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The Multifractal Formalism for Measures, Review and Extension to Mixed Cases

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Abstract. The multifractal formalism for single measure is reviewed. Next, a mixed generalized multifractal formalism is introduced which extends the multifractal formalism of a single measure based on generalizations of the Hausdorff and packing measures to a vector of simultaneously many measures. Borel-Cantelli and Large deviations Theorems are extended to higher orders and thus applied for the validity of the new variant of the multifractal formalism for some special cases of multi-doubling type measures.

Key Words: Hausdorff measures, packing measures, Hausdorff dimension, packing dimension, renyi dimension, multifractal formalism, vector valued measures, mixed cases, Holderian measures, doubling measures, Borel-Cantelli, large deviations.

AMS Subject Classifications: 28A78, 28A80

1 Introduction

In the present work, we are concerned with the whole topic of multifractal analysis of measures and the validity of multifractal formalisms. We aim to consider some cases of simultaneous behaviors of measures instead of a single measure as in the classic or original multifractal analysis of measures. We call such a study mixed multifractal analysis. Such a mixed analysis has been generating a great attention recently and thus

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proved to be powerful in describing the local behavior of measures especially fractal ones (see [1,2,9–14]).

In this paper, multi purposes will be done. Firstly we review the classical multifractal analysis of measures and recall all basics about fractal measures as well as fractal dimensions. We review Hausdorff measures, Packing measures, Hausdorff dimensions, Packing dimensions as well as Renyi dimensions and we recall the eventual relations linking these notions. A second aim is to develop a type of multifractal analysis, multifractal spectra, multifractal formalism which permit to study simultaneously a higher number of measures. As it is noticed from the literature on multifractal analysis of measures, this latter always considered a single measure and studies its scaling behavior as well as the multifractal formalism associated. Recently, many works have been focused on the study of simultaneous behaviors of finitely many measures. In [9], a mixed multifractal analysis is developed dealing with a generalization of Rényi dimensions for finitely many self similar measures. This was one of the motivations leading to our present paper. Secondly, we intend to combine the generalized Hausdorff and packing measures and dimensions recalled after with Olsen's results in [14] to define and develop a more general multifractal analysis for finitely many measures by studying their simultaneous regularity, spectrum and to define a mixed multifractal formalism which may describe better the geometry of the singularities's sets of these measures. We apply the techniques of L. Olsen especially in [9] and [14] with the necessary modifications to give a detailed study of computing general mixed multifractal dimensions of simultaneously many finite number of measures and try to project our results for the case of a single measure to show the generecity of our's.

The first point to check in multifractal analysis of a measure is its singularity on its spectrum. Given a measure μ eventually Borel and finite, for $x\in\operatorname{supp}(\mu)$, the singularity of μ is estimated via $\mu(B(x,r))$ as $r\to 0$. If $\mu(B(x,r))\sim r^{\alpha}$, the measure μ is said to be α-Hölder at x. The local lower dimension and the local upper dimension of μ at the point x are respectively defined by

$$\underline{\alpha}_{\mu}(x)=\liminf_{r\downarrow 0}\frac{\log(\mu(B(x,r)))}{\log r}\quad\text{and}\quad\overline{\alpha}_{\mu}(x)=\limsup_{r\downarrow 0}\frac{\log(\mu(B(x,r)))}{\log r}.$$

When these quantities are equal we call their common value the local dimension, denoted by $\alpha_\mu(x)$ of μ at x. Next, the α-singularity set is $X(\alpha)=\{x\in\operatorname{supp}(\mu);\ \alpha_\mu(x)=\alpha\}$ and finally, the spectrum of singularities is the mapping defined by $d(\alpha)=\dim X(\alpha)$ where dim stands for the Hausdorff dimension.

The computation of such a spectrum is the delicate point and the most principal aim in the whole multifractal study of the measure. Its computation needs more efforts and special techniques based on the characteristics of the measure, such that self similarity, scalings. In multifractal analysis, it is related to multifractal dimensions and in some cases it is computed by means of the Legendre transform of such dimensions. This fact constitutes the so-called multifractal formalism for measures.

The present work will be organized as follows. The next section concerns a review of Hausdorff and packing measures and dimensions. Section 3 is concerned to Multifractal generalizations of Hausdorff and packing measures as well as the associated dimensions. In Section 4, the mixed multifractal generalizations of Hausdorff and packing measures and dimensions are introduced. Section 5 is devoted to the mixed multifractal generalization of Bouligand-Minkowsky or Rényi dimension inspired from Olsen in [14]. In Section 6, a mixed multifractal formalism associated to the mixed multifractal generalizations of Hausdorff and packing measures and dimensions is proved in some case based on a generalization of the well known large deviation formalism.

2 Hausdorff and packing measures and dimensions

Given a subset $E\subseteq\mathbb{R}$, and $\epsilon>0$, we call an ϵ-covering of E, any countable set $(U_i)_i$ of non-empty subsets $U_i\subseteq\mathbb{R}$ satisfying

$$E\subseteq\bigcup_i U_i \quad \text{and} \quad |U_i|=\operatorname{diam}(U_i)\leq\epsilon, \tag{2.1}$$

where for any subset $U\subseteq\mathbb{R}$, $|U|=\operatorname{diam}(U)$ is the diameter defined by

$$|U|=\operatorname{diam}(U)=\sup_{x,y\in U}|x-y|.$$

Remark here that for $\epsilon_1<\epsilon_2$, any ϵ_1-covering of E is obviously an ϵ_2-covering of E. This implies that the quantity

$$\mathcal{H}^s_\epsilon(E)=\inf\Big\{\sum_i|U_i|^s;(U_i)\text{ satisfying (2.1)}\Big\}$$

is a non increasing function in ϵ. Its limit

$$\mathcal{H}^s(E)=\lim_{\epsilon\downarrow 0}\mathcal{H}^s_\epsilon(E)$$

defines the so-called s-dimensional Hausdorff measure of E. It holds that for any set $E\subseteq\mathbb{R}$ there exists a critical value s_E in the sense that

$$\mathcal{H}^s(E)=0,\quad\forall s<s_E\quad\text{and}\quad\mathcal{H}^s(E)=+\infty,\quad\forall s>s_E,$$

or otherwise,

$$s_E=\sup\{s>0;\,\mathcal{H}^s(E)=0\}=\inf\{s>0;\,\mathcal{H}^s(E)=+\infty\}.$$

Such a value is called the Hausdorff dimension of the set E and is usually denoted by $\dim_H E$ or simply $\dim E$. When $U_i=B(x_i,r_i)$ is a ball centered at $x_i\in E$ and with diameter $r_i<\epsilon$, the covering $(B(x_i,r_i))_i$ is called an ϵ-centered covering of E. However, surprisingly, the quantity $\mathcal{H}^s$ restricted only on centered coverings does not define a measure. To obtain a good measure with centered coverings one should do more. Denote

$$\overline{\mathcal{C}}_\epsilon^s(E)=\inf\Big\{\sum_i|2r_i|^s;\ (B(x_i,r_i))_i \text{ an } \epsilon-\text{centered covering of } E\Big\}$$

and similarly as above,

$$\overline{\mathcal{C}}^s(E)=\lim_{\epsilon\downarrow 0}\overline{\mathcal{C}}_\epsilon^s(E).$$

As stated previously, this is not a good measure. So, to obtain a good candidate, we set for $E\subseteq\mathbb{R}$,

$$\mathcal{C}^s(E)=\sup_{F\subseteq E}\overline{\mathcal{C}}^s(F).$$

It is called the centered Hausdorff s-dimensional measure of E. But, although a fascinating relation to the Hausdorff measure exists. It holds that

$$2^{-s}\mathcal{C}^s(E)\le\mathcal{H}^s(E)\le\mathcal{C}^s(E);\quad \forall E\subseteq\mathbb{R}^d. \tag{2.2}$$

Indeed, let $F\subseteq E$ be subsets of $\mathbb{R}^d$. It follows from the definition of $\mathcal{H}^s$ and $\overline{\mathcal{C}}^s$ that $\mathcal{H}^s(F)\le\overline{\mathcal{C}}^s(F)$. Next, from the fact that $\mathcal{H}^s$ is an outer metric measure on $\mathbb{R}^d$, and the definition of $\mathcal{C}^s$, il results that $\mathcal{H}^s(E)\le\mathcal{C}^s(E)$. Next, let $\{U_j\}_j$ be an ϵ-covering of F and $r_j=\mathrm{diam}(U_j)$. For each i fixed, consider a point $x_i\in U_i\cap F$. This results in a centered ϵ-covering $\{B(x_i,r_i)\}_i$ of F. Consequently,

$$\overline{\mathcal{C}}_\epsilon^s(F)\le\sum_i(2r_i)^s=2^s\sum_i(\mathrm{diam}(U_i))^s.$$

