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On the Green Function of the Annulus

Massimo Grossi[1,*] and Djordjije Vujadinović

[1] *Massimo Grossi, Dipartimento di Matematica, Università di Roma La Sapienza, P.le A. Moro 2–00185 Roma, Italy*
[2] *Djordjije Vujadinović, Faculty of Mathematics, University of Montenegro, Dzordza Văsingtona bb, 81000 Podgorica, Montenegro*

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Abstract. Using the Gegenbauer polynomials and the zonal harmonics functions we give some representation formula of the Green function in the annulus. We apply this result to prove some uniqueness results for some nonlinear elliptic problems.

Key Words: Green's function, symmetries, uniqueness.

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1 Introduction and statement of the main results

The classical Green function of the operator $-\Delta$ with Dirichlet boundary conditions is defined by

$$\begin{cases} -\Delta_x G(x,y)=\delta_y(x) & \text{in } \Omega, \\ G(x,y)=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where δ_y is the Dirac function centered at y and Ω is a bounded domain of $\mathbb{R}^n$, $n\geq 2$.

It is well known that the Green function can be written as

$$G(x,y)=\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}+H(x,y), \tag{1.2}$$

where $H(x,y)$ is a smooth function in $\Omega\times\Omega$ which is harmonic in both variables x and y. Finally the Robin function is defined as

$$R(x)=H(x,x). \tag{1.3}$$

The knowledge of the Green (or the Robin) function is of great importance in applications (we mention the paper [2] and the rich list of references therein). Indeed the explicit

*Corresponding author. *Email addresses:* massimo.grossi@uniroma1.it (M. Grossi), djordjijevuj@t-com.me (D. Vujadinović)

calculation of the Green function is an old problem (see for example the book by Courant and Hilbert, [5]) but it can be solved only in special cases (like the ball or half-space).

For these reason, even if it is not possible to have the explicit expression, it is very important to deduce any properties of the Green function.

In this paper we are interested to the case where the domain is the *annulus* in $\mathbb{R}^n$, namely $\Omega=\{x\in\mathbb{R}^n:\ a<|x|<b\}$ (in the rest of the paper by simplicity we assume that $b=1$). Even if the annuls possesses many symmetries, you can not explicitly write the Green function. If $n=2$ in [7] it was given a representation for the Green function as trigonometrical series. In this paper we give a representation formula of the Green function when $n\geq 3$ using the *zonal spherical harmonics*. Our first result is the following,

Theorem 1.1. *Let A be the annulus $A=\{x\in\mathbb{R}^n:\ a<|x|<1\}$. Then we have that the Green function in A is given by,*

$$\begin{aligned} &G_A(x,y)\\ =&\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}\\ &-\frac{1}{\omega_{n-1}}\sum_{m=0}^{\infty}\frac{|x|^{2m+n-2}|y|^{2m+n-2}-a^{2m+n-2}(|x|^{2m+n-2}+|y|^{2m+n-2})+a^{2m+n-2}}{(2m+n-2)(|x||y|)^{n+m-2}(1-a^{2m+n-2})}Z_m\Big(\frac{x}{|x|},\frac{y}{|y|}\Big). \end{aligned}\tag{1.4}$$

Moreover the Robin function is given by, setting $d_0=1$ and

$$d_m=\binom{n+m-2}{n-2}-\binom{n+m-3}{n-2,},$$

for $m\geq 1$,

$$R_A(x)=-\frac{1}{\omega_{n-1}}\sum_{m=0}^{\infty}d_m\frac{a^{2m+n-2}-2a^{2m+n-2}|x|^{2m+n-2}+|x|^{4m+2n-4}}{(2m+n-2)|x|^{2m+2n-4}(1-a^{2m+n-2})}.\tag{1.5}$$

Here $Z_m(x,y)$ are the zonal spherical harmonics (see Section 2 or [1] for the definition and main properties).

Next corollary gives an alternative expression of the Green function which does not involve the Newtonian potential.

