = \binom{t}{j}, \phi(t) = \binom{n-t}{n} \right\}, \\ T_6 &= \{\alpha = n, \psi(t) \equiv 1, \phi(t) = 1/(t+a)\}, \\ T_7 &= \left\{ \alpha = n, \psi(t) = \binom{x+n-t}{n+m}, \phi(t) = \binom{m+t}{n} \right\}, \\ T_8 &= \left\{ \alpha = n, \psi(t) = \binom{m+t}{m}, \phi(t) = \binom{p+m+t}{p} \right\}, \\ T_9 &= \left\{ \alpha \mapsto -(\alpha+1), \psi(t) = t^r, \phi(t) = \binom{n-t}{n} \right\}, \\ T_{10} &= \{\alpha = m, \psi(t) = a/(a+t), \phi(t) = (-1)^t\}, \quad (a = -(2m+1)), \end{aligned}$$

yield, respectively, the following 10 notable formulas (cf. [9]).

1 Euler's formula for series transform (with $\psi(k) \downarrow 0$ as $k \rightarrow \infty$)

$$\sum_{k=0}^{\infty} (-1)^k \psi(k) = \sum_{j=0}^{\infty} (-1)^j \Delta^j \psi(0) / 2^{j+1}.$$

2 Vandermonde's convolution formula (with $x, y \in \mathbf{R}$)

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}.$$

3 Euler's formula for the arithmetic-geometric series

$$\sum_{k=0}^{\infty} k^r x^k = \sum_{j=0}^r \frac{j! x^j}{(1-x)^{j+1}} \left\{ \begin{matrix} r \\ j \end{matrix} \right\}, \quad (|x| < 1).$$

4 Stirling's formula for the arithmetic series of higher order

$$\sum_{k=0}^m k^r = \sum_{j=0}^r j! \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \binom{m+1}{j+1}.$$

5 Knuth's combinatorial identity (with $s \in \mathbf{R}$)

$$\sum_{k=j}^n \binom{s+k}{k} \binom{k}{j} = \binom{s+j}{j} \binom{n+s+1}{n-j}.$$

6 A formula stated as a theorem in Wilf's [13] (with $a \in \mathbf{R}$)

$$\sum_{k=0}^n (-1)^k \binom{n}{k} / \binom{k+a}{k} = \frac{a}{n+a}, \quad (n+a \neq 0).$$

7 Riordan's identity (with $x \in \mathbf{R}$)

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{n-k} \binom{x+n-k}{n+m} = \binom{x}{m} \binom{x}{n}.$$

8 An identity due to Li Shanlai

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k} \binom{p+n+m-k}{n+m} = \binom{p+n}{n} \binom{p+m}{m}.$$

9 An extended Stirling summation formula (with $\alpha \in \mathbf{R}$)

$$\sum_{k=0}^n \binom{\alpha+k}{k} k^r = \sum_{j=0}^r \binom{\alpha+j}{j} \binom{\alpha+n+1}{n-j} j! \left\{ \begin{matrix} r \\ j \end{matrix} \right\}.$$

10 A so-called miraculous formula treated in Graham-Knuth-Patashnik [5]

$$\sum_{k=0}^m \binom{m}{k} \frac{2m+1}{2m+1-k} (-2)^k = 1 / \binom{-1/2}{m}.$$

The above instances suggest that (4.1) may be continually used to deduce more formulas of some interest via various suitable choices of the triplet $\{\alpha, \psi, \phi\}$. Indeed, as further consequences of (4.1) we may mention in succession the following 14 verifiable special formulas.