Hence,

$$\overline{\mathcal{C}}_\epsilon^s(F)\le 2^s\mathcal{H}_\epsilon^s(F).$$

Next, as $\epsilon\downarrow 0$, we obtain

$$\overline{\mathcal{C}}^s(F)\le 2^s\mathcal{H}^s(F),\quad \forall F\subseteq E,$$

which guaranties that

$$\mathcal{C}^s(E)\le 2^s\mathcal{H}^s(E).$$

It holds that these measures give rise to some critical values in the sense that, for any set $E\subseteq\mathbb{R}$ there exists a critical value h_E and c_E for which

$$\mathcal{H}^s(E)=0,\quad \forall s<h_E\quad\text{and}\quad \mathcal{H}^s(E)=+\infty,\quad \forall s>h_E,$$

and similarly

$$\mathcal{C}^s(E)=0,\quad \forall s<c_E\quad\text{and}\quad \mathcal{C}^s(E)=+\infty,\quad \forall s>c_E.$$

But using Eq. (2.2) above, it proved that $h_E=c_E$ and otherwise,

$$h_E=\sup\{s>0;\ \mathcal{H}^s(E)=0\}=\inf\{s>0;\ \mathcal{H}^s(E)=+\infty\}.$$

Such a value is called the Hausdorff dimension of the set E and is usually denoted by $\dim_H E$ or simply $\dim E$.

Similarly, we call a centered ϵ-packing of $E\subseteq\mathbb{R}^d$, any countable set $(B(x_i,r_i))_i$ of disjoint balls centered at points $x_i\in E$ and with diameters $r_i<\epsilon$. The packing measure and dimension are defined as follows

$$\overline{\mathcal{P}}^s(E)=\lim_{\varepsilon\downarrow 0}\Big(\sup\Big\{\sum_i (2r_i)^s;\ (B(x_i,r_i))_i\epsilon-\text{packing of }E\Big\}\Big),$$
$$\mathcal{P}^s(E)=\inf\Big\{\sum_i \overline{\mathcal{P}}^s(E_i);\ E\subseteq\cup_i E_i\Big\}.$$

It holds as for the Hausdorff measure that there exists critical values Δ_E and p_E satisfying respectively

$$\overline{\mathcal{P}}^s(E)=+\infty\quad\text{for }\ s<\Delta(E)\quad\text{and}\quad \overline{\mathcal{P}}^s(E)=0\quad\text{for }\ \alpha>\Delta(E),$$

and respectively

$$\mathcal{P}^s(E)=\infty\quad\text{for }\ s<p_E\quad\text{and}\quad \mathcal{P}^s(E)=0\quad\text{for }\ s>p_E.$$

The critical value $\Delta(E)$ is called the logarithmic index of E and p_E is called the packing dimension of E denote by $\mathrm{Dim}_P(E)$ or simply $\mathrm{Dim}(E)$. These quantities may be shown as

$$\Delta(E)=\sup\{s;\overline{\mathcal{P}}^s(E)=0\}=\inf\{s;\overline{\mathcal{P}}^s(E)=+\infty\}$$

and respectively

$$\mathrm{Dim}(E)=\sup\{s;\mathcal{P}^s(E)=0\}=\inf\{s;\mathcal{P}^s(E)=+\infty\}.$$

Usually, we have the inequality

$$\dim(E)\le\mathrm{Dim}(E)\le\Delta(E),\quad \forall E\subseteq\mathbb{R}^d.$$

Definition 2.1. A set $E\subseteq\mathbb{R}^d$ is said to be fractal in the sense of Taylor iff $\dim(E)=\mathrm{Dim}(E)$.

3 Multifractal generalizations of Hausdorff and packing measures

Let μ be a Borel probability measure on $\mathbb{R}^d$, a nonempty set $E\subseteq\mathbb{R}^d$ and $\epsilon>0$. Let also q, t be real numbers. We will recall hereafter the steps leading to the multifractal generalizations of the Hausdorff and packing measures due to L. Olsen in [9]. Denote

$$\overline{\mathcal{H}}^{q,t}_{\mu,\epsilon}(E)=\inf\Big\{\sum_i (\mu(B(x_i,r_i)))^q(2r_i)^t\Big\},$$

where the inf is taken over the set of all centered ϵ-coverings of E, and for the empty set, $\overline{\mathcal{H}}^{q,t}_{\mu,\epsilon}(\emptyset)=0$. As for the preceding cases of Hausdorff and packing measures, it consists of a non increasing quantity as a function of ε. We then consider its limit

$$\overline{\mathcal{H}}^{q,t}_{\mu}(E)=\lim_{\epsilon\downarrow 0}\overline{\mathcal{H}}^{q,t}_{\mu,\epsilon}(E)=\sup_{\delta>0}\overline{\mathcal{H}}^{q,t}_{\mu,\epsilon}(E)$$

and finally, the multifractal generalization of the s-dimensional Huasdorrf measure

$$\mathcal{H}^{q,t}_{\mu}(E)=\sup_{F\subseteq E}\overline{\mathcal{H}}^{q,t}_{\mu}(F).$$

Similarly, we define the multifractal generalization of the packing measure as follows

$$\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(E)=\sup\Big\{\sum_i(\mu(B(x_i,r_i)))^q(2r_i)^t\Big\},$$

where the sup is taken over the set of all centered ϵ-packings of E. For the empty set, we set as usual $\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(\emptyset)=0$. Next,

$$\overline{\mathcal{P}}^{q,t}_{\mu}(E)=\lim_{\epsilon\downarrow 0}\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(E)=\inf_{\delta>0}\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(E)$$

and finally,

$$\mathcal{P}^{q,t}_{\mu}(E)=\inf_{E\subseteq\cup_i E_i}\sum_i\overline{\mathcal{P}}^{q,t}_{\mu}(E_i).$$

In [9], it has been proved that the measures $\mathcal{H}^{q,t}_{\mu}$, $\mathcal{P}^{q,t}_{\mu}$ and the pre-measure $\overline{\mathcal{P}}^{q,t}_{\mu}$ assign in a usual way a dimension to every set $E\subseteq\mathbb{R}^d$ as resumed in the following proposition.

Proposition 3.1 (see [9]). Given a subset $E\subseteq\mathbb{R}^d$,

1. There exists a unique number $\dim^q_{\mu}(E)\in[-\infty,+\infty]$ such that

$$\mathcal{H}^{q,t}_{\mu}(E)=\begin{cases}+\infty, & \text{for } t<\dim^q_{\mu}(E),\\ 0, & \text{for } t>\dim^q_{\mu}(E).\end{cases}$$

2. There exists a unique number $\mathrm{Dim}^q_{\mu}(E)\in[-\infty,+\infty]$ such that

$$\mathcal{P}^{q,t}_{\mu}(E)=\begin{cases}+\infty, & \text{for } t<\mathrm{Dim}^q_{\mu}(E),\\ 0, & \text{for } t>\mathrm{Dim}^q_{\mu}(E).\end{cases}$$

3. There exists a unique number $\Delta^q_{\mu}(E)\in[-\infty,+\infty]$ such that

$$\overline{\mathcal{P}}^{q,t}_{\mu}(E)=\begin{cases}+\infty, & \text{for } t<\Delta^q_{\mu}(E),\\ 0, & \text{for } t>\Delta^q_{\mu}(E).\end{cases}$$

The quantities $\dim_\mu^q(E)$, $\mathrm{Dim}_\mu^q(E)$ and $\Delta_\mu^q(E)$ defines the so-called multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of the set E. More precisely, one has

$$\dim_\mu^0(E)=\dim(E),\quad \mathrm{Dim}_\mu^0(E)=\mathrm{Dim}(E)\quad\text{and}\quad \Delta_\mu^0(E)=\Delta(E).$$

The characteristics of these functions have been studied completely by L. Olsen. He proved among author results that $\dim_\mu^q$ and Dim_μ^q are monotones and σ-stables. Furthermore, if $E=\mathrm{supp}(\mu)$ is the support of the measure μ, one obtains

a. The functions $q\longmapsto \mathrm{Dim}_\mu^q(E)$ and $q\longmapsto \Delta_\mu^q(E)$ are convex non increasing.

b. $q\longmapsto \dim_\mu^q(E)$ is non increasing.

c. i. For $q<1$; $0\le \dim_\mu^q(E)\le \mathrm{Dim}_\mu^q(E)\le \Delta_\mu^q(E)$.
   ii. $\dim_\mu^1(E)=Dim_\mu^1(E)=\Delta_\mu^1(E)=0$.
   iii. For $q>1$; $\dim_\mu^q(E)\le \mathrm{Dim}_\mu^q(E)\le \Delta_\mu^q(E)\le 0$.