Corollary 1.1. Let A be the annulus $A=\{x\in\mathbb{R}^n: a<|x|<1\}$. Then we have that the Green function is given by,

$$G_A(x,y)=\begin{cases}\dfrac{1}{\omega_{n-1}}\displaystyle\sum_{m=0}^{\infty}\frac{(|x|^{2m+n-2}-a^{2m+n-2})(1-|y|^{2m+n-2})}{(2m+n-2)(|x||y|)^{n+m-2}(1-a^{2m+n-2})}Z_m\Big(\frac{x}{|x|},\frac{y}{|y|}\Big), & \text{if } |x|<|y|,\\ \dfrac{1}{2^{n-2}|x|^{n-2}|1-\frac{x\cdot y}{|x|^2}|}+R_A(x), & \text{if } |x|=|y|,\quad x\neq y,\\ \dfrac{1}{\omega_{n-1}}\displaystyle\sum_{m=0}^{\infty}\frac{(|y|^{2m+n-2}-a^{2m+n-2})(1-|x|^{2m+n-2})}{(2m+n-2)(|x||y|)^{n+m-2}(1-a^{2m+n-2})}Z_m\Big(\frac{x}{|x|},\frac{y}{|y|}\Big), & \text{if } |x|>|y|.\end{cases}\tag{1.6}$$

These results are useful to derive some properties of the Robin function of the annulus. Actually we will show that the Robin function is a radial function which admits only one critical point which is *nondegenerate* in the radial direction.

Theorem 1.2. *Let A be the annulus $\{x\in R^n \ |a<|x|<1\}$ for $n\geq 2$ and $R_A(x)$ the corresponding Robin function. Then, if $r=|x|$, we have that $R_A(x)=R_A(r)$ and $R_A(r)$ has a unique critical point r_0, which satisfies $R''_A(r_0)>0$.*

Note that this result was proved, when $n=2$, in [4] using different techniques. In Proposition 3.1 we give an alternative proof of this result.

Finally we apply these results to deduce some properties of nonlinear PDE's problem. A straightforward application is a *uniqueness* results of concentrating solutions. Let us consider the problem

$$\begin{cases} -\Delta u = N(N-2)u^{\frac{n+2}{n-2}}+\varepsilon u & \text{in } A, \\ u>0 & \text{in } A, \\ u=0 & \text{on } \partial A, \end{cases} \tag{1.7}$$

and

$$S_\varepsilon = \inf_{\substack{u\in H^1_0(A)\\ u\not\equiv 0}} \frac{\int_A(|\nabla u|^2-\varepsilon u^2)}{\left(\int_A |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}. \tag{1.8}$$

It is well known (see [3]) that there exists solutions u_ε which achieve S_ε and satisfying

$$S_\varepsilon \to S, \tag{1.9}$$

where S is the best constant in Sobolev inequalities. In the next theorem we show the uniqueness of this solution (up to a suitable rotation).

Theorem 1.3. *Let us suppose that $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ are two solutions of* (1.7) *satisfying* (1.9). *Then, up to a suitable rotation, we have that*

$$u_{1,\varepsilon} \equiv u_{2,\varepsilon} \tag{1.10}$$

for ε small enough.

When Ω is a generic domain of $\mathbb{R}^n$, Theorem 1.3 was proved by Glangetas (see [6]) under the assumption that the critical point of the Robin function is nondegenerate. Of course, due to the rotationally invariance of the annulus, any critical point is degenerate and so Glangetas' result is therefore not applicable (note that the author conjectured the uniqueness result in the annulus at page 573 in [6]). Indeed the meaning of Theorem 1.3 is that just the nondegeneracy in the radial direction is necessary to have the uniqueness of the solution up to a suitable rotation.

The paper is organized as follows: in Section 2 we recall some preliminaries about the zonal harmonics and the Gegenbauer polynomials. In Section 3 we prove the Theorem 1.1, Corollary 1.1 and some properties of the Robin function (proof of Theorem 1.2). Finally in Section 4 we prove Theorem 1.3.

2 Preliminaries

In this Section we would like to point out the basic properties of zonal harmonics which are going to be used trough the paper. A good reference for the interested reader is the book [1].

By $H_m(R^n)$ we are going to denote the finite dimensional Hilbert space of all harmonic homogeneous polynomials of degree m.

Let us denote by S the unit sphere of $\mathbb{R}^n$. A *spherical harmonic* of degree m is the restriction to S of an element of $H_m(\mathbb{R}^n)$. The collection of all spherical harmonics of degree m will be denoted by $H_m(S)$.

Now we consider an important subset of $H_m(S)$, the so-called *zonal harmonics*. They can be defined in different ways. We choose the equivalent definition given in Theorem 5.38 in [1].