11 The (GF) for harmonic numbers $H_0=0$, $H_k=1+\frac{1}{2}+\dots+\frac{1}{k}$, ($k \geq 1$)

$$\sum_{k=1}^{\infty} H_k x^k = \frac{1}{1-x} \log \frac{1}{1-x}, \quad (|x| < 1).$$

12 A pair of identities involving Fibonacci numbers defined by $F_0=0$, $F_1=1$, $F_n=F_{n-1}+F_{n-2}$, $n \in \{2, 3, \dots\}$

$$(i) \sum_{k=0}^n \binom{n}{k} k^r F_{s-k} = \sum_{j=0}^r j! \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \binom{n}{j} F_{s+n-j}, \quad (s \geq k).$$

$$(ii) \sum_{k=0}^n \binom{n}{k} k^r F_{s+k} = \sum_{j=0}^r j! \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \binom{n}{j} F_{s+2n-j}, \quad (s \geq 0).$$

13 Stanley's identity (with $x, y \in \mathbf{R}$)

$$\sum_{k=0}^{\min(a,b)} \binom{x+y+k}{k} \binom{y}{a-k} \binom{x}{b-k} = \binom{x+a}{b} \binom{x+b}{a}.$$

14 A summation formula for a kind of trigonometric series involving 4 parameters $\alpha, \beta, x \in \mathbf{R}$ and $m \in \mathbf{N}$ (with $0 < |\alpha| < \pi/3$)

$$\begin{aligned} & \sum_{k=0}^{\infty} \binom{x}{k} k^m \left(2 \sin \frac{\alpha}{2} \right)^k \cos \left(\beta + \frac{k}{2} (\alpha + \pi) \right) \\ &= \sum_{j=0}^m \binom{x}{j} j! \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \left(2 \sin \frac{\alpha}{2} \right)^j \cos \left(\alpha x + \beta + \frac{j}{2} (\pi - \alpha) \right). \end{aligned}$$

15 Montmort's series transform formula (with $|x| < 1$, $|x/(1-x)| < 1$)

$$\sum_{k=1}^{\infty} \psi(k)x^k = \sum_{j=0}^{\infty} \Delta^j \psi(1) \cdot \left(\frac{x}{1-x}\right)^{j+1}.$$

16 A formula due to D. A. Zave [4] (with $m \in \mathbf{N}$, $|x| < 1$)

$$\sum_{k=1}^{\infty} \binom{k+m}{m} (H_{k+m} - H_m)x^k = (1-x)^{-m-1} \log\left(\frac{1}{1-x}\right).$$

17 An equivalent form of Abel's identity

$$\sum_{k \geq 0} (-1)^k \binom{n}{k} [a(kx-a)^{k-1} (kx+b)^{n-k} + (-a)^k b^{n-k}] \equiv 0.$$

18 A pre-limit form of Dobinski's formula for Bell numbers (with $\alpha > 1$)

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\frac{1}{\alpha}\right)^k k^n = \sum_{j=0}^n \binom{\alpha}{j} \left(\frac{1}{\alpha}\right)^j j! \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \left(1 + \frac{1}{\alpha}\right)^{\alpha-j}.$$

19 Grosswald's formula (with $n > r > 0$ and $n+r=2m$)

$$\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \binom{2m+k}{n} 2^{-k} = (-1)^{(n-r)/2} 2^{r-n} \binom{n}{m} \binom{2m}{n} \binom{n}{r}^{-1}.$$

20 A generalized Stirling formula (with $\alpha \in \mathbf{R}$)

$$\sum_{k=0}^m \binom{\alpha}{k} (-1)^k k^r = \sum_{j=0}^r (-1)^j \binom{\alpha}{j} \binom{m-\alpha}{m-j} j! \left\{ \begin{matrix} r \\ j \end{matrix} \right\}.$$

21 Two identities involving harmonic numbers

$$(i) \sum_{k=0}^n (-1)^{k-1} \binom{n}{k} H_{a+k} = \frac{1}{a+1} / \binom{a+n}{n-1}, \quad (n \geq 1, a \in \mathbf{N}).$$

$$(ii) \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} / \binom{a+k}{k-1} = (a+1)(H_{a+n} - H_a).$$

22 A general combinatorial identity involving harmonic numbers

$$\sum_{k=0}^n \binom{n}{k} \binom{r+n-k}{r} (-1)^{k-1} H_{a+k} = \frac{1}{a+1} \sum_{j=0}^r \binom{r}{j} \binom{n}{j} / \binom{a+n-j}{a+1}.$$

23 A summation formula for a kind of combinatorial series

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \binom{\lambda+k}{k}^{-1} k^r = \frac{\lambda}{\alpha+\lambda} \sum_{j=0}^r (-1)^j \binom{\alpha}{j} j! \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \binom{\alpha+\lambda-1}{j}^{-1},$$

where $\alpha, \lambda \in \mathbf{R}$ and $r \in \mathbf{N}$ such that $\alpha \geq -1$ and $\lambda \geq r+2$.