4 Mixed multifractal generalizations of Hausdorff and packing measures and dimensions

The purpose of this section is to present our ideas about mixed multifractal generalizations of Hausdorff and packing measures and dimensions. Let $\mu=(\mu_1,\mu_2,\cdots,\mu_k)$ a vector valued measure composed of probability measures on $\mathbb{R}^d$. We aim to study the simultaneous scaling behavior of μ, which we denote

$$\lim_{r\downarrow 0}\frac{\log\mu(B(x,r))}{\log r}\equiv\Big(\lim_{r\downarrow 0}\frac{\log\mu_1(B(x,r))}{\log r},\cdots,\lim_{r\downarrow 0}\frac{\log\mu_k(B(x,r))}{\log r}\Big).$$

Let $E\subseteq\mathbb{R}^d$ be a nonempty set and $\epsilon>0$. Let also $q=(q_1,q_2,\cdots,q_k)\in\mathbb{R}^k$ and $t\in\mathbb{R}$. The mixed generalized multifractal Hausdorff measure is defined as follows. Denote

$$\mu(B(x,r))\equiv\big(\mu_1(B(x,r)),\cdots,\mu_k(B(x,r))\big)$$

and the product

$$(\mu(B(x,r)))^q\equiv(\mu_1(B(x,r)))^{q_1}\cdots(\mu_k(B(x,r)))^{q_k}.$$

Denote next,

$$\overline{\mathcal{H}}_{\mu,\epsilon}^{q,t}(E)=\inf\Big\{\sum_i(\mu(B(x_i,r_i)))^q(2r_i)^t\Big\},$$

where the inf is taken over the set of all centered ϵ-coverings of E, and for the empty set, $\overline{\mathcal{H}}_{\mu,\epsilon}^{q,t}(\emptyset)=0$. As for the single case, of Hausdorff measure, it consists of a non increasing function of the variable ε. So that, its limit as $\epsilon\downarrow 0$ exists. Let

$$\overline{\mathcal{H}}_{\mu}^{q,t}(E)=\lim_{\epsilon\downarrow 0}\overline{\mathcal{H}}_{\mu,\epsilon}^{q,t}(E)=\sup_{\delta>0}\overline{\mathcal{H}}_{\mu,\epsilon}^{q,t}(E).$$

Let finally

$$\mathcal{H}_{\mu}^{q,t}(E)=\sup_{F\subseteq E}\overline{\mathcal{H}}_{\mu}^{q,t}(F).$$

Lemma 4.1. *$\mathcal{H}_{\mu}^{q,t}$ is an outer metric measure on $\mathbb{R}^d$.*

The proof of this lemma is technic and follows carefully analogous steps as the single case.

Definition 4.1. The restriction of $\mathcal{H}_{\mu}^{q,t}$ on Borel sets is called the mixed generalized Hausdorff measure on $\mathbb{R}^d$.

Now, we define the mixed generalized multifractal packing measure. We use already the same notations as previously. Let

$$\overline{\mathcal{P}}_{\mu,\epsilon}^{q,t}(E)=\sup\Big\{\sum_i(\mu(B(x_i,r_i)))^q(2r_i)^t\Big\},$$

where the sup is taken over the set of all centered ϵ-packings of E. For the empty set, we set as usual $\overline{\mathcal{P}}_{\mu,\epsilon}^{q,t}(\emptyset)=0$. Next, we consider the limit as $\epsilon\downarrow 0$,

$$\overline{\mathcal{P}}_{\mu}^{q,t}(E)=\lim_{\epsilon\downarrow 0}\overline{\mathcal{P}}_{\mu,\epsilon}^{q,t}(E)=\inf_{\delta>0}\overline{\mathcal{P}}_{\mu,\epsilon}^{q,t}(E)$$

and finally,

$$\mathcal{P}_{\mu}^{q,t}(E)=\inf_{E\subseteq\cup_i E_i}\sum_i\overline{\mathcal{P}}_{\mu}^{q,t}(E_i).$$

Lemma 4.2. *$\mathcal{P}_{\mu}^{q,t}$ is an outer metric measure on $\mathbb{R}^d$.*

The proof of this lemma is more specific than Lemma 4.1 and uses the following result.

$$\overline{\mathcal{P}}_{\mu}^{q,t}(A\cup B)=\overline{\mathcal{P}}_{\mu}^{q,t}(A)+\overline{\mathcal{P}}_{\mu}^{q,t}(B),\quad\text{whenever }\ d(A,B)>0. \tag{4.1}$$

Indeed, let

$$0<\epsilon<\frac{1}{2}d(A,B)$$

and $(B(x_i,r_i))_i$ be a centered ϵ-packing of the union $A\cup B$. It can be divided into two parts I and J,

$$(B(x_i,r_i))_i=\big(B(x_i,r_i)\big)_{i\in I}\bigcup\big(B(x_i,r_i)\big)_{i\in J},$$

where

$$\forall i\in I,\quad B(x_i,r_i)\cap B=\emptyset\quad\text{and}\quad\forall i\in J,\quad B(x_i,r_i)\cap A=\emptyset.$$

Therefore, $(B(x_i,r_i))_{i\in I}$ is a centered ϵ-packing of A and $(B(x_i,r_i))_{i\in J}$ is a centered ϵ-packing of the union B. Hence,

$$\sum_i (\mu(B(x_i,r_i)))^q(2r_i)^t = \underbrace{\sum_{i\in I}(\mu(B(x_i,r_i)))^q(2r_i)^t}_{\leq \overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(A)} + \underbrace{\sum_{i\in I}(\mu(B(x_i,r_i)))^q(2r_i)^t}_{\leq \overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(B)}.$$

Consequently,

$$\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(A\cup B)\leq \overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(A)+\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(B)$$

and thus the limit for $\epsilon\downarrow 0$ gives

$$\overline{\mathcal{P}}^{q,t}_{\mu}(A\cup B)\leq \overline{\mathcal{P}}^{q,t}_{\mu}(A)+\overline{\mathcal{P}}^{q,t}_{\mu}(B).$$

The converse is more easier and it states that $\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}$ and next $\overline{\mathcal{P}}^{q,t}_{\mu}$ are sub-additive. Let $(B(x_i,r_i))_i$ be a centered ϵ-packing of A and $(B(y_i,r_i))_i$ be a centered ϵ-packing of B. The union $\big(B(x_i,r_i)\big)_i\bigcup\big(B(y_i,r_i)\big)_i$ is a centered ϵ-packing of $A\cup B$. So that

$$\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(A\cup B)\geq \sum_i (\mu(B(x_i,r_i)))^q(2r_i)^t+\sum_i (\mu(B(y_i,r_i)))^q(2r_i)^t.$$

Taking the sup on $(B(x_i,r_i))_i$ as a centered ϵ-packing of A and next the sup on $(B(y_i,r_i))_i$ as a centered ϵ-packing of B, we obtain

$$\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(A\cup B)\geq \overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(A)+\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(B)$$

and thus the limit for $\epsilon\downarrow 0$ gives

$$\overline{\mathcal{P}}^{q,t}_{\mu}(A\cup B)\geq \overline{\mathcal{P}}^{q,t}_{\mu}(A)+\overline{\mathcal{P}}^{q,t}_{\mu}(B).$$

Definition 4.2. The restriction of $\mathcal{P}^{q,t}_{\mu}$ on Borel sets is called the mixed generalized packing measure on $\mathbb{R}^d$.

It holds as for the case of the multifractal analysis of a single measure that the measures $\mathcal{H}^{q,t}_{\mu}$, $\mathcal{P}^{q,t}_{\mu}$ and the pre-measure $\overline{\mathcal{P}}^{q,t}_{\mu}$ assign a dimension to every set $E\subseteq\mathbb{R}^d$.