For $x\in\mathbb{R}^n$ with $n\geq 2$ and $\xi\in S$ we define the zonal harmonic $Z_m(x,\xi)$ of degree m as

$$Z_0(x,\xi)=1, \tag{2.1a}$$

$$Z_m(x,\xi)=(n+2m-2)\sum_{k=0}^{[m/2]}(-1)^k\frac{n(n+2)\cdots(n+2m-2k-4)}{2^k k!(m-2k)!}(x\cdot\xi)^{m-2k}|x|^{2k}, \tag{2.1b}$$

as $m>0$. Several properties of the zonal harmonic can be found in Chapter 5 of [1]. Let us emphasize that there is an explicit formula for the zonal harmonic as $n=2$,

$$Z_m(e^{i\theta},e^{i\phi})=2\cos(m(\theta-\phi)).$$

The zonal harmonics have a particularly simple expression in terms of the Gegenbauer (or ultraspherical) polynomials P_m^λ. The latter can be defined in terms of generating functions. If we write (see [11, pp. 148])

$$(1-2rt+r^2)^{-\lambda}=\sum_{m=0}^{\infty}P_m^\lambda(t)r^m, \tag{2.2}$$

where $0\leq|r|<1$, $|t|\leq 1$ and $\lambda>0$, then the coefficient P_m^λ is called Gegenbauer polynomial of degree m associated with λ.

The next theorem (see [11, pages 146–150]) is related to representation of the zonal harmonics.

Theorem 2.1. *If $n>2$ is an integer, $\lambda=\frac{n-2}{2}$ and $k=0,1,2,\cdots$, then we have that for all $x',y'\in S$ it holds*

$$Z_m(x',y')=\frac{2m+n-2}{n-2}P_m^\lambda(x'\cdot y'). \tag{2.3}$$

Proof. In Corollary 2.13 in [11] it is proved that

$$Z_m(x',y')=c_{n,m}P_m^\lambda(x'\cdot y'). \tag{2.4}$$

Let us compute the constant $c_{n,m}$. If we put $x'=y'\in S$ in (2.3) we get

$$Z_m(x',x')=c_{n,m}P_m^{\lambda}(1). \tag{2.5}$$

In [1], Proposition 5.27 and Proposition 5.8 showed that

$$Z_m(x',x')=\binom{n+m-2}{n-2}-\binom{n+m-3}{n-2}=\binom{n+m-3}{m}\frac{2m+n-2}{n-2}. \tag{2.6}$$

On the other hand in [9], it was shown that

$$P_m^{\frac{n-2}{2}}(1)=\binom{n+m-3}{m}. \tag{2.7}$$

By (2.5)-(2.7) we deduce that $c_{n,m}=\frac{2m+n-2}{n-2}$. □

We end this section pointing out the result (see [1, pp. 217, Theorem 10.13]) related to the solution of Dirichlet problem in annulus. Recall that $A=\{x\in R^n|a<|x|<1\}$ and set

$$P_A[f](x)=\int_S f(\xi)P_A(x,\xi)d\sigma(\xi)+\int_S f(a\xi)P_A(x,a\xi)d\sigma(\xi),$$

where

$$P_A(x,\xi)=\sum_{m=0}^{\infty}b_m(x)Z_m(x,\xi),b_m(x)=\frac{1-(a/|x|)^{2m+n-2}}{1-a^{2m+n-2}},$$

and

$$P_A(x,a\xi)=\sum_{m=0}^{\infty}c_m(x)Z_m(x,\xi),c_m(x)=|x|^{-m}\Big(\frac{a}{|x|}\Big)^{m+n-2}\frac{1-|x|^{2m+n-2}}{1-a^{2m+n-2}}.$$

Both series $P_A(x,\xi)$, $P_A(x,a\xi)$ converge absolutely and uniformly on $K\times S$, $K\subset A$ (K is some compact subset). We have the following

Theorem 2.2. *Suppose $n>2$ and that f is continuous function on ∂A. Define u on $\bar{A}$ by*

$$u(x)=\begin{cases} P_A[f](x), & x\in A,\\ f(x), & x\in\partial A.\end{cases}$$

Then u is continuous on $\bar{A}$ and harmonic on A.

3 The representation formula for the Green function

Our first aim is to write the Green function for the annulus in terms of $Z_m(x,y)$. The starting point for our results is going to be the next easy lemma which will play an important role in proving Theorem 1.1.

