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A Cyclic Probabilistic C-Contraction Results using Hadzic and Lukasiewicz T-Norms in Menger Spaces

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Abstract. In this paper, we introduce generalized cyclic C-contractions through p number of subsets of a probabilistic metric space and establish two fixed point results for such contractions. In our first theorem we use the Hadzic type t-norm. In our next theorem we use Lukasiewicz t-norm. Our results generalize the results of Choudhury and Bhandari [11]. A control function [3] has been utilized in our second theorem. The results are illustrated with some examples.

Key Words: Menger space, Cauchy sequence, fixed point, ϕ-function, ψ-function.

AMS Subject Classifications: 54E40, 54H25

1 Introduction and mathematical preliminaries

The notion of probabilistic metric spaces was introduced by Menger in 1942 by way of a probabilistic generalization of the notion of metric spaces [23]. Several aspects of this space have developed by many workers over the years which followed. A comprehensive development of this space is given in the book [29] by Schweizer and Sklar. Fixed point theory has developed vastly in this space which in the present time forms a branch of study by itself. The richness of this theory is due to the inherent flexibility possessed by the structure of the space itself. An example is that the Banach contractions were extended to this space in more than one inequivalent ways. The first probabilistic contraction was defined by Sehgal and Bharucha-Reid in 1972 [30] which is different from the contraction of Hick's contraction [18]. The former is called Sehgal contraction or B-contraction and the latter is known as C-contraction. Both B-contraction and C-contraction have been

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generalized and fixed point results for these generalized probabilistic contractions have been obtained in a large number of works as, for instances, in [2–7, 11, 14, 15, 34].

Cyclic contractive mappings were first introduced by Kirk et al. in [21]. These are mappings for which the contraction inequality is valid for choices of points from two different subsets of a metric space. Kirk et al. proved fixed point results for such mappings in metric spaces. The problems of cyclic contractions have been strongly associated with proximity point problems. Some other results dealing with cyclic contractions and proximity point problems are noted in [1, 6–8, 12, 16, 19, 32] and [33].

Cyclic contractions have been extended to p-cyclic contractions involving p subsets of a metric space [9, 19, 20, 31].

In this paper we extend the notion of C-contraction to p-cyclic C-contraction in Menger space which are varieties of probabilistic metric spaces in which the probabilistic triangular inequality is obtained with the help of t-norms. We prove unique fixed point results for these mappings in complete Menger spaces. We use Hadzic type and Lukasiewicz t-norms in our respective two theorems. There are supportive illustrations of the theorems we prove here.

In the following we begin with the description of mathematical preliminaries for our discussions in this paper.

Kirk, Srinivasan and Veeramani established the following results.

Theorem 1.1 (see [21])**.** *Let A and B be two non-empty closed subsets of a complete metric space X, and suppose $f: A\bigcup B \to A\bigcup B$ satisfies:*

(1) $fA\subseteq B$ and $fB\subseteq A$,

(2) $d(fx,fy)\le kd(x,y)$ for all $x\in A$ and $y\in B$, where $k\in(0,1)$.

Then f has a unique fixed point in $A\bigcap B$.

A generalization of cyclic mapping is p-cyclic mapping. The definition is the following:

Definition 1.1 (see [21])**.** Let $\{A_i\}_{i=1}^p$ be non-empty sets. A p-cyclic mapping is a mapping $T:\bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$, which satisfies the following conditions:

$$TA_i\subseteq A_{i+1} \quad \text{for } 1\le i<p, \quad TA_p\subseteq A_1. \tag{1.1}$$

In this case where $p=2$, this reduces to cyclic mappings. In this paper we are interested in the fixed point properties of p-cyclic mappings of C-contraction in probabilistic metric spaces. In the following we describe the space briefly and to the extent of our requirement. Several aspects of this space has been described comprehensively by Schweizer and Sklar [29].

The following are the mathematical descriptions of some concepts and results which are needed in this paper.

Definition 1.2 (see [17, 29])**.** A mapping $F: R\to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R}F(t)=0$ and $\sup_{t\in R}F(t)=1$, where R is the set of all real numbers and R^+ denotes the set of all non-negative real numbers.

Definition 1.3 (t-norm [17,29])**.** A t-norm is a function $\Delta:[0,1]\times[0,1]\to[0,1]$ which satisfies the following conditions for all $a,b,c,d\in[0,1]$, $\Delta(1,a)=a$, $\Delta(a,b)=\Delta(b,a)$, $\Delta(c,d)\geq\Delta(a,b)$ whenever $c\geq a$, and $d\geq b$, $\Delta(\Delta(a,b),c)=\Delta(a,\Delta(b,c))$.

The following are three examples of t-norm:

(i) The minimum t-norm, $\Delta=T_m$, defined by $T_m(a,b)=\min\{a,b\}$.
(ii) The product t-norm, $\Delta=T_p$, defined by $T_p(a,b)=a\cdot b$.
(iii) The Lukasiewicz t-norm, $\Delta=T_L$, defined by $T_L(a,b)=\max\{a+b-1,0\}$.