Proposition 4.1. Given a subset $E\subseteq\mathbb{R}^d$,

1. There exists a unique number $\dim^q_{\mu}(E)\in[-\infty,+\infty]$ such that

$$\mathcal{H}^{q,t}_{\mu}(E)=\begin{cases} +\infty, & \text{for } t<\dim^q_{\mu}(E),\\ 0, & \text{for } t>\dim^q_{\mu}(E).\end{cases}$$

2. There exists a unique number $\mathrm{Dim}_\mu^q(E)\in[-\infty,+\infty]$ such that

$$\mathcal{P}_\mu^{q,t}(E)=\begin{cases} +\infty, & \text{for } t<\mathrm{Dim}_\mu^q(E),\\ 0, & \text{for } t>\mathrm{Dim}_\mu^q(E).\end{cases}$$

3. There exists a unique number $\Delta_\mu^q(E)\in[-\infty,+\infty]$ such that

$$\overline{\mathcal{P}}_\mu^{q,t}(E)=\begin{cases} +\infty, & \text{for } t<\Delta_\mu^q(E),\\ 0, & \text{for } t>\Delta_\mu^q(E).\end{cases}$$

Definition 4.3. The quantities $\dim_\mu^q(E)$, $\mathrm{Dim}_\mu^q(E)$ and $\Delta_\mu^q(E)$ define the so-called mixed multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of the set E.

Remark that if we denote $Q_i=(0,0,\cdots,q_i,0,\cdots,0)$ the vector with zero coordinates except the ith one which equals q_i, we obtain the multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of the set E for the single measure μ_i,

$$\dim_\mu^{Q_i}(E)=\dim_{\mu_i}^{q_i}(E),\quad \mathrm{Dim}_\mu^{Q_i}(E)=\mathrm{Dim}_{\mu_i}^{q_i}(E)\quad\text{and}\quad \Delta_\mu^{Q_i}(E)=\Delta_{\mu_i}^{q_i}(E).$$

Similarly, for the null vector of $\mathbb{R}^k$, we obtain

$$\dim_\mu^0(E)=\dim(E),\quad \mathrm{Dim}_\mu^0(E)=\mathrm{Dim}(E)\quad\text{and}\quad \Delta_\mu^0(E)=\Delta(E).$$

Proof of Proposition 4.1. We will sketch only the proof of the first point. The rest is analogous.

First, we claim that $\forall t\in\mathbb{R}$ such that $\mathcal{H}_\mu^{q,t}(E)<\infty$ it holds that $\mathcal{H}_\mu^{q,t'}(E)=0$ for any $t'>t$. Indeed, let $\epsilon>0$, $F\subseteq E$ and $(B(x_i,r_i))_i$ be a centered ϵ-covering of F. We have

$$\overline{\mathcal{H}}_{\mu,\epsilon}^{q,t'}(F)\le\sum_i(\mu(B(x_i,r_i)))^q(2r_i)^{t'}\le\delta^{t'-t}\sum_i(\mu(B(x_i,r_i)))^q(2r_i)^t.$$

Consequently,

$$\overline{H}_{\mu,\epsilon}^{q,t'}(F)\le\epsilon^{t'-t}\overline{H}_{\mu,\epsilon}^{q,t'}(F).$$

Hence,

$$\overline{\mathcal{H}}_\mu^{q,t'}(F)=0,\quad\forall F\subseteq E.$$

As a result, $\mathcal{H}_\mu^{q,t'}(E)=0$. We then set

$$\dim_\mu^q(E)=\inf\{t\in\mathbb{R};\ \mathcal{H}_\mu^{q,t'}(E)=0\}.$$

One can proceed otherwise by claiming that $\forall t\in\mathbb{R}$ such that $\mathcal{H}_\mu^{q,t}(E)>0$ it holds that $\mathcal{H}_\mu^{q,t'}(E)=+\infty$ for any $t'<t$. Indeed, proceeding as previously, we obtain for $\epsilon>0$,

$$\epsilon^{t'-t}\overline{H}_{\mu,\epsilon}^{q,t}(F)\leq\overline{H}_{\mu,\epsilon}^{q,t'}(F).$$

Hence,

$$\overline{\mathcal{H}}_\mu^{q,t'}(F)=+\infty,\quad \forall F\subseteq E.$$

As a result, $\mathcal{H}_\mu^{q,t'}(E)=+\infty$. We then set

$$\dim_\mu^q(E)=\sup\{t\in\mathbb{R};\,\mathcal{H}_\mu^{q,t'}(E)=+\infty\}.$$

Next, we aim to study the characteristics of the mixed multifractal generalizations of dimensions. To do this we will adapt the following notations. For $q=(q_1,\cdots,q_k)\in\mathbb{R}^k$,

$$b_{\mu,E}(q)=\dim_\mu^q(E),\quad B_{\mu,E}(q)=\mathrm{Dim}_\mu^q(E)\quad \text{and}\quad \Lambda_{\mu,E}(q)=\Delta_\mu^q(E).$$

When $E=\mathrm{supp}(\mu)$ is the support of the measure μ, we will omit the indexation with E and denote simply

$$b_\mu(q),\ B_\mu(q)\ \text{ and }\ \Lambda_\mu(q).$$

Thus, we complete the proof. $\square$

The following propositions resume the characteristics of these functions and extends the results of L. Olsen [9] for our case.

Proposition 4.2. (a) $b_{\mu,\cdot}(q)$ and $B_{\mu,\cdot}(q)$ are non decreasing with respect to the inclusion property in $\mathbb{R}^d$.

(b) $b_{\mu,\cdot}(q)$ and $B_{\mu,\cdot}(q)$ are σ-stable.

Proof. (a) Let $E\subseteq F$ be subsets of $\mathbb{R}^d$. We have

$$\mathcal{H}_\mu^{q,t}(E)=\sup_{A\subseteq E}\overline{\mathcal{H}}_\mu^{q,t}(A)\leq\sup_{A\subseteq F}\overline{\mathcal{H}}_\mu^{q,t}(A)=\mathcal{H}_\mu^{q,t}(F).$$

So for the monotony of $b_{\mu,\cdot}(q)$.

(b) Let $(A_n)_n$ be a countable set of subsets $A_n\subseteq\mathbb{R}^d$ and denote $A=\bigcup_n A_n$. It holds from the monotony of $b_{\mu,\cdot}(q)$ that

$$b_{\mu,A_n}(q)\leq b_{\mu,A}(q),\quad \forall n.$$

Hence,

$$\sup_n b_{\mu,A_n}(q)\leq b_{\mu,A}(q).$$

Next, for any $t>\sup_n b_{\mu,A_n}(q)$, there holds that

$$\mathcal{H}_\mu^{q,t}(A_n)=0,\quad \forall n.$$

Consequently, from the sub-additivity property of $\mathcal{H}_\mu^{q,t}$, it holds that

$$\mathcal{H}_\mu^{q,t}\Big(\bigcup_n A_n\Big)=0, \quad \forall t>\sup_n b_{\mu,A_n}(q).$$

Which means that

$$b_{\mu,A}(q)\leq t, \quad \forall t>\sup_n b_{\mu,A_n}(q).$$

Hence,

$$b_{\mu,A}(q)\leq \sup_n b_{\mu,A_n}(q).$$

Similar arguments permit to prove the properties of $B_{\mu,A}(q)$. $\square$

Next, we continue to study the characteristics of the mixed generalized multifractal dimensions. The following result is obtained.

Proposition 4.3. (a) The functions $q\longmapsto B_\mu(q)$ and $q\longmapsto \Lambda_\mu(q)$ are convex.

(b) For $i=1,2,\cdots,k$, the functions $q_i\longmapsto b_\mu(q)$, $q_i\longmapsto B_\mu(q)$ and $q_i\longmapsto \Lambda_\mu(q)$, ($\widehat{q_i}=(q_1,\cdots,q_{i-1},q_{i+1},\cdots,q_k)$ fixed), are non increasing.