24 A miraculous formula involving Fibonacci numbers (with $\alpha \in \mathbf{R}, r \in \mathbf{Z}$)

$$\sum_{k=0}^n \binom{\alpha}{k} \binom{\alpha-k}{n-k} (-1)^k F_{r+3k} = \binom{\alpha}{n} (-2)^n F_{r+n}.$$

Note that suitable specializations of 22-24 could lead to several known formulas and identities, and that the convergence conditions $|x| < 1$ and $|\alpha| < \pi/3$ involved in 3 and 14, respectively, are both deducible from the condition $\theta_1 < 1$ given by Corollary 4.1.

Actually, 11-24 are deducible from (4.1) via the following special triplets $T_{11} - T_{24}$ respectively

- $T_{11} = \{ \alpha = -1, \psi(t) = H_t, \phi(t) = (1-x)^t \},$
- $T_{12(i)} = \{ \alpha = n, \psi(t) = t^r, \phi(t) = F_{s+t} \},$
- $T_{12(ii)} = \{ \alpha = n, \psi(t) = t^r, \phi(t) = F_{s+2t} \},$
- $T_{13} = \left\{ \alpha = x, \psi(t) = \binom{t+y}{y+b-a}, \phi(t) = \binom{y+b+t}{b} \right\}$
 (with $b = \min(a, b), x, y \in \mathbf{N}_j$ making $b \rightarrow (b-k)$, betting finally $x, y \in \mathbf{R}$),
- $T_{14} = \{ \alpha = x, \psi(t) = t^m, \phi(t) = \cos(\alpha t + \beta) \},$
- $T_{15} = \{ \alpha = -1, \psi(t) \mapsto \psi(t+1), \phi(t) = (1-x)^t \},$
- $T_{16} = \{ \alpha = -m-1, \psi(t) = H_{m+t} - H_m, \phi(t) = (1-x)^t \},$
- $T_{17} = \left\{ \alpha = n, \psi(k) = a(kx-a)^{k-1} (kx+b)^{n-k} + (-a)^k b^{n-k}, \phi(t) = \binom{n-t}{n} \right\},$
- $T_{18} = \left\{ \alpha, \psi(t) = t^n, \phi(t) = \left(1 + \frac{1}{\alpha}\right) (\dots) t \right\},$
- $T_{19} = \left\{ \alpha = n-r, \psi(t) = \binom{2m+t}{n}, \phi(t) = 2^{-t} \right\},$
- $T_{20} = \left\{ \alpha, \psi(t) = t^r, \phi(t) = \binom{m-t}{m} \right\},$
- $T_{21(i)} = \left\{ \alpha = n, \psi(t) = H_{a+t}, \phi(t) = \binom{n-t}{n} \right\},$
- $T_{21(ii)} = \left\{ \alpha = n, \psi(t) = H_{a+t}, \phi(t) = \binom{t}{n} \right\},$

$$\begin{aligned}
T_{22} &= \left\{ \alpha = n, \psi(t) = H_{a+t}, \phi(t) = \binom{r+n-t}{n} \right\}, \\
T_{23} &= \left\{ \alpha, \psi(t) = t^r, \phi(t) = \frac{1}{t+\lambda} \right\}, \\
T_{24} &= \left\{ \alpha, \psi(t) = F_{r+3t}, \phi(t) = \binom{\alpha-t}{n} \right\}.
\end{aligned}$$

Surely, for the sake of immediate verification, the reader may find that the following short table of difference formulas is both needful and helpful.