Lemma 3.1. *We have that, for any* $|\xi|=1$, $|y|\leq 1$ *and* $y\neq\xi$,

$$\frac{1}{|\xi-y|^{n-2}}=\sum_{m=0}^{\infty}\frac{n-2}{2m+n-2}|y|^m Z_m\Big(\xi,\frac{y}{|y|}\Big) \tag{3.1}$$

and

$$\frac{1}{|a\xi-y|^{n-2}}=\sum_{m=0}^{\infty}\frac{n-2}{2m+n-2}\frac{a^m}{|y|^{n+m-2}}Z_m\Big(\xi,\frac{y}{|y|}\Big). \tag{3.2}$$

Proof. Since $|\xi|=1$, by using the formula (2.2) we obtain

$$\begin{aligned}\frac{1}{|\xi-y|^{n-2}}&=\Big(1-2|y|\Big(\frac{y}{|y|}\cdot\xi\Big)+|y|^2\Big)^{-\frac{n-2}{2}}\\&=\sum_{m=0}^{\infty}|y|^m P_m^{\frac{n-2}{2}}\Big(\xi\cdot\frac{y}{|y|}\Big)\\&=\sum_{m=0}^{\infty}\frac{n-2}{2m+n-2}|y|^m Z_m\Big(\xi,\frac{y}{|y|}\Big).\end{aligned} \tag{3.3}$$

In a similar way we prove (3.2):

$$\begin{aligned}\frac{1}{|a\xi-y|^{n-2}}&=|y|^{2-n}\Big(1-2\frac{a}{|y|}\Big(\frac{y}{|y|}\cdot\xi\Big)+\frac{a^2}{|y|^2}\Big)^{-\frac{n-2}{2}}\\&=|y|^{2-n}\sum_{m=0}^{\infty}\frac{a^m}{|y|^m}P_m^{\frac{n-2}{2}}\Big(\xi\cdot\frac{y}{|y|}\Big)\\&=\sum_{m=0}^{\infty}\frac{n-2}{2m+n-2}\frac{a^m}{|y|^{n+m-2}}Z_m\Big(\xi,\frac{y}{|y|}\Big).\end{aligned} \tag{3.4}$$

Thus, we complete the proof. $\square$

Now we are in position to prove our representation formula for the Green function.

Proof of Theorem 1.1. By (1.2) we have to write $H(x,y)$, where $H(x,y)$ is an harmonic function satisfying

$$H(x,y)=-\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}$$

on ∂A. By Theorem 2.2 we have that $H(x,y)=P_A[f_y](x)$ with

$$f_y(x)=-\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}},$$

where $y\in A$ is fixed. Using Lemma 3.1, we obtain

$$\begin{aligned} P_A[f_y](x) &= -\frac{1}{(n-2)\omega_{n-1}}\int_S \frac{1}{|\xi-y|^{n-2}}P_A(x,\xi)d\sigma(\xi) \\ &\quad -\frac{1}{(n-2)\omega_{n-1}}\int_S \frac{1}{|a\xi-y|^{n-2}}P_A(x,a\xi)d\sigma(\xi) \\ &= -\frac{1}{(n-2)\omega_{n-1}}I_1 - \frac{1}{(n-2)\omega_{n-1}}I_2. \end{aligned} \tag{3.5}$$

So we have that

$$\begin{aligned} I_1 &= \int_S \frac{1}{|\xi-y|^{n-2}}P_A(x,\xi)d\sigma(\xi) = \sum_{m=0}^{\infty} b_m(x)\int_S \frac{Z_m(x,\xi)}{|\xi-y|^{n-2}}d\sigma(\xi) \\ &= \sum_{m=0}^{\infty} b_m(x)\int_S \sum_{k=0}^{\infty}\frac{n-2}{2k+n-2}|y|^k Z_k\Big(\frac{y}{|y|},\xi\Big)Z_m(x,\xi)d\sigma(\xi) \\ &= \sum_{m=0}^{\infty} b_m(x)\frac{n-2}{2m+n-2}|y|^m\int_S Z_m\Big(\frac{y}{|y|},\xi\Big)Z_m(x,\xi)d\sigma(\xi) \\ &= \sum_{m=0}^{\infty} b_m(x)\frac{n-2}{2m+n-2}|y|^m\int_S Z_m(x,\xi)Z_m\Big(\xi,\frac{y}{|y|}\Big)d\sigma(\xi) \\ &= \sum_{m=0}^{\infty} b_m(x)\frac{n-2}{2m+n-2}|y|^m Z_m\Big(x,\frac{y}{|y|}\Big). \end{aligned} \tag{3.6}$$