Here we use the Hadzic type t-norm and Lukasiewicz t-norm.

Definition 1.4 (Hadzic type t-norm [17])**.** A t-norm Δ is said to be Hadzic type t-norm if the family $\{\Delta^p\}_{p\in N}$ of its iterates, defined for each $s\in(0,1)$ as $\Delta^0(s)=1$, $\Delta^{p+1}(s)=\Delta(\Delta^p(s),s)$ for all integer $p\geq 0$, is equi-continuous at $s=1$, that is, given $\lambda>0$ there exits $\eta(\lambda)\in(0,1)$ such that $1\geq s>\eta(\lambda)\Rightarrow\Delta^p(s)>1-\lambda$ for all integer $p\geq 0$.

Definition 1.5 (Menger space [17, 29])**.** A Menger space is a triplet (X,F,Δ), where X is a non empty set, F is a function defined on $X\times X$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied: $F_{x,y}(0)=0$ for all $x,y\in X$, $F_{x,y}(s)=1$ for all $s>0$ and $x,y\in X$ if and only if $x=y$, $F_{x,y}(s)=F_{y,x}(s)$ for all $x,y\in X$, $s>0$ and $F_{x,y}(u+v)\geq\Delta(F_{x,z}(u),F_{z,y}(v))$ for all $u,v\geq 0$ and $x,y,z\in X$.

Definition 1.6 (see [17,29])**.** A sequence $\{x_n\}\subset X$ is said to converge to some point $x\in X$ if given $\epsilon>0$, $\lambda>0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that for all $n>N_{\epsilon,\lambda}$,

$$F_{x_n,x}(\epsilon)>1-\lambda. \tag{1.2}$$

Definition 1.7 (see [17,29])**.** A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if given $\epsilon>0$, $\lambda>0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $F_{x_n,x_m}(\epsilon)>1-\lambda$ for all

$$m,n>N_{\epsilon,\lambda}. \tag{1.3}$$

Definition 1.6 and 1.7 can be equivalently written by replacing "$>$" with "$\geq$" in (1.2) and (1.3), respectively. More often than not, they are written in that way. We have given them in the present form for our convenience in the proofs of our theorems.

Definition 1.8 (see [17, 29])**.** A Menger space (X,F,Δ) is said to be complete if every Cauchy sequence is convergent in X.

Definition 1.9 (C-contraction [18])**.** Let (X,F) be a probabilistic metric space. A mapping $f:X\to X$ is called a C-contraction if there exists $k\in(0,1)$ such that for every $p,q\in X$ and $t>0$ the following implication holds: $F_{p,q}(t)>1-t$ implies

$$F_{fp,fq}(kt)>1-kt. \tag{1.4}$$

In 1984, Khan, Swaleh and Sessa [22] introduced a new type of contractive fixed point problems in metric spaces. They introduced "Altering distance function", which is a control function that alters the distance between two points in a metric space. The idea of altering distance function was further generalized in a number of works. After introducing this function many fixed point results of functions satisfying various types of contractive conditions involving altering distances have been proved. Some of these results may be noted in [13,25,27] and [28].

Choudhury and Das [3] extended the concept of "Altering distance function" to the context of Menger spaces which are special types of probabilistic metric spaces in which the triangular inequality is obtained with the application of a t-norm. This idea of control function has opened the possibility of proving new fixed point results in Menger spaces. Some recent results on fixed point and coincidence point problems using this control function in Menger spaces have appeared in works like [4,5,10,14,15] and [24].

It has been established in [3] that the "Altering distance function" can be generated from the Φ-function. We will also utilize the following function in one of our theorems. The following is an extension of "Altering distance function" to Menger spaces.

Definition 1.10 (Φ-Function [3])**.** A function $\phi: R\to R^+$ is said to be a Φ-function if it satisfies the following conditions: $\phi(t)=0$ if and only if $t=0$, $\phi(t)$ is strictly monotone increasing and $\phi(t)\to\infty$ as $t\to\infty$, ϕ is left continuous in $(0,\infty)$, ϕ is continuous at 0.

Here we also use the following function.

Definition 1.11 (Ψ-Function)**.** A function $\psi:[0,1]\to[0,1]$ is said to be a Ψ-function if it satisfies the following conditions: ψ is strictly monotone increasing, $\psi^n(s)\to 0$ as $n\to\infty$ for all $s\geq 0$, $\psi(s)<s$, ψ is continuous.

2 Main results

In this section we have established one lemma, two different types definitions and two theorems using two different types t-norm.