- (i) $\Delta^k a^t = (a-1)^k a^t$, ($a \neq 0$); $(\Delta^k a^t)_0 = (-a)^k$;
- (ii) $\Delta^k \binom{a+t}{n} = \binom{a+t}{n-k}$, ($k \leq n$); $\Delta^k \binom{a+t}{n}_0 = \binom{a}{n-k}$;
- (iii) $\Delta^k \binom{a-t}{n} = (-1)^k \binom{a-t-k}{n-k}$, ($k \leq n$); $\Delta^k \binom{a-t}{n}_0 = (-1)^k \binom{a-k}{n-k}$;
- (iv) $\Delta^k \left(\frac{1}{t+a}\right) = \frac{(-1)^k k!}{(t+a)(t+a+1)\cdots(t+a+k)}$, $\Delta^k \left(\frac{1}{t+a}\right)_0 = \frac{(-1)^k}{a} / \binom{k+a}{k}$, ($a \neq 0$);
- (v) $\Delta^k \cos(at+b) = (2 \cdot \sin \frac{a}{2})^k \cos(at+b + \frac{k}{2}(a+\pi))$, $\Delta^k \sin(at+b) = (2 \cdot \sin \frac{a}{2})^k \sin(at+b + \frac{k}{2}(a+\pi))$;
- (vi) $(\Delta^k t^n)_0 = k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k! S(n, k)$;
- (vii) $\Delta^k H_t = \Delta^{k-1} \frac{1}{t+1} = \frac{(-1)^{k-1} (k-1)!}{(t+k)_k}$, $(\Delta^k H_t)_0 = \frac{(-1)^{k-1}}{k}$;
- (viii) $\Delta^k F_t = F_{t-k}$, ($k \in \mathbf{N}$, $t \geq k$).

6 Lifting process and formula chains

For brevity, the formulas (2.1a)-(2.1b) may be rewritten as a single formula

$$A(\delta)f(g(\delta))\phi(t) = \sum_{k \geq 0} (p_k(D)f(0))\delta^k \phi(t), \quad (6.1)$$

where δ denotes either D or Δ . Also, we will make use of (2.7) with $x \mapsto t$

$$A(t)f(g(t)) = \sum_{k \geq 0} (p_k(D)f(0))t^k. \quad (6.2)$$

In what follows we will show that both (6.2) and (6.1) could be used as basic formulas to produce chains of formulas having freedom degrees in increasing orders. The basic idea is to construct a kind of iteration process starting from a given initial formula with a form of either (6.2) or (6.1).

Let $\{A_m(t)\}_1^\infty, \{g_m(t)\}_1^\infty$ and $\{f_m(t)\}_1^\infty$ be arbitrary fps or functions of the class C^∞ such that $A_m(0) = 1$, $g_m(0) = 0$, $g'_m(0) = Dg_m(0) \neq 0$, ($m = 1, 2, \dots$). Accordingly, for each $m \geq 1$ we may construct a Riordan array $(d_{kj}) = (A_m(t), g_m(t))$ with d_{kj} being defined by

$$d_{kj} = [t^k] A_m(t) (g_m(t))^j, \quad (j \geq k < \infty).$$

Consequently, we have Sheffer-type operators $p_k^{(m)}(D)$ defined by

$$p_k^{(m)}(D) = \sum_{j=0}^k (d_{kj}/j!) D^j.$$

This leads to the expansion formula

$$A_m(t) f_m(g_m(t)) = \sum_{k \geq 0} (p_k^{(m)}(D) f_m(0)) t^k \quad (6.3)$$

and its allied operator series expansion formula for $\phi_m(t) \in C^\infty$

$$A_m(\delta) f_m(g_m(\delta)) \phi_m(t) = \sum_{k \geq 0} (p_k^{(m)}(D) f_m(0)) \delta^k \phi_m(t). \quad (6.4)$$

Now, suppose that for each $m \geq 1$, the RHS of (6.3) defines a function $f_{m+1}(t)$ iteratively, viz

$$\sum_{k \geq 0} (p_k^{(m)}(D) f_m(0)) t^k = f_{m+1}(t). \quad (6.5)$$

Accordingly, the sequence $\{f_m(t)\}_2^\infty$ is created by the iteration process of (6.5) starting from A_1, g_1 and f_1 . In this way we see that (6.3) gives a (∞^{2m+1}) degree formula, since it consists of arbitrary functions $A_j(t), g_j(t), (j=1, 2, \dots, m)$ and $f_1(t)$.