Similarly, for the second integral, we get

$$\begin{aligned} I_2 &= \int_S \frac{1}{|a\xi-y|^{n-2}}P_A(x,a\xi)d\sigma(\xi) \\ &= \sum_{m=0}^{\infty} c_m(x)\int_S \sum_{k=0}^{\infty}\frac{n-2}{2k+n-2}\frac{a^k}{|y|^{n+k-2}}Z_k\Big(\xi,\frac{y}{|y|}\Big)Z_m(x,\xi)d\sigma(\xi) \\ &= \sum_{m=0}^{\infty} c_m(x)\frac{n-2}{2m+n-2}\frac{a^m}{|y|^{n+m-2}}\int_S Z_m\Big(\frac{y}{|y|},\xi\Big)Z_m(x,\xi)d\sigma(\xi) \\ &= \sum_{m=0}^{\infty} c_m(x)\frac{n-2}{2m+n-2}\frac{a^m}{|y|^{n+m-2}}Z_m\Big(x,\frac{y}{|y|}\Big). \end{aligned} \tag{3.7}$$

So, we obtain

$$\begin{aligned} &G_A(x,y) \\ =&\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}} - \frac{1}{(n-2)\omega_{n-1}}\sum_{m=0}^{\infty}\frac{\frac{c_m(x)a^m}{|y|^{n+m-2}}+|y|^m b_m(x)}{c_{n,m}}Z_m\Big(x,\frac{y}{|y|}\Big) \\ =&\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}} \end{aligned}$$

$$-\frac{1}{(n-2)\omega_{n-1}}\sum_{m=0}^{\infty}\frac{a^{2m+n-2}(1-|x|^{2m+n-2})+|y|^{2m+n-2}(|x|^{2m+n-2}-a^{2m+n-2})}{c_{n,m}(|x||y|)^{n+m-2}(1-a^{2m+n-2})}Z_m\Big(\frac{x}{|x|},\frac{y}{|y|}\Big). \tag{3.8}$$

The Robin function is

$$\begin{aligned}R_A(y)=&\lim_{x\to y}\Big(G_A(x,y)-\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}\Big)\\=&-\sum_{m=0}^{\infty}d_m\frac{\frac{a^{2m+n-2}}{|y|^{2m+2n-4}}+|y|^{2m}-2\frac{a^{2m+n-2}}{|y|^{n-2}}}{(2m+n-2)(1-a^{2m+n-2})}.\end{aligned}\tag{3.9}$$

By direct calculation we get the formulas (1.4) and (1.5). □

Proof of Corollary 1.1. As in the proof of Lemma 3.1 we have, for $|x|>|y|$,

$$\begin{aligned}\frac{1}{|x-y|^{n-2}}=&\frac{1}{|x|^{n-2}\Big(1-2\frac{x\cdot y}{|x||y|}\frac{|y|}{|x|}+\Big(\frac{|y|}{|x|}\Big)^2\Big)}\\=&\frac{1}{|x|^{n-2}}\sum_{m=0}^{\infty}\Big(\frac{|y|}{|x|}\Big)^m P_k^{\frac{n-2}{2}}\Big(\frac{x\cdot y}{|x||y|}\Big)\\=&\frac{1}{|x|^{n-2}}\sum_{m=0}^{\infty}\frac{n-2}{2m+n-2}\Big(\frac{|y|}{|x|}\Big)^m Z_k\Big(\frac{x}{|x|},\frac{y}{|y|}\Big).\end{aligned}\tag{3.10}$$

From (3.10) and (1.4) the claim follows. □

We have the following,

Corollary 3.1. We have that

$$\begin{aligned}&\nabla R_A(x)\cdot x\\=&-\frac{2}{\omega_{n-1}}\sum_{m=0}^{\infty}d_m\frac{\frac{(2-m-n)a^{2m+n-2}}{|x|^{2m+2n-4}}+m|x|^{2m}+\frac{(n-2)a^{2m+n-2}}{|x|^{n-2}}}{(2m+n-2)(1-a^{2m+n-2})}.\end{aligned}\tag{3.11}$$

Proof. It follows directly by Theorem 1.1. □

We end this section by proving Theorem 1.2. As we mention in the introduction this result generalizes that of Chen and Lin [4] to higher dimensions.

Proof of Theorem 1.2. By (3.11) we have the the Robin function is radial.