Lemma 2.1. *Let (X,F,Δ) be a Menger space. Assume T is a p-cyclic mapping (Definition 1.1) on X. Let $\{x_n\}$ be a sequence satisfying the following condition:*

$$F_{x_{n+1},x_{n+2}}(t)\geq F_{x_n,x_{n+1}}\Big(\frac{t}{k}\Big), \tag{2.1}$$

whenever $x_n\in A_{n+1}$, $x_{n+1}\in A_{n+2}$, $n\geq 0$, $k\in(0,1)$, $t>0$, then for $i\geq 1$

$$F_{x_{n+i},x_{n+i+1}}(t)\geq F_{x_n,x_{n+1}}\Big(\frac{t}{k^i}\Big). \tag{2.2}$$

Proof. Let x_0 be any point in A_1. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}, n\in N$ where N is the set of natural numbers. By (1.1), we have $x_o \in A_1$, $x_1\in A_2$, $x_2\in A_3$, $\cdots$, $x_{p-1}\in A_p$ and in general

$$x_{np}\in A_1,\ x_{np+1}\in A_2,\cdots,x_{np+(p-1)}\in A_p \quad \text{for all} \quad n\geq 0. \tag{2.3}$$

For any $n\geq 1$ and $t>0$, we have

$$F_{x_n,x_{n+1}}(t)=F_{Tx_{n-1},Tx_n}(t)\geq F_{x_{n-1},x_n}\Big(\frac{t}{k}\Big), \quad (x_{n-1}\in A_n, x_n\in A_{n+1}). \tag{2.4}$$

Again, by repeated applications of (2.4), it follows that for all $t>0$ and $n\geq 0$ and each $i\geq 1$,

$$F_{x_{n+i},x_{n+i+1}}(t)\geq F_{x_n,x_{n+1}}\Big(\frac{t}{k^i}\Big). \tag{2.5}$$

So, we complete the proof. □

In our first theorem we use the following definition.

Definition 2.1 (*P*-Cyclic *C*-Contraction Result of Type-I)**.** A P-cyclic mapping T is called a P-cyclic C-contraction result of type-I if for any $r>0$ and $0<\lambda<1$, $F_{x,y}(r)>1-\lambda$ implies

$$F_{Tx,Ty}(\psi_1(r))>1-\psi_2(\lambda), \tag{2.6}$$

whenever $x\in A_i$, $y\in A_{i+1}$, and $A_{p+1}=A_1$, for any $r>0$, $0<\lambda<1$, where ψ_1, ψ_2 are Ψ-functions.

Theorem 2.1. *Let (X,F,Δ) be a complete Menger space with Hadzic type t-norm Δ and T be a P-cyclic C-contraction result of type-I (Definition 2.1), then T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.*

Proof. Let x_0 be any point in A_1. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}, n\in N$ where N is the set of natural numbers. By (1.1), we have $x_o \in A_1$, $x_1\in A_2$, $x_2\in A_3$, $\cdots$, $x_{p-1}\in A_p$ and in general

$$x_{np}\in A_1,\ x_{np+1}\in A_2,\cdots,x_{np+(p-1)}\in A_p \quad \text{for all} \quad n\geq 0. \tag{2.7}$$

Let $0<\eta<1$ be given. We can find $r>0$ such that

$$F_{x_0,x_1}(r)>1-\eta.$$

Then, by an application of (2.6), we get

$$F_{Tx_0,Tx_1}(\psi_1(r))>1-\psi_2(\eta),$$

where $x_0\in A_1$ and $x_1\in A_2$, that is,

$$F_{x_1,x_2}(\psi_1(r))>1-\psi_2(\eta).$$

Continuing this process, we obtain

$$F_{x_2,x_3}(\psi_1^2(r))=F_{Tx_1,Tx_2}(\psi_1^2(r))>1-\psi_2^2(\eta),$$

where ($x_1\in A_2$ and $x_2\in A_3$) and, in general, for all $n\in N$, we obtain

$$F_{x_n,x_{n+1}}(\psi_1^n(r))>1-\psi_2^n(\eta),$$

where $x_n\in A_{n+1}$ and $x_{n+1}\in A_{n+2}$.

Let $\epsilon>0$ be arbitrary. By the properties of Ψ-function we can find a positive integer N such that for all integer $n>N$,

$$\psi_1^n(r)<\epsilon.$$

Consequently, for all $n>N$, we get

$$F_{x_n,x_{n+1}}(\epsilon)\geq F_{x_n,x_{n+1}}(\psi_1^n(r))>1-\psi_2^n(\eta)\to 1 \quad \text{as} \quad n\to\infty. \tag{2.8}$$

Thus, for arbitrary $\epsilon>0$, we get

$$F_{x_n,x_{n+1}}(\epsilon)\to 1 \quad \text{as} \quad n\to\infty. \tag{2.9}$$

We next prove that $\{x_n\}$ is a Cauchy sequence (Definition 1.7), that is, we prove that for arbitrary $\epsilon>0$ and $0<\lambda<1$, there exists $N(\epsilon,\lambda)$ such that

$$F_{x_n,x_m}(\epsilon)>1-\lambda$$

for all $m,n\geq N$ (ϵ,λ). Without loss of generality we can assume that $m>n$.