Proof. (a) We start by proving that $\Lambda_{\mu,E}$ is convex. Let $p,q\in\mathbb{R}^k$, $\alpha\in[0,1]$, $s>\Lambda_{\mu,E}(p)$ and $t>\Lambda_{\mu,E}(q)$. Consider next a centered ϵ-packing $(B_i=B(x_i,r_i))_i$ of E. Applying Hölder's inequality, it holds that

$$\sum_i(\mu(B_i))^{\alpha q+(1-\alpha)p}(2r_i)^{\alpha t+(1-\alpha)s}\leq\Big(\sum_i(\mu(B_i))^q(2r_i)^t\Big)^\alpha\Big(\sum_i(\mu(B_i))^p(2r_i)^s\Big)^{1-\alpha}.$$

Hence,

$$\overline{\mathcal{P}}_{\mu,\epsilon}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s}(E)\leq\big(\overline{\mathcal{P}}_{\mu,\epsilon}^{q,t}(E)\big)^\alpha\big(\overline{\mathcal{P}}_{\mu,\epsilon}^{p,s}(E)\big)^{1-\alpha}.$$

The limit on $\epsilon\downarrow 0$ gives

$$\overline{\mathcal{P}}_{\mu}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s}(E)\leq\big(\overline{\mathcal{P}}_{\mu}^{q,t}(E)\big)^\alpha\big(\overline{\mathcal{P}}_{\mu}^{p,s}(E)\big)^{1-\alpha}.$$

Consequently,

$$\overline{\mathcal{P}}_{\mu}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s}(E)=0, \quad \forall s>\Lambda_{\mu,E}(p) \quad \text{and} \quad t>\Lambda_{\mu,E}(q).$$

It results that

$$\Lambda_{\mu,E}(\alpha q+(1-\alpha)p)\leq\alpha\Lambda_{\mu,E}(q)+(1-\alpha)\Lambda_{\mu,E}(p).$$

We now prove the convexity of $B_{\mu,E}$. We set in this case $t=B_{\mu,E}(q)$ and $s=B_{\mu,E}(p)$. We have

$$\mathcal{P}_\mu^{q,t+\varepsilon}(E)=\mathcal{P}_\mu^{p,s+\varepsilon}(E)=0.$$

Therefore, there exists $(H_i)_i$ and $(K_i)_i$ coverings of the set E for which

$$\sum_i \overline{\mathcal{P}}_{\mu}^{q,t+\varepsilon}(H_i) \leq 1 \quad \text{and} \quad \sum_i \overline{\mathcal{P}}_{\mu}^{p,s+\varepsilon}(K_i) \leq 1.$$

Denote for $n \in \mathbb{N}$, $E_n = \bigcup_{1\leq i,j\leq n}(H_i \cap K_j)$. Thus, $(E_n)_n$ is a covering of E. So that,

$$\begin{aligned}
&\mathcal{P}_{\mu}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s+\varepsilon}(E_n)\\
&\leq \sum_{i,j=1}^{n} \mathcal{P}_{\mu}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s+\varepsilon}(H_i \cap K_j)\\
&\leq \sum_{i,j=1}^{n} \overline{\mathcal{P}}_{\mu}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s+\varepsilon}(H_i \cap K_j)\\
&\leq \Big(\sum_{i,j=1}^{n} \overline{\mathcal{P}}_{\mu}^{q,t+\varepsilon}(H_i \cap K_j)\Big)^{\alpha} \Big(\sum_{i,j=1}^{n} \overline{\mathcal{P}}_{\mu}^{p,s+\varepsilon}(H_i \cap K_j)\Big)^{1-\alpha}\\
&\leq n^{\alpha} n^{1-\alpha} = n < \infty.
\end{aligned}$$

Consequently,

$$B_{\mu,E_n}(\alpha q+(1-\alpha)p) \leq \alpha t+(1-\alpha)s+\varepsilon, \quad \forall \varepsilon>0.$$

Hence,

$$B_{\mu,E}(\alpha q+(1-\alpha)p) \leq \alpha B_{\mu,E}(q)+(1-\alpha)B_{\mu,E}(p).$$

(b) For $i=1,2,\cdots,k$, let $\widehat{q_i}$ fixed and $p_i \leq q_i$ reel numbers. Denote next $q=(q_1,\cdots,q_{i-1},q_i,q_{i+1},\cdots,q_k)$ and $p=(q_1,\cdots,q_{i-1},p_i,q_{i+1},\cdots,q_k)$. Let finally $A \subseteq E$. For a centered ϵ-covering $(B(x_i,r_i))_i$ of A, we have immediately

$$\mu(B(x_i,r_i))^q (2r_i)^t \leq \mu(B(x_i,r_i))^p (2r_i)^t, \quad \forall t \in \mathbb{R}.$$

Hence,

$$\overline{H}_{\mu,\epsilon}^{q,t}(A) \leq \overline{H}_{\mu,\epsilon}^{p,t}(A).$$

When $\epsilon \downarrow 0$, we obtain

$$\overline{H}_{\mu}^{q,t}(A) \leq \overline{H}_{\mu}^{p,t}(A).$$

Therefore,

$$\mathcal{H}_{\mu}^{q,t}(E) = \sup_{A\subseteq E} \overline{\mathcal{H}}_{\mu}^{q,t}(A) \leq \sup_{A\subseteq E} \overline{\mathcal{H}}_{\mu}^{p,t}(A) = \mathcal{H}_{\mu}^{p,t}(E).$$

This induces the fact that

$$\mathcal{H}_{\mu}^{q,t}(E)=0, \quad \forall t > b_{\mu,E}(p).$$

Consequently

$$b_{\mu,E}(q) < t, \quad \forall t > b_{\mu,E}(p).$$

Hence,

$$b_{\mu,E}(q) \leq b_{\mu,E}(p).$$

The remaining part to prove the monotony $\Lambda_{\mu,E}$ and $B_{\mu,E}$ is analogous. $\square$

Proposition 4.4. (a) $0\leq b_\mu(q)\leq B_\mu(q)\leq \Lambda_\mu(q)$, whenever $q_i<1$ for all $i=1,2,\cdots,k$.

(b) $b_\mu(\P_i)=B_\mu(\P_i)=\Lambda_\mu(\P_i)=0$, where $\P_i=(0,0,\cdots,1,0,\cdots,0)$.

(c) $b_\mu(q)\leq B_\mu(q)\leq \Lambda_\mu(q)\leq 0$ whenever $q_i>1$ for all $i=1,2,\cdots,k$.

The proof of this results reposes on the following intermediate ones.

Lemma 4.3. *There exists a constant $\xi\in[0,+\infty]$ satisfying for any $E\subseteq\mathbb{R}^d$,*

$$\mathcal{H}_\mu^{q,t}(E)\leq \xi\mathcal{P}_\mu^{q,t}(E)\leq \xi\overline{\mathcal{P}}_\mu^{q,t}(E),\quad \forall q,t.$$

More precisely, ξ is the number related to the Besicovitch covering theorem.

Theorem 4.1 (Besicovitch Covering Theorem)**.** *There exists a constant $\xi\in\mathbb{N}$ satisfying: For any $E\in\mathbb{R}^d$ and $(r_x)_{x\in E}$ a bounded set of positive real numbers, there exists ξ sets $B_1,B_2,\cdots,B_\xi$, that are finite or countable composed of balls $B(x,r_x)$, $x\in E$ such that*

- $E\subseteq \bigcup\limits_{1\leq i\leq\xi}\bigcup\limits_{B\in B_i} B.$
- *each B_i is composed of disjoint balls.*

Proof of Lemma 4.3. It suffices to prove the first inequality. The second is always true for all $\xi>0$. Let $F\subseteq\mathbb{R}^d$, $\epsilon>0$ and $\mathcal{V}=\{B(x,\epsilon/2);\ x\in F\}$. Let next $\big((B_{ij})_j\big)_{1\leq i\leq\xi}$ be the ξ sets of $\mathcal{V}$ obtained by the Besicovitch covering theorem. So that, $(B_{ij})_{i,j}$ is a centered ϵ-covering of the set F and for each i, $(B_{ij})_j$ is a centered ϵ-packing of F. Therefore,

$$\overline{\mathcal{H}}_{\mu,\epsilon}^{q,t}(F)\leq\sum_{i=1}^{\xi}\sum_j\big(\mu(B_{ij})\big)^q(2r_{ij})^t\leq\sum_{i=1}^{\xi}\overline{\mathcal{P}}_{\mu,\epsilon}^{q,t}(F)=\xi\overline{\mathcal{P}}_{\mu,\epsilon}^{q,t}(F).$$

Hence,

$$\overline{\mathcal{H}}_\mu^{q,t}(F)\leq\xi\overline{\mathcal{P}}_\mu^{q,t}(F).$$

Consequently, for $E\subseteq\bigcup_i E_i$, we obtain

$$\begin{aligned}\mathcal{H}_\mu^{q,t}(E)=&\mathcal{H}_\mu^{q,t}\Big(\bigcup_i(E_i\cap E)\Big)\leq\sum_i\mathcal{H}_\mu^{q,t}(E_i\cap E)\\ \leq&\sum_i\sup_{F\subseteq E_i\cap E}\overline{\mathcal{H}}_\mu^{q,t}(F)\leq\xi\sum_i\sup_{F\subseteq E_i\cap E}\overline{\mathcal{P}}_\mu^{q,t}(F)\\ \leq&\xi\sum_i\overline{\mathcal{P}}_\mu^{q,t}(E_i).\end{aligned}$$