Let us set $f(r)=rR_A'(r)$. Then, by (3.11) we get

$$\begin{aligned}f'(r)=&-\frac{2}{\omega_{n-1}}\sum_{m=0}^{\infty}d_m\frac{\frac{2(n+m-2)^2a^{2m+n-2}}{r^{2m+2n-3}}+2m^2r^{2m-1}-\frac{(n-2)^2a^{2m+n-2}}{r^{n-1}}}{(2m+n-2)(1-a^{2m+n-2})}\\<&-\sum_{m=0}^{\infty}d_m\frac{\frac{2(n-2)^2a^{2m+n-2}}{r^{2m+2n-3}}-\frac{(n-2)^2a^{2m+n-2}}{r^{n-1}}}{(2m+n-2)(1-a^{2m+n-2})}\\<&0.\end{aligned}\tag{3.12}$$

The inequality (3.12) implies that the function $f(r)$ is strictly decreasing. Since we have that

$$\lim_{r\to 1^-} f(r)=\lim_{r\to 1^-} rR'_A(r)=-\frac{2}{\omega_{n-1}}\sum_{m=0}^{\infty}\frac{m}{2m+n-2}=-\infty \tag{3.13}$$

and

$$\lim_{r\to a^+} f(r)=\lim_{r\to a^+} rR'_A(r)=\frac{2a^2}{\omega_{n-1}}\sum_{m=0}^{\infty}\frac{(m+n-2)(1-a^{2m-2})}{(2m+n-2)(1-a^{2m+n-2})}=+\infty. \tag{3.14}$$

we conclude that there exists exactly one r_0, $a<r_0<1$, such that $f(r_0)=0$ and then $R'_A(r_0)=0$.

On the other hand,

$$0>f'(r_0)=r_0R''_A(r_0)+R'_A(r_0)=r_0R''_A(r_0).$$

So $R''_A(r_0)<0$. $\square$

Following the line of the proof of the previous theorem we have the following alternative proof to the result in [4].

Proposition 3.1. The Robin function of the 2-dimensional annulus has a unique nondegenerate critical point.

Proof. Let us recall the formula for the Robin function in the plane (see [7])

$$R(y)=-\frac{\log^2|y|}{\log a}+\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1-a^{2m}}(|y|^{2m}-2a^{2m}+a^{2m}|y|^{-2m}).$$

As in the previous theorem we have,

$$R'(r)=-2\frac{\log r}{r\log a}+\sum_{m=1}^{\infty}\frac{2}{1-a^{2m}}(r^{2m-1}-a^{2m}r^{-2m-1}).$$

So,

$$R''(r)=-2\frac{1-\log r}{r^2\log a}+\sum_{m=1}^{\infty}\frac{2}{1-a^{2m}}\Big((2m-1)r^{2m-2}+\frac{(2m+1)a^{2m}}{r^{2m+2}}\Big)>0,\quad (a<r<1).$$

On the other hand we get

$$\begin{aligned}\lim_{r\to a}R'(r)&=\lim_{|y|\to a}-2\frac{\log|y|}{|y|\log a}+\sum_{m=1}^{\infty}\frac{2}{1-a^{2m}}(|y|^{2m-1}-a^{2m}|y|^{-2m-1})\\&=-\frac{2}{a}+\sum_{m=1}^{N}\frac{2}{1-a^{2m}}(a^{2m-1}-a^{-1})=-\infty.\end{aligned} \tag{3.15}$$

On the other hand, while

$$\lim_{r\to 1} R'(r) = \lim_{|y|\to 1} -2\frac{\log|y|}{|y|\log a} + \sum_{m=1}^{\infty} \frac{2}{1-a^{2m}}\left(|y|^{2m-1} - a^{2m}|y|^{-2m-1}\right)$$
$$= \sum_{m=1}^{N} 2 = +\infty. \tag{3.16}$$

So, we can conclude that there exist unique r_0, $a<r_0<1$, for which $R'(r_0)=0$ and $R''(r_0)>0$. □

4 A uniqueness result for a nonlinear elliptic equation

Let us consider the problem

$$\begin{cases} -\Delta u = N(N-2)u^{\frac{n+2}{n-2}} + \varepsilon u & \text{in } A, \\ u>0 & \text{in } A, \\ u=0 & \text{on } \partial A, \end{cases} \tag{4.1}$$

and solutions satisfying

$$S_\varepsilon \to S, \tag{4.2}$$

where S is the best constant in Sobolev inequalities and

$$S_\varepsilon = \inf_{\substack{u\in H_0^1(A) \\ u\not\equiv 0}} \frac{\int_A (|\nabla u|^2 - \varepsilon u^2)}{\left(\int_A |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}. \tag{4.3}$$

It is well known that a family of solutions satisfying (1.7) and (1.9) concentrates at one point, i.e.,

$$|\nabla u_e|^2 \rightharpoonup \delta_P \tag{4.4}$$

weakly in the sense of measure as $\varepsilon\to 0$. It was proved by Han (see [8]) that P is a critical point of the Robin function.