Now,

$$\epsilon=\epsilon\frac{1-k}{1-k}>\epsilon(1-k)(1+k+k^2+\cdots+k^{m-n-1}).$$

Then, by the monotone increasing property of F, we have

$$F_{x_n,x_m}(\epsilon)\geq F_{x_n,x_m}(\epsilon(1-k)(1+k+k^2+\cdots+k^{m-n-1})),$$

that is,

$$\begin{aligned}F_{x_n,x_m}(\epsilon)\geq&\Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)),\Delta(F_{x_{n+1},x_{n+2}}(\epsilon k(1-k)),\Delta(\cdots,\\&\Delta(F_{x_{m-2},x_{m-1}}(\epsilon k^{m-n-2}(1-k)),F_{x_{m-1},x_m}(\epsilon k^{m-n-1}(1-k)))\cdots))).\end{aligned} \tag{2.10}$$

Putting $t=(1-k)\epsilon k^i$ in (2.5), we get

$$F_{x_{n+i},x_{n+i+1}}((1-k)\epsilon k^i)\geq F_{x_n,x_{n+1}}((1-k)\epsilon).$$

Then, by (2.10), we have

$$\begin{aligned}F_{x_n,x_m}(\epsilon)\geq&\Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)),\Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)),\Delta(\cdots,\\&\Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)),F_{x_n,x_{n+1}}(\epsilon(1-k)))\cdots))),\end{aligned}$$

that is,

$$F_{x_n,x_m}(\epsilon)\geq\Delta^{(m-n)}F_{x_n,x_{n+1}}(\epsilon(1-k)). \tag{2.11}$$

Since the t-norm Δ is a Hadzic type t-norm, the family $\{\Delta^p\}$ of its iterates is equicontinuous at the point $s=1$, that is, there exists $\eta(\lambda)\in(0,1)$ such that for all $m>n$, $\Delta^{(m-n)}(s)>1-\lambda$ whenever

$$\eta(\lambda)<s\leq 1. \tag{2.12}$$

Since, $F_{x_0,x_1}(t)\to 1$ as $t\to\infty$ and $0<k<1$, there exists an positive integer $N(\epsilon,\lambda)$ such that

$$F_{x_0,x_1}\Big(\frac{(1-k)\epsilon}{k^n}\Big)>\eta(\lambda)\quad\text{for all}\ \ n\geq N(\epsilon,\lambda). \tag{2.13}$$

From (2.13) and (2.5), with $n=0$, $i=n$ and $t=(1-k)\epsilon$, we get

$$F_{x_n,x_{n+1}}(\epsilon(1-k))\geq F_{x_0,x_1}\Big(\frac{(1-k)\epsilon}{k^n}\Big)>\eta(\lambda)$$

for all $n\geq N(\epsilon,\lambda)$. Then, from (2.12) with $s=F_{x_n,x_{n+1}}(\epsilon(1-k))$, we have

$$\Delta^{(m-n)}(F_{x_n,x_{n+1}}(\epsilon(1-k)))>1-\lambda.$$

It then follows from (2.11) that $F_{x_n,x_m}(\epsilon)>1-\lambda$ for all $m,n\geq N(\epsilon,\lambda)$. Thus $\{x_n\}$ is a Cauchy sequence.

By the completeness of X, there exists $z\in X$ such that

$$x_n\to z\quad\text{ as }\ \ n\to\infty. \tag{2.14}$$

By the construction of the sequence $\{x_n\}$, we have $x_p\in A_1$, $x_{2p}\in A_1$, $\cdots$, $x_{np}\in A_1$. Therefore the subsequence $\{x_{np}\}$ of $\{x_n\}$ which belongs to A_1 also converges to z in A_1, since A_1 is closed. Similarly subsequence $\{x_{np+1}\}$ belongs to A_2 also converges to z in A_2. Since A_3, A_4, $\cdots$, A_p are closed sets, similarly we get $z\in A_3$, A_4, $\cdots$, A_p. Therefore $z\in A_1\bigcap A_2\bigcap A_3\cdots\bigcap A_p$.

Now, we prove that $Tz=z$.

By (2.14), for all $t>0$, we have

$$F_{x_n,z}(\psi_1^{-1}(t))\to 1\quad\text{ as }\ \ n\to\infty,$$

that is, for arbitrary $0<\lambda<1$, we can find $N_1>0$ such that for all $n>N_1$, we have

$$F_{x_n,z}(\psi_1^{-1}(t))>1-\lambda. \tag{2.15}$$

By virtue of (2.6), we get from (2.15),

$$F_{Tx_n,Tz}(\psi_1(\psi_1^{-1}(t)))>1-\psi_2(\lambda)>1-\lambda,$$

(since $\psi_2(\lambda)<\lambda$) which implies that,

$$F_{x_{n+1},Tz}(t)>1-\lambda. \tag{2.16}$$

Now, taking limit as $n\to\infty$ on both sides of (2.16), for all $t>0$, we have

$$F_{z,Tz}(t)\geq 1-\lambda.$$

Since λ is arbitrary, and for $t>0$, we obtain

$$F_{z,Tz}(t)=1,$$

that is, $z=Tz$.