Moreover, start from (6.4) with $m=1$ and the initial choices A_1, g_1, f_1 and ϕ_1 and let f_{m+1} be defined iteratively by the RHS of (6.4), viz

$$\sum_{k \geq 0} (p_k^{(m)}(D) f_m(0)) \delta^k \phi_m(t) = f_{m+1}(t), \quad (m \geq 1). \quad (6.6)$$

Then we see that (6.4) is a formula of (∞^{3m+1}) degree, as it consists of arbitrary functions $A_j, g_j, \phi_j, (j=1, 2, \dots, m)$ and f_1 .

From that given above we may conclude that the iteration processes defined by (6.5)-(6.6) could produce two chains of formulas with increasing freedom degrees $(2m+1)$ and $(3m+1), (m=1, 2, \dots)$, respectively. The iterations process may also be called lifting process, for it could lift the freedom degrees of formulas successively.

Note that every member of the formula chains could be employed as a source formula to yield a formula family. Thus we may state a set-theoretic proposition belonging Cantor's category: "There exist countably infinitely many formula families (with various degrees)".

Certainly, the rather special and quite interesting formula family is the so-called $(\Sigma\Delta)$ class that consists of all exact formulas (series summations and expansions) deducible from (4.1) via proper triplets. Moreover, the set of all exact formulas deducible from (6.1) also gives an interesting family, say $[\Sigma\delta]$ family, which is much more comprehensive than $(\Sigma\Delta)$ class. Surely, more fruitful applications of both (4.1) and (6.1) to Computational Analysis are worthy of further investigation.

References

- [1] R. P. Boas and R. C. Buck, *Polynomial Expansions of Analytic Functions*, New York, Springer, 1964.
- [2] L. Carlitz, Weighted stirling numbers, I, II, *Fibonacci Quarterly*, 18 (1980), 147–162, 242–257.
- [3] A. Erdelyi, *Higher Transcendental Functions*, New York, McGraw-Hill, 1953.
- [4] D. H. Greene and D. E. Knuth, *Mathematics for the Analysis of Algorithms*, Basel: Birkhäuser, 1981, Boston, Birkhäuser, 1982.
- [5] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Reading, MA: Addison-Wesley, 1994.
- [6] T. X. He, L. C. Hsu and P. J.-S. Shiue, Multivariate expansions associated with Sheffer-type polynomials and operators, *Bull. Inst. Acad. Sinica (N. S.)*, 1(4) (2006), 451–473.
- [7] T. X. He, L. C. Hsu and X. R. Ma, On an extension of riordan array and its applications in the construction of convolution-type and abel-type identities, *Euro. J. Combinatorics*, 42 (2014), 112–134.
- [8] T. X. He, L. C. Hsu and D. S. Yin, A pair of operator summation formulas and their applications, *Comput. Math. Appl.*, 58 (2009), 1340–1348.
- [9] L. C. Hsu, Concerning two formulaic classes in computational combinatorics, *J. Math. Res. Appl.*, 32(2) (2012), 127–142.
- [10] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Athemathical Physics*, Berline, New York, Springer, 1966.
- [11] S. M. Roman, *The Umbral Calculus*, New York, Acad. Press, 1984.
- [12] L. W. Shapiro, S. Getu, W. J. Woan and L. Woodson, The riordan group, *Discrete Appl. Math.*, 34 (1991), 229–239.
- [13] H. S. Wilf, *Generating Functionology*, New York, Acad. Press, 1994.