So as Lemma 4.3. ☐

Proof of Proposition 4.4. It follows from Propositions 4.2, 4.3 and Lemma 4.3. ☐

5 Mixed multifractal generalization of Bouligand-Minkowski's dimension

In this section, we propose to develop mixed multifractal generalization of Bouligand-Minkowski's dimension. Such a dimension is sometimes called the box-dimension or the Renyi dimension. Some mixed generalizations are already introduced in [15]. We will see hereafter that the mixed generalizations to be provided resemble to those in [15]. We will prove that in the mixed case, these dimensions remain strongly related to the mixed multifractal generalizations of the Hausdorff and packing dimensions. In the case of a single measure μ, the Bouligand-Minkowski dimensions are introduced as follows. For $E\subseteq \mathrm{supp}(\mu)$, $\delta>0$ and $q\in\mathbb{R}$, let

$$\mathcal{T}^q_{\mu,\delta}(E)=\inf\Big\{\sum_i \big(\mu\big(B(x_i,\delta)\big)\big)^q\Big\},$$

where the inf is over the set of all centered δ-coverings $\big(B(x_i,\delta)\big)_i$ of the set E. The Bouligand-Minkowski dimensions are

$$\overline{L}^q_\mu(E)=\limsup_{\delta\downarrow 0}\frac{\log\big(\mathcal{T}^q_{\mu,\delta}(E)\big)}{-\log\delta}$$

for the upper one and

$$\underline{L}^q_\mu(E)=\liminf_{\delta\downarrow 0}\frac{\log\big(\mathcal{T}^q_{\mu,\delta}(E)\big)}{-\log\delta}$$

for the lower. In the case of equality, the common value is denoted $L^q_\mu(E)$ and is called the Bouligand-Minkowski dimension of the set E. We can equivalently define these dimensions via the δ-packings as follows. For $\delta>0$ and $q\in\mathbb{R}$, we set

$$\mathcal{S}^q_{\mu,\delta}(E)=\sup\Big\{\sum_i \big(\mu\big(B(x_i,\delta)\big)\big)^q\Big\},$$

where the sup is taken over all the centered δ-packings $\big(B(x_i,\delta)\big)_i$ of the set E. The upper dimension is

$$\overline{C}^q_\mu(E)=\limsup_{\delta\downarrow 0}\frac{\log\big(\mathcal{S}^q_{\mu,\delta}(E)\big)}{-\log\delta}$$

and the lower is

$$\underline{C}^q_\mu(E)=\liminf_{\delta\downarrow 0}\frac{\log\big(\mathcal{S}^q_{\mu,\delta}(E)\big)}{-\log\delta}$$

and similarly, when these are equal, the common value will be denoted $C^q_\mu(E)$ and it defines the dimension of E. We now introduce the mixed multifractal generalization of the Bouligand-Minkowski dimensions. As we have noticed, our idea here is quite the same as the one in [15]. Let $\mu=(\mu_1,\mu_2,\cdots,\mu_k)$ be a vector valued measure composed of probability measures on $\mathbb{R}^d$. Denote as previously

$$\mu(B(x,r))\equiv\big(\mu_1(B(x,r)),\cdots,\mu_k(B(x,r))\big)$$

and for $q=(q_1,q_2,\cdots,q_k)\in\mathbb{R}^k$,

$$(\mu(B(x,r)))^q\equiv(\mu_1(B(x,r)))^{q_1}\cdots(\mu_k(B(x,r)))^{q_k}.$$

Next, for a nonempty subset $E\subseteq\mathbb{R}^d$ and $\delta>0$, we will use the same notations for $\mathcal{T}^q_{\mu,\delta}(E)$, $\overline{C}^q_\mu(E)$ and $\underline{C}^q_\mu(E)$ but without forgetting that we use the new product for the measure μ. Similarly for $\mathcal{S}^q_{\mu,\delta}(E)$, $\overline{L}^q_\mu(E)$ and $\underline{L}^q_\mu(E)$.

Definition 5.1. For $E\subseteq\operatorname{supp}(\mu)$ and $q=(q_1,q_2,\cdots,q_k)\in\mathbb{R}^k$, we will call

(a) $\overline{C}^q_\mu(E)$ and $\overline{L}^q_\mu(E)$ the upper mixed multifractal generalizations of the Bouligand Minkowski dimension of E.

(b) $\underline{C}^q_\mu(E)$ and $\underline{L}^q_\mu(E)$ the lower mixed multifractal generalizations of the Bouligand Minkowski dimension of E.

(c) $C^q_\mu(E)$ and $L^q_\mu(E)$ the mixed multifractal generalizations of the Bouligand Minkowski dimension of E.

Remark 5.1. We stress the fact that each quantity defines in fact a mixed generalization that can be different from the other. That is, we did not mean that $\overline{C}^q_\mu(E)$ and $\overline{L}^q_\mu(E)$ are the same (equal) and similarly for the lower ones. We will prove in the contrary that as for the single case, they can be different.

Theorem 5.1. *For*

1). For all $q\in\mathbb{R}^k$, we have

$$\underline{L}^q_\mu(E)\leq\underline{C}^q_\mu(E)\quad\textit{and}\quad\overline{L}^q_\mu(E)\leq\overline{C}^q_\mu(E).$$

2). For any $q\in\mathbb{R}^{*k}_-$, we have:

i). $b_{\mu,E}(q)\leq\underline{L}^q_\mu(E)=\underline{C}^q_\mu(E)$ and

ii). $\overline{L}_{\mu,E}(q)=\overline{C}^q_\mu(E)=\Lambda_{\mu,E}(q)$.

3). For any $q\in\mathbb{R}^{*k}_+$, we have

$$\overline{L}_{\mu,E}(q)\leq\overline{C}^q_\mu(E)\leq\Lambda_{\mu,E}(q).$$

Proof. 1). Using Besicovitch covering theorem we get

$$\mathcal{T}^q_{\mu,\delta}(E)\leq C\mathcal{S}^q_{\mu,\delta}(E),$$

with some constant C fixed. So as 1) is proved.

2). We firstly prove that

$$\underline{L}_\mu^q(E) \geq \underline{C}_\mu^q(E) \quad \text{and} \quad \overline{L}_\mu^q(E) \geq \overline{C}_\mu^q(E).$$

Indeed, let $\big(B(x_i,\delta)\big)_i$ be a centered δ-packing of E and $\big(B(y_i,\delta/2)\big)$ be a centered $\delta/2$-covering of E. Consider for each i, the integer k_i such that $x_i \in B(y_{k_i},\delta/2)$. It is straightforward that for $i \neq j$ we have $k_i \neq k_j$. Consequently, for $q \in \mathbb{R}_-^{*k}$, there holds that

$$\begin{aligned}\sum_i \big(\mu(B(x_i,\delta))\big)^q &= \sum_i \Big(\frac{\mu(B(x_i,\delta))}{\mu(B(y_{k_i},\delta/2))}\Big)^q \Big(\mu\Big(B\Big(y_{k_i},\frac{\delta}{2}\Big)\Big)\Big)^q \\ &\leq \sum_i \Big(\mu\Big(B\Big(y_i,\frac{\delta}{2}\Big)\Big)\Big)^q.\end{aligned}$$

Which means that

$$\mathcal{S}_{\mu,\delta}^q(E) \leq \mathcal{T}_{\mu,\frac{\delta}{2}}^q(E)$$

and thus, for any $q \in \mathbb{R}_-^{*k}$,

$$\underline{L}_\mu^q(E) \geq \underline{C}_\mu^q(E) \quad \text{and} \quad \overline{L}_\mu^q(E) \geq \overline{C}_\mu^q(E).$$

Using the assertion 1), we obtain the equalities

$$\underline{L}_\mu^q(E) = \underline{C}_\mu^q(E) \quad \text{and} \quad \overline{L}_\mu^q(E) = \overline{C}_\mu^q(E)$$

for all $q \in \mathbb{R}_-^{*k}$. Therefore, to prove i), it remains to prove the inequality of the left hand side. So, let $t > \underline{L}_\mu^q(E)$ and $F \subseteq E$. Consider next a sequence $(\delta_n)_n \subseteq [0,1]$ to be $\downarrow 0$, and satisfying

$$t > \frac{\log(\mathcal{T}_{\mu,\delta_n}^q(E))}{-\log\delta_n}, \quad \forall n \in \mathbb{N}.$$