To prove the uniqueness of the fixed point, let v be another fixed point of T in $A_1\bigcap A_2\bigcap A_3\cdots\bigcap A_p$, that is, $Tv=v$.

We can get $\epsilon_1>0$ such that

$$F_{z,v}(\epsilon_1)>1-\lambda,$$

where $0<\lambda<1$. Then, by the inequality (2.6), we have

$$F_{Tz,Tv}(\psi_1(\epsilon_1))>1-\psi_2(\lambda),$$

that is,

$$F_{z,v}(\psi_1(\epsilon_1))>1-\psi_2(\lambda).$$

Continuing this process n times, we obtain

$$F_{z,v}(\psi_1^n(\epsilon_1))>1-\psi_2^n(\lambda). \tag{2.17}$$

For arbitrary $\mu>0$, by virtue of properties of ψ-function it is possible to find $N>0$ such that

$$\psi_1^n(\epsilon_1)<\mu \quad \text{for all} \quad n>N. \tag{2.18}$$

Combining (2.17) and (2.18), we have

$$F_{z,v}(\mu)\geq F_{z,v}(\psi_1^n(\epsilon_1))>1-\psi_2^n(\lambda) \quad \text{for all} \quad n>N.$$

Taking $n\to\infty$ both sides of the above inequality, and for all $\mu>0$, we have

$$F_{z,v}(\mu)=1,$$

that is, $z=v$. Hence T have a unique fixed point in $A_1\bigcap A_2\bigcap A_3\cdots\bigcap A_p$. □

Example 2.1. Let $X=R$ and

$$F_{x,y}(t)=\begin{cases} \dfrac{t}{t+|x-y|}, & \text{if } t>0, \\ 0, & \text{if } t\leq 0, \end{cases}$$

we take $A_1=[-3,0]=A_3$ and $A_2=[0,3]=A_4$ of X and $Y=\bigcup_{i=1}^4 A_i$. Define $T:Y\to Y$ by $Tx=-x$ for all $x\in Y$. It is easily verified that $T(A_i)\subset A_{i+1}$ for $i=1,2,3$ and $T(A_4)\subset A_1$, hence T is a p-cyclic mapping.

We consider the following possible cases:

Case-I: For $x\in A_1=A_3$ and $y\in A_2=A_4$, $x=y$.

$$\begin{aligned} &F_{x,y}(r)=\frac{r}{r+|x-y|}=1>1-\lambda\Rightarrow\lambda>0, \\ &F_{Tx,Ty}(\psi_1(r))=F_{Tx,Ty}(r)=\frac{r}{r+|Tx-Ty|}=1>1-\psi_2(\lambda)=1-\lambda\Rightarrow\lambda>0. \end{aligned}$$

Hence $F_{x,y}(r)>1-\lambda\Rightarrow F_{Tx,Ty}(\psi_1(r))>1-\psi_2(\lambda)$.

Case-II: For $x\in A_1=A_3$ and $y\in A_2=A_4$, $x\neq y$.

$$\begin{aligned} F_{x,y}(r)&=\frac{r}{r+|x-y|}>1-\lambda\Rightarrow\frac{|x-y|}{r+|x-y|}<\lambda, \\ F_{Tx,Ty}(\psi_1(r))&=F_{Tx,Ty}(r)=\frac{r}{r+|Tx-Ty|}=\frac{r}{r+|-x+y|} \\ &=\frac{r}{r+|x-y|}>1-\psi_2(\lambda)=1-\lambda\Rightarrow\frac{|x-y|}{r+|x-y|}<\lambda. \end{aligned}$$

Hence $F_{x,y}(r)>1-\lambda\Rightarrow F_{Tx,Ty}(\psi_1(r))>1-\psi_2(\lambda)$.

Case-III: For $x\in A_2=A_4$ and $y\in A_3=A_1$, $x\neq y$. The proof is similar as in Case-II.

Case-IV: For $x\in A_2=A_4$ and $y\in A_3=A_1$, $x=y$. The proof is similar as in Case-I.

Following the above four cases it is easily verified that if we take $\psi_1(t)=\psi_2(t)=t$ and Δ is the minimum t-norm, then the mapping T satisfies the inequality (2.6) of Theorem 2.1. Taking $\phi(t)=2t$, $\psi_1(t)=t^2$, $\psi_2(t)=5t/7$, T does not satisfy the inequality described in Theorem 2.2.