Using Theorem 1.2 we have the following "asymptotic" uniqueness result.

Theorem 4.1. *Let $A=\{x\in R^n \ |a<|x|<1\}$ and u_ε a family of solutions to* (1.7) *satisfying* (1.9). *Then*

$$|\nabla u|^2 \rightharpoonup \delta_{r_0}, \tag{4.5}$$

where r_0 is the **unique** *root of the equation*

$$\sum_{m=0}^{\infty} d_m \frac{\frac{(2-m-n)a^{2m+n-2}}{r^{2m+2n-4}} + mr^{2m} + \frac{(n-2)a^{2m+n-2}}{r^{n-2}}}{(2m+n-2)(1-a^{2m+n-2})} = 0. \tag{4.6}$$

Next aim is to improve the previous result. Indeed we will show that not only our problem has a unique point of concentration, but even it has a unique solution for ϵ small. Of course, since the problem is rotationally invariant we can have uniqueness only up to a suitable rotation.

Now let us consider a solution to (1.7) satisfying (1.9). Up to a suitable rotation we can assume that its maximum point is given by $(y_1,0,\cdots,0)$ with $y_1\in(a,1)$. Then we want to give a representation formula for this solution. This involves classical results which we recall below. Basically we follow the line of the proof of Theorem A in [6].

Let us introduce some notations. Set, for $y=(y_1,0,\cdots,0)$ with $y_1\in(a,1)$,

$$U_{y,\lambda}(x)=\frac{\lambda^{\frac{n-2}{2}}}{(1+\lambda^2|x-y|^2)^{\frac{n-2}{2}}}, \tag{4.7}$$

which is the only positive solution to

$$-\Delta u=N(N-2)u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n. \tag{4.8}$$

Let $PU_{y,\lambda}$ be the projection of $U_{y,\lambda}$ into $H^1_0(A)$, i.e.,

$$\begin{cases} \Delta PU_{y,\lambda}=\Delta U_{y,\lambda} & \text{in } A, \\ PU_{y,\lambda}=0 & \text{on } \partial A, \end{cases} \tag{4.9}$$

and for $\lambda>0$ let us define

$$\begin{aligned} E_{y,\lambda}=\Big\{v\in H^1_0(A):\langle v,PU_{y,\lambda}\rangle=\Big\langle v,\frac{\partial PU_{y,\lambda}}{\partial\lambda}\Big\rangle=\Big\langle v,\frac{\partial PU_{y,\lambda}}{\partial x_i}\Big\rangle=0 \\ \text{for any} \quad i=1,\cdots,n\Big\}, \end{aligned} \tag{4.10}$$

where $\langle u_1,u_2\rangle=\int_A\nabla u_1\nabla u_2$ is the scalar product in $H^1_0(A)$.

Finally set for some $\delta>0$, $\omega_0=(a+\delta,1-\delta)\subset(a,1)$ such that $|\nabla R|>1$ in $(a,a+\delta)\cup(1-\delta,1)$.

The following proposition is classical for concentration problems like (1.7) as $\varepsilon\to0$ (see [10] or [6] for example).

Proposition 4.1. There exist $\varepsilon_0>0$, $\lambda_0>0$ and $\eta_0>0$ such that, for $\varepsilon\in(0,\varepsilon_0)$ and for $(x,\lambda)\in\omega_0\times[\lambda_0,+\infty)$, there exists a unique $v_{y,\lambda}\in E_{y,\lambda}$ such that $\|v_{y,\lambda}\|_{H^1_0(A)}<\eta_0$ and for any $w\in E_{y,\lambda}$,

$$-\int_A\nabla(PU_{y,\lambda}+v_{y,\lambda})\nabla w=N(N-2)\int_A(PU_{y,\lambda}+v_{y,\lambda})^{\frac{n+2}{n-2}}w+\varepsilon(PU_{y,\lambda}+v_{y,\lambda})w. \tag{4.11}$$

Moreover, the function $(x,\lambda)\to v_{y,\lambda}$ is C^1 and there exists a constant $a_2>0$ such that

$$\|v_{y,\lambda}\|_{H^1_0(A)}<\frac{a_2}{\lambda^{\frac{n+2}{2}}}. \tag{4.12}$$

Now we are in position to prove Theorem 1.3.