This means that for each $n \in \mathbb{N}$, there exists a centered δ_n-covering $\big(B(x_{ni},\delta_n)\big)_i$ of E such that

$$\sum_i \big(\mu(B(x_{ni},\delta_n))\big)^q < \delta_n^{-t}.$$

There balls may be considered to be intersecting the set F. Next, for each i, choose an element $y_i \in B(x_{ni},\delta_n) \cap F$. This results on a centered $2\delta_n$-covering $\big(B(y_i,2\delta_n)\big)_i$ of F. Therefore,

$$\begin{aligned}\overline{\mathcal{H}}_{\mu,2\delta_n}^{q,t}(F) &\leq \sum_i \big(\mu(B(x_{ni},\delta_n))\big)^q (4\delta_n)^t \\ &= 4^t \sum_i \Big(\frac{\mu(B(y_i,2\delta_n))}{\mu(B(x_{ni},\delta_n))}\Big)^q \big(\mu(B(x_{ni},\delta_n))\big)^q \delta_n^t \\ &\leq 4^t \sum_i \big(\mu(B(x_{ni},\delta_n))\big)^q \delta_n^t \\ &\leq 4^t \delta_n^{-t} \delta_n^t = 4^t.\end{aligned}$$

Hence,

$$\overline{\mathcal{H}}_{\mu}^{q,t}(F) \leq 4^t, \quad \forall F \subseteq E, \quad t > \underline{L}_{\mu}^{q}(E).$$

So that,

$$\mathcal{H}_{\mu}^{q,t}(E) \leq 4^t < \infty, \quad \forall t > \underline{L}_{\mu}^{q}(E).$$

Consequently,

$$b_{\mu,E}(q) \leq t, \quad \forall t > \underline{L}_{\mu}^{q}(E) \Rightarrow b_{\mu,E}(q) \leq \underline{L}_{\mu}^{q}(E).$$

The remaining part can be proved by following similar techniques. $\square$

Next we need to introduce the following quantities which will be useful later. Let $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$ be a vector valued measure composed of probability measures on $\mathbb{R}^d$. For $j = 1,2,\cdots,k$, $a > 1$ and $E \subseteq \mathrm{supp}(\mu)$, denote

$$T_a^j(E) = \limsup_{r \downarrow 0} \Big(\sup_{x \in E} \frac{\mu_j\big(B(x,ar)\big)}{\mu_j\big(B(x,r)\big)} \Big)$$

and for $x \in \mathrm{supp}(\mu)$, $T_a^j(x) = T_a^j(\{x\})$. Denote also

$$\begin{aligned}
&P_0(\mathbb{R}^d, E) = \{\mu;\ \exists a > 1;\ \forall x \in E,\ T_a^j(x) < \infty,\ \forall j\},\\
&P_1(\mathbb{R}^d, E) = \{\mu;\ \exists a > 1;\ T_a^j(E) < \infty,\ \forall j\},\\
&P_0(\mathbb{R}^d) = P_0(\mathbb{R}^d, \mathrm{supp}(\mu)) \quad \text{and} \quad P_1(\mathbb{R}^d) = P_1(\mathbb{R}^d, \mathrm{supp}(\mu)).
\end{aligned}$$

Theorem 5.2. *For*

1). For $\mu \in P_0(\mathbb{R}^d)$ and $q \in \mathbb{R}_+^{*k}$, there holds that

$$b_{\mu,E}(q) \leq \overline{L}_{\mu}^{q}(E).$$

2). For $\mu \in P_1(\mathbb{R}^d)$ and $q \in \mathbb{R}_+^{*k}$, there holds that

i) $\underline{L}_{\mu}^{q}(E) = \underline{C}_{\mu}^{q}(E)$.

ii) $\overline{L}_{\mu,E}(q) = \overline{C}_{\mu}^{q}(E) = \Lambda_{\mu,E}(q)$.

Proof. 1). The vector valued measure $\mu \in P_0(\mathbb{R}^d)$ yields that

$$E = \bigcup_{m \in \mathbb{N}} E_m,$$

where

$$E_m = \Big\{ x \in E;\ \frac{\mu_j(B(x_i, 4r))}{\mu_j(B(x_i, r))} < m,\ 0 < r < \frac{1}{m},\ \forall j \Big\}.$$

Next, remark that for $t > \overline{L}_{\mu}^{q}(E)$ and $F \subseteq E_m$, there exists a sequence $(\delta_n)_n \in [0,1] \downarrow 0$ for which

$$t < \frac{\log(\mathcal{T}_{\mu,\delta_n}^{q}(F))}{-\log \delta_n}, \quad \forall n \in \mathbb{N}.$$

Therefore, there exists a centered δ_n-covering $(B(x_{ni},\delta_n))_i$ of F satisfying

$$\sum_i \big(\mu(B(x_{ni},\delta_n))\big)^q < \delta_n^{-t}.$$

Let next $y_{ni}\in B(x_{ni},\delta_n)$. Then, $(B(x_{ni},2\delta_n))_i$ is a centered $2\delta_n$-covering of F. Hence,

$$\begin{aligned}
\overline{\mathcal{H}}^{q,t}_{\mu,2\delta_n}(F) &\leq \sum_i \big(\mu(B(y_{ni},2\delta_n))\big)^q (4\delta_n)^t \\
&\leq 4^t \sum_i \Big(\frac{\mu(B(y_{ni},2\delta_n))}{\mu(B(x_{ni},\delta_n))}\Big)^q \big(\mu(B(x_{ni},\delta_n))\big)^q \delta_n^t \\
&\leq 4^t \sum_i \Big(\frac{\mu(B(x_{ni},4\delta_n))}{\mu(B(x_{ni},\delta_n))}\Big)^q \big(\mu(B(x_{ni},\delta_n))\big)^q \delta_n^t \\
&\leq 4^t m^{|q|} \sum_i \big(\mu(B(x_{ni},\delta_n))\big)^q \delta_n^t \\
&\leq 4^t m^{|q|},
\end{aligned}$$

where $|q|=q_1+q_2+\cdots+q_k$. Thus,

$$\overline{\mathcal{H}}^{q,t}_{\mu}(F)\leq 4^t m^{|q|}, \quad \forall m, \quad \text{and} \quad F\subseteq E_m.$$

Which means that

$$\mathcal{H}^{q,t}_{\mu}(E_m)\leq 4^t m^{|q|}<\infty, \quad \forall m, \quad \text{and} \quad t>\underline{L}^q_\mu(E).$$

Consequently,

$$b_{\mu,E_m}(q)\leq t, \quad \forall m, \quad \text{and} \quad t>\underline{L}^q_\mu(E).$$

Using the σ-stability of $b_{\mu,\cdot}(q)$ (see Proposition 4.2), it results that

$$b_{\mu,E}(q)\leq t, \quad \forall t>\underline{L}^q_\mu(E) \Rightarrow b_{\mu,E}(q)\leq \underline{L}^q_\mu(E).$$

Assertion 2 is left to the reader. $\square$

We now re-introduce the mixed multifractal generalization of the L^q-dimensions called also Renyi dimensions based on integral representations. See [15] for more details and other results. For $q\in\mathbb{R}^{*,k}$, $\mu=(\mu_1,\mu_2,\cdots,\mu_k)$ and $\delta>0$, we set

$$I^q_{\mu,\delta}=\int_{S_\mu}\Big(\mu(B(t,\delta))\Big)^q d\mu(t),$$

where, in this case,

$$\begin{aligned}
&S_\mu=\operatorname{supp}(\mu_1)\times\operatorname{supp}(\mu_2)\times\cdots\times\operatorname{supp}(\mu_k),\\
&\Big(\mu(B(t,\delta))\Big)^q=\Big(\mu_1(B(t_1,\delta))\Big)^{q_1}\Big(\mu_2(B(t_2,\delta))\Big)^{q_2}\cdots\Big(\mu_k(B(t_k,\delta))\Big)^{q_k},
\end{aligned}$$

and

$$d\mu(t)=d\mu_1(t_1)d\mu_2(t_2)\cdots d\mu_k(t_k).$$

The mixed multifractal generalizations of the Renyi dimensions are

$$\overline{I}_\mu^q=\limsup_{\delta\downarrow 0}\frac{\log I_{\mu,\delta}^q}{-\log\delta} \quad \text{and} \quad \underline{I}_\mu^q=\liminf_{\delta\downarrow 0}\frac{\log I_{\mu,\delta}^q}{-\log\delta}.$$

We now propose to relate these dimensions to the quantities $\underline{C}_\mu^q$, $\overline{C}_\mu^q$, $\underline{L}_\mu^q$, $\overline{L}_\mu^q$ introduced previously.