Example 2.2. Let $X=\{\frac{1}{n}\}\bigcup\{0\}$ with F defined as in Example 2.1. Now we consider the following subsets of X:

$$A_1=\Big\{\frac{1}{n}\Big\backslash n \text{ is odd}\Big\}\bigcup\{0\}$$

and

$$A_2=\Big\{\frac{1}{n}\Big\backslash n \text{ is even}\Big\}\bigcup\{0\}.$$

Consider the mapping $T:X\to X$ given by

$$Tx=\begin{cases} 0, & \text{if } x=0,\\ \dfrac{1}{n+1}, & \text{if } x=\dfrac{1}{n}, \quad n\in N. \end{cases}$$

Now A_1 and A_2 are closed and $X=\bigcup_{i=1}^{2}A_i$ is a cyclic representation of X with respect to T. Following the result of [26] it may be easily examined that the mapping T is an p-cyclic mapping where $p=2$.

We use the control function ϕ (Definition 1.10) in our next theorem in the inequality (2.6). Here we use the Lukasiewicz t-norm Δ (Defined as $\Delta(a,b)=\max\{a+b-1,0\}$ for all $a,b\in[0,1]$. We also prove our second theorem by different arguments from the first theorem.

In our next theorem we use the following definition.

Definition 2.2 (P-Cyclic C-Contraction Result of type-II)**.** A P-cyclic mapping T is called a P-cyclic C-contraction result of type-II if for any $r>0$ and $0<\lambda<1$, $F_{x,y}(\phi(r))>1-\lambda$ implies

$$F_{Tx,Ty}(\phi(\psi_1(r)))>1-\psi_2(\lambda), \tag{2.19}$$

whenever $x\in A_i$, $y\in A_j$, $i\neq j$ and $A_{p+1}=A_1$, where ϕ is a Φ-function and ψ_1, ψ_2 are Ψ-functions.

In particular, if $\phi(t)=t$, $\psi_1(t)=\psi_2(t)=kt$, for all $t\geq 0$ and $p=2$, we call the mapping T as cyclic C-contraction.

Theorem 2.2. *Let (X,F,Δ) be a complete Menger space with the Lukasiewicz t-norm Δ (Defined as $\Delta(a,b)=\max\{a+b-1,0\}$ for all $a,b\in[0,1]$) and T be a P-cyclic C-contraction result of type-II (Definition 2.2) on X. Then T has a unique fixed point in $\bigcap_{i=1}^{p}A_i$.*

Proof. Let x_0 be any arbitrary point in A_1. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n=Tx_{n-1}$, $n\in N$ where N is the set of natural numbers.

By (1.1), we have $x_o\in A_1$, $x_1\in A_2$, $x_2\in A_3$, $\cdots$, $x_{p-1}\in A_p$ and in general

$$x_{np}\in A_1, x_{np+1}\in A_2,\cdots,x_{np+(p-1)}\in A_p \quad \text{for all} \quad n\geq 0. \tag{2.20}$$

Let $0<\eta<1$ be given. By the property that $\phi(t)\to\infty$ as $t\to\infty$, we can find $r>0$ such that $F_{x_0,x_1}(\phi(r))>1-\eta$.

Then, by an application of (2.19), we get $F_{Tx_0,Tx_1}(\phi(\psi_1(r)))>1-\psi_2(\eta)$, (where $x_0\in A_1$ and $x_1\in A_2$) that is,

$$F_{x_1,x_2}(\phi(\psi_1(r)))>1-\psi_2(\eta).$$

Continuing this process, we obtain $F_{x_2,x_3}(\phi(\psi_1^2(r)))=F_{Tx_1,Tx_2}(\phi(\psi_1^2(r)))>1-\psi_2^2(\eta)$, (where $x_1\in A_2$ and $x_2\in A_3$) and, in general, for all $n\in N$, we obtain

$$F_{x_n,x_{n+1}}(\phi(\psi_1^n(r)))>1-\psi_2^n(\eta),$$

where $x_n \in A_{n+1}$ and $x_{n+1} \in A_{n+2}$.

Let $\epsilon > 0$ be arbitrary. By the properties of Φ-function and Ψ-function we can find a positive integer N such that for all integer $n > N$,

$$\phi(\psi_1^n(r)) < \epsilon.$$

Consequently, for all $n > N$, we get

$$F_{x_n,x_{n+1}}(\epsilon) \geq F_{x_n,x_{n+1}}(\phi(\psi_1^n(r))) > 1-\psi_2^n(\eta) \to 1 \quad \text{as} \quad n \to \infty. \tag{2.21}$$

Thus, for arbitrary $\epsilon > 0$, we get

$$F_{x_n,x_{n+1}}(\epsilon) \to 1 \quad \text{as} \quad n \to \infty. \tag{2.22}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then, there exist $\epsilon > 0$ and $0 < \lambda < 1$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\epsilon) \leq 1-\lambda. \tag{2.23}$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (2.23), so that

$$F_{x_{m(k)},x_{n(k)-1}}(\epsilon) > 1-\lambda. \tag{2.24}$$

If $\epsilon_1 < \epsilon$ then, we have

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_1) \leq F_{x_{m(k)},x_{n(k)}}(\epsilon).$$