Proof of Theorem 1.3. Let us consider two solutions $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$. Up to a suitable rotation we can assume that $\|u_{1,\varepsilon}\|_\infty = u_{1,\varepsilon}(\tilde{y}_{1,\varepsilon})$ and $\|u_{2,\varepsilon}\|_\infty = u_{2,\varepsilon}(\tilde{y}_{2,\varepsilon})$ with $\tilde{y}_{1,\varepsilon} = (|\tilde{y}_{1,\varepsilon}|,0,\cdots,0)$ and $\tilde{y}_{2,\varepsilon} = (|\tilde{y}_{2,\varepsilon}|,0,\cdots,0)$ with $|\tilde{y}_{1,\varepsilon}|,|\tilde{y}_{2,\varepsilon}| \to r_0$.

In this setting we can apply Proposition 4.1 which gives that $u_{1,\varepsilon} = PU_{y_{1,\varepsilon},\lambda_{1,\varepsilon}} + v_{y_{1,\varepsilon},\lambda_{1,\varepsilon}}$ and $u_{2,\varepsilon} = PU_{y_{2,\varepsilon},\lambda_{2,\varepsilon}} + v_{y_{2,\varepsilon},\lambda_{2,\varepsilon}}$ for some $(y_{1,\varepsilon},\lambda_{1,\varepsilon})$, $(y_{2,\varepsilon},\lambda_{2,\varepsilon}) \in \omega_0 \times (0,+\infty)$. If we show that $y_{1,\varepsilon} = y_{2,\varepsilon}$ and $\lambda_{1,\varepsilon} = \lambda_{2,\varepsilon}$ then by Proposition 4.1 we derive $v_{y_{1,\varepsilon},\lambda_{1,\varepsilon}} = v_{y_{2,\varepsilon},\lambda_{2,\varepsilon}}$.

Now we use the crucial fact that $u = PU_{y,\lambda} + v_{y,\lambda}$ is a solution to (1.7) if and only if the pair (y,λ) is a critical point of the reduced functional

$$K_\varepsilon(y,\lambda) = \frac{\int_A \left(|\nabla PU_{y,\lambda} + v_{y,\lambda}|^2 - \varepsilon (PU_{y,\lambda} + v_{y,\lambda})^2\right)}{\left(\int_A |PU_{y,\lambda} + v_{y,\lambda}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}. \tag{4.13}$$

So our claim is equivalent to show that the functional $K_\varepsilon(y,\lambda) : \omega_0 \times (0,+\infty) \to \mathbb{R}$ has exactly one critical point.

Let us introduce the function $\tilde{K}_\varepsilon(y,\lambda) : \omega_0 \times (0,+\infty) \to \mathbb{R}$ defined as

$$\tilde{K}_\varepsilon(y_1,\lambda) = A^{-\frac{n-2}{n}} \Big(n(n-2)A + \frac{n(n-2)B}{\lambda^{n-2}} R_A(y_1,0,\cdots,0) - C\frac{\varepsilon}{\lambda^2} \Big). \tag{4.14}$$

In [6, page 576], it was proved that there exist constants $A,B,C \in \mathbb{R}$ such that

$$\tilde{K}_\varepsilon(y,\lambda) \to K_\varepsilon(y,\lambda) \quad \text{in} \quad C^2(\omega_0 \times (0,+\infty)). \tag{4.15}$$

By Theorem 1.2 we have that $\nabla RA(r_0) = 0$ and $\frac{\partial^2 R_A}{\partial x_1^2}(r_0) \neq 0$. This means that r_0 is a non-degenerate critical point for the function $R_A(y_1,0,\cdots,0)$. Hence Step 1 in [6] applies and then we have the uniqueness of the critical point of $K_\varepsilon(y_1,\lambda)$. Then $y_{1,\varepsilon} = y_{2,\varepsilon}$ and $\lambda_{1,\varepsilon} = \lambda_{2,\varepsilon}$ and the claim follows. $\square$

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