From the above, we conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $n(k) > m(k) > k$ and satisfying (2.23), (2.24) whenever ϵ is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0)=0$, it is possible to obtain $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2)) \leq 1-\lambda, \tag{2.25}$$

and

$$F_{x_{m(k)},x_{n(k)-1}}(\phi(\in_2)) > 1-\lambda. \tag{2.26}$$

From the definition of ψ-function ψ_1^{-1} is continuous at ϵ_2 and $\epsilon_2 < \psi_1^{-1}(\epsilon_2)$. So there exists $\eta_1 > 0$ such that

$$\psi_1^{-1}(\epsilon_2 - \eta_1) > \epsilon_2. \tag{2.27}$$

We take

$$0<\eta_2=\phi(\epsilon_2)-\phi(\epsilon_2-\eta_1). \tag{2.28}$$

Which is possible as ϕ is strictly increasing. Then, from (2.25), we have

$$\begin{aligned}1-\lambda\geq F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2))&\geq\Delta(F_{x_{m(k)},x_{m(k)+1}}(\eta_2),F_{x_{m(k)+1},x_{n(k)}}(\phi(\epsilon_2)-\eta_2))\\&=\Delta(F_{x_{m(k)},x_{m(k)+1}}(\eta_2),F_{x_{m(k)+1},x_{n(k)}}(\phi(\epsilon_2-\eta_1)))\quad\text{by (2.28).}\end{aligned} \tag{2.29}$$

Now, by (2.26) and (2.27)

$$F_{x_{m(k)},x_{n(k)-1}}(\phi(\psi_1^{-1}(\epsilon_2-\eta_1)))\geq F_{x_{m(k)},x_{n(k)-1}}(\phi(\epsilon_2))>1-\lambda.$$

Then, using (2.19), we have

$$F_{x_{m(k)+1},x_{n(k)}}(\phi(\epsilon_2-\eta_1))=F_{Tx_{m(k)},Tx_{n(k)-1}}(\phi(\psi_1(\psi_1^{-1}(\epsilon_2-\eta_1))))>1-\psi_2(\lambda),$$

that is,

$$F_{x_{m(k)+1},x_{n(k)}}(\phi(\epsilon_2-\eta_1))>1-\psi_2(\lambda). \tag{2.30}$$

Then, from (2.29), we have

$$1-\lambda\geq\Delta(F_{x_{m(k)},x_{m(k)+1}}(\eta_2),1-\psi_2(\lambda))\quad\text{by (2.30).} \tag{2.31}$$

Taking $k\to\infty$ in (2.31), using (2.22), and the continuity of Δ, we have

$$1-\lambda\geq\Delta(1,1-\psi_2(\lambda))=\max\{1-\psi_2(\lambda),0\}=1-\psi_2(\lambda),$$

(since Δ is a Lukasiewicz t-norm Δ (Defined as $\Delta(a,b)=\max\{a+b-1,0\}$ for all $a,b\in[0,1]$), which implies $\psi_2(\lambda)\geq\lambda$, which contradicts the fact that $\psi_2(\lambda)<\lambda$. Thus $\{x_n\}$ is a Cauchy sequence.

By the completeness of X, there exists $z\in X$ such that

$$x_n\to z\quad\text{as}\quad n\to\infty. \tag{2.32}$$

By the construction of the sequence $\{x_n\}$, we have $x_p\in A_1$, $x_{2p}\in A_1$, $\cdots$, $x_{np}\in A_1$. Therefore the subsequence $\{x_{np}\}$ of $\{x_n\}$ which belongs to A_1 also converges to z in A_1, since A_1 is closed. Similarly subsequence $\{x_{np+1}\}$ belongs to A_2 also converges to z in A_2. Since A_3, A_4, $\cdots$, A_p are closed sets, similarly we get $z\in A_3$, A_4, $\cdots$, A_p. Therefore, $z\in A_1\bigcap A_2\bigcap A_3\cdots\bigcap A_p$.

Now, we prove that $Tz=z$.

By (2.32), for all $t>0$, we have

$$F_{x_n,z}(\phi(\psi_1^{-1}(t)))\to 1\quad\text{as}\quad n\to\infty,$$

that is, for arbitrary $0<\lambda<1$, we can find $N_1>0$ such that for all $n>N_1$, we have

$$F_{x_n,z}(\phi(\psi_1^{-1}(t)))>1-\lambda. \tag{2.33}$$

By virtue of (2.19), we get from (2.33),

$$F_{Tx_n,Tz}(\phi(\psi_1(\psi_1^{-1}(t))))>1-\psi_2(\lambda)>1-\lambda,$$

since $\psi_2(\lambda)<\lambda$, which implies that,

$$F_{x_{n+1},Tz}(\phi(t))>1-\lambda. \tag{2.34}$$

Now, taking limit as $n\to\infty$ on both sides of (2.34), for all $t>0$, we have

$$F_{z,Tz}(\phi(t))\geq 1-\lambda.$$

Since λ is arbitrary, for all $t>0$, we obtain

$$F_{z,Tz}(\phi(t))=1.$$

But a property of ϕ implies that given $s>0$ we can find $t>0$ such that $\phi(t)<s$. Then it follows that for all $s>0$, $F_{z,Tz}(s)=1$, that is, $z=Tz$.

To prove the uniqueness of the fixed point, let v be another fixed point of T, that is, $Tv=v$. By the properties of ϕ-function, we can get $\epsilon_1>0$ such that

$$F_{z,v}(\phi(\epsilon_1))>1-\lambda,$$

where $0<\lambda<1$. Then, by the inequality (2.19), we have

$$F_{Tz,Tv}(\phi(\psi_1(\epsilon_1)))>1-\psi_2(\lambda),$$

that is,

$$F_{z,v}(\phi(\psi_1(\epsilon_1)))>1-\psi_2(\lambda).$$

Continuing this process n times, we obtain

$$F_{z,v}(\phi(\psi_1^n(\epsilon_1)))>1-\psi_2^n(\lambda). \tag{2.35}$$

For arbitrary $\mu>0$, by virtue of properties of ϕ-function and ψ-function it is possible to find $N>0$ such that

$$\phi(\psi_1^n(\epsilon_1))<\mu \quad \text{for all } \ n>N. \tag{2.36}$$

Combining (2.35) and (2.36), we have

$$F_{z,v}(\mu)\geq F_{z,v}(\phi(\psi_1^n(\epsilon_1)))>1-\psi_2^n(\lambda) \quad \text{for all } \ n>N.$$

Taking $n\to\infty$ both sides of the above inequality, and for all $\mu>0$, we have

$$F_{z,v}(\mu)=1,$$

that is, $z=v$. Hence T have a unique fixed point in $A_1\bigcap A_2\bigcap A_3\cdots\bigcap A_p$. ☐

Taking $p=2$, we get the following corollary.

Corollary 2.1. Let (X,F,Δ) be a complete Menger space, where Δ is the minimum t-norm. Let A and B be two non-empty closed subsets of X. Let a mapping $T: A\bigcup B\to A\bigcup B$ satisfies the following conditions:
(i) $TA\subseteq B$ and $TB\subseteq A$,
(ii) $F_{x,y}(t)>1-t$ implies $F_{Tx,Ty}(kt)>1-kt$,
for all $x\in A$ and $y\in B$, where $0<k<1$ and $t>0$. Then $A\bigcap B$ is non empty and T has a unique fixed point in $A\bigcap B$.

If we take $\phi(t)=t$ and $\psi_1(t)=\psi_2(t)=kt$, in Theorem 2.2, where $0<k<1$, Δ is a minimum t-norm, we obtain the following corollary.

Corollary 2.2. Let (X,F,Δ) be a complete Menger space with minimum t-norm Δ and $0<k<1$. Let $T:X\to X$ be a P-cyclic mapping such that for all $r>0$, $0<\lambda<1$ and for all $x\in A_i$, $y\in A_{i+1}$,

$$F_{x,y}(r)>1-\lambda$$

implies

$$F_{Tx,Ty}(kr)>1-k\lambda.$$

Then T has a unique fixed point in $A_1\bigcap A_2\bigcap A_3\cdots\bigcap A_p$.

The above corollary is actually the extension of C-contraction in Menger spaces. Taking $p=2$ we get the following example.

Example 2.3. Let $X=\{x_1,x_2,x_3,x_4\}$, $A=\{x_1,x_2,x_4\}$ and $B=\{x_2,x_3\}$. Here the t-norm $\Delta(a,b)=\min(a,b)$ and $F_{x,y}(t)$ be defined as

$$F_{x_1,x_2}(t)=F_{x_1,x_3}(t)=F_{x_1,x_4}(t)=F_{x_2,x_4}(t)=F_{x_3,x_4}(t)=\begin{cases}0, & \text{if } t\le 0,\\ 0.4, & \text{if } 0<t<4,\\ 1, & \text{if } t\ge 4,\end{cases}$$

$$F_{x_2,x_3}(t)=\begin{cases}0, & \text{if } t\le 0,\\ 0.60, & \text{if } 0<t\le 7,\\ 1, & \text{if } t>7.\end{cases}$$

It may be easily verified that (X,F,Δ) is a complete Menger space. If we define $T:A\bigcup B\to A\bigcup B$ as follows: $Tx_1=x_2$, $Tx_2=x_2$, $Tx_3=x_2$, $Tx_4=x_3$, then it satisfies all the conditions of the Theorem 2.2 where $\phi(t)=2t$, $\psi_1(t)=t^2$, $\psi_2(t)=5t/7$ and x_2 is the unique fixed point of T.

Remark 2.1. It is to be noted that the method of proof of Theorem 2.2 is different from that of Theorem 2.1. This is due to the use of the control function in Theorem 2.2 that the method in the proof of Theorem 2.1 can not be adopted here. Also the Theorem 2.2 could be proved here only with Lukasiewicz t-norm. It remains an open problem whether the proof of Theorem 2.2 can be accomplished with Hadzic type t-norm as in Theorem 2.1.

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