Proposition 5.1. The following results hold:

a). $\forall q\in\mathbb{R}_-^{*,k}$,

$$\underline{C}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu))\geq \underline{I}_\mu^q \quad \textit{and} \quad \overline{C}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu))\geq \overline{I}_\mu^q.$$

b). $\forall q\in\mathbb{R}_+^{*,k}$,

$$\underline{C}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu))\leq \underline{I}_\mu^q \qquad \textit{and} \qquad \overline{C}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu))\leq \overline{I}_\mu^q.$$

c). $\forall q\in\mathbb{R}^{*,k}$, $\mu\in P_1(\mathbb{R}^d)$,

$$\underline{C}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu))= \underline{I}_\mu^q \qquad \textit{and} \qquad \overline{C}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu))= \overline{I}_\mu^q.$$

d). $\forall q\in\mathbb{R}_-^{*,k}$,

$$\underline{I}_\mu^q\leq \underline{L}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu)) \qquad \textit{and} \qquad \overline{I}_\mu^q\leq \overline{L}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu)).$$

Proof. We only prove a). The remaining proofs of points b), c) and d) follow the same ideas.

For $\delta>0$, let $(B(x_i,\delta))_i$ be a centered δ-covering of $\mathrm{supp}(\mu)$ and let next $(B(x_{ij},\delta))_j$, $1\leq i\leq\xi$ the ξ sets defined in Besicovitch covering theorem. It holds that

$$\begin{aligned}
\sum_{i,j}\Big(\mu(B(x_{ij},\delta))\Big)^{q+\mathbb{I}} &=\sum_{i,j}\Big(\mu(B(x_{ij},\delta))\Big)^q\int_{B(x_{ij},\delta)^k}d\mu(t)\\
&\geq\sum_{i,j}\int_{B(x_{ij},\delta)^k}\Big(\mu(B(t,2\delta))\Big)^q d\mu(t)\\
&\geq\int_{S_\mu}\Big(\mu(B(t,2\delta))\Big)^q d\mu(t).
\end{aligned}$$

As a results,

$$\xi\mathcal{S}_{\mu,\delta}^{q+\mathbb{I}}(\mathrm{supp}(\mu))\geq I_{\mu,2\delta}^q.$$

Which implies that

$$\underline{C}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu))\geq \underline{I}_\mu^q \quad \text{and} \quad \overline{C}_\mu^{q+\mathbb{I}}(\mathrm{supp}(\mu))\geq \overline{I}_\mu^q.$$

Thus, we complete the proof. ∎

6 A mixed multifractal formalism for vector valued measures

Let $\mu=(\mu_1,\mu_2,\cdots,\mu_k)$ be a vector valued probability measure on $\mathbb{R}^d$. For $x\in\mathbb{R}^d$ and $j=1,2,\cdots,k$, we denote

$$\underline{\alpha}_{\mu_j}(x)=\liminf_{r\downarrow 0}\frac{\log(\mu_j(B(x,r)))}{\log r} \quad\text{and}\quad \overline{\alpha}_{\mu_j}(x)=\limsup_{r\downarrow 0}\frac{\log(\mu_j(B(x,r)))}{\log r},$$

respectively the local lower dimension and the local upper dimension of μ_j at the point x and as usually the local dimension $\alpha_{\mu_j}(x)$ of μ_j at x will be the common value when these are equal. Next for $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_k)\in\mathbb{R}_+^k$, let

$$\begin{aligned}\underline{X}_\alpha&=\{x\in\operatorname{supp}(\mu);\ \underline{\alpha}_{\mu_j}(x)\geq\alpha_j,\ \forall j=1,2,\cdots,k\},\\ \overline{X}^\alpha&=\{x\in\operatorname{supp}(\mu);\ \overline{\alpha}_{\mu_j}(x)\leq\alpha_j,\ \forall j=1,2,\cdots,k\},\end{aligned}$$

and

$$X(\alpha)=\underline{X}_\alpha\cap\overline{X}^\alpha.$$

The mixed multifractal spectrum of the vector valued measure μ is defined by

$$\alpha\longmapsto\dim X(\alpha),$$

where dim stands for the Hausdorff dimension.

In this section, we propose to compute such a spectrum for some cases of measures that resemble to the situation raised by Olsen in [9] but in the mixed case. This will permit to describe better the simultaneous behavior of finitely many measures. We intend precisely to compute the mixed spectrum based on the mixed multifractal generalizations of the Haudorff and packing dimensions b_μ, B_μ and Λ_μ. We start with the following technic results.

Lemma 6.1. *For*

1). $\forall\delta>0$, $t\in\mathbb{R}$ and $q\in\mathbb{R}_+^k$, $\alpha\in\mathbb{R}^k$ such that $\langle\alpha,q\rangle+t\geq 0$, we have

i). $\mathcal{H}^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}^\alpha)\leq 2^{\langle\alpha,q\rangle+k\delta}\mathcal{H}_\mu^{q,t}(\overline{X}^\alpha)$.

ii). $\mathcal{P}^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}^\alpha)\leq 2^{\langle\alpha,q\rangle+k\delta}\mathcal{P}_\mu^{q,t}(\overline{X}^\alpha)$.

2). $\forall\delta>0$, $t\in\mathbb{R}$ and $q\in\mathbb{R}_-^k$, $\alpha\in\mathbb{R}^k$ such that $\langle\alpha,q\rangle+t\geq 0$, we have

i). $\mathcal{H}^{\langle\alpha,q\rangle+t+k\delta}(\underline{X}_\alpha)\leq 2^{\langle\alpha,q\rangle+k\delta}\mathcal{H}_\mu^{q,t}(\underline{X}_\alpha)$.

ii). $\mathcal{P}^{\langle\alpha,q\rangle+t+k\delta}(\underline{X}_\alpha)\leq 2^{\langle\alpha,q\rangle+k\delta}\mathcal{P}_\mu^{q,t}(\underline{X}_\alpha)$.

Proof. 1). For i), We prove the first part. For $m\in\mathbb{N}^*$, consider the set

$$\overline{X}_m^\alpha=\Big\{x\in\overline{X}^\alpha;\frac{\log(\mu_j(B(x,r)))}{\log r}\le\alpha_j+\frac{\delta}{q_j};\,0<r<\frac{1}{m},\,1\le j\le k\Big\}.$$

Let next $0<\eta<1/m$ and $(B(x_i,r_i))_i$ a centered η-covering of $\overline{X}_m^\alpha$. It holds that

$$(\mu(B(x_i,r_i)))^q\ge r_i^{\langle\alpha,q\rangle+k\delta}.$$

Consequently,

$$\mathcal{H}_\eta^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}_m^\alpha)\le\sum_i(2r_i)^{\langle\alpha,q\rangle+t+k\delta}\le 2^{\langle\alpha,q\rangle+k\delta}\sum_i(\mu(B(x_i,r_i)))^q(2r_i)^t.$$

Hence, $\forall\eta>0$, there holds that

$$\mathcal{H}_\eta^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}_m^\alpha)\le 2^{\langle\alpha,q\rangle+k\delta}\overline{\mathcal{H}}_{\mu,\eta}^{q,t}(\overline{X}_m^\alpha).$$

Which means that

$$\mathcal{H}^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}_m^\alpha)\le 2^{\langle\alpha,q\rangle+k\delta}\overline{\mathcal{H}}_{\mu}^{q,t}(\overline{X}_m^\alpha)\le 2^{\langle\alpha,q\rangle+k\delta}\mathcal{H}_{\mu}^{q,t}(\overline{X}_m^\alpha).$$

Next, observing that $\overline{X}^\alpha=\bigcup_m\overline{X}_m^\alpha$, we obtain

$$\mathcal{H}^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}^\alpha)\le 2^{\langle\alpha,q\rangle+k\delta}\mathcal{H}_{\mu}^{q,t}(\overline{X}^\alpha).$$

ii). For $q\in\mathbb{R}_+^{*,k}$ and $m\in\mathbb{N}^*$, consider the set $\overline{X}_m^\alpha$ defined previously and let $E\subseteq\overline{X}_m^\alpha$, $0<\eta<1/m$ and $\big(B(x_i,r_i)\big)_i$ a centered η-packing of E. We have

$$\sum_i(2r_i)^{\langle\alpha,q\rangle+t+k\delta}\le 2^{\langle\alpha,q\rangle+k\delta}\sum_i(\mu(B(x_i,r_i)))^q(2r_i)^t\le 2^{\langle\alpha,q\rangle+k\delta}\overline{\mathcal{P}}_{\mu,\eta}^{q,t}(E).$$

Consequently, $\forall\eta>0$,

$$\overline{\mathcal{P}}_\eta^{\langle\alpha,q\rangle+t+k\delta}(E)\le 2^{\langle\alpha,q\rangle+k\delta}\overline{\mathcal{P}}_{\mu,\eta}^{q,t}(E).$$

Hence, $\forall E\subseteq\overline{X}_m^\alpha$,

$$\overline{\mathcal{P}}^{\langle\alpha,q\rangle+t+k\delta}(E)\le 2^{\langle\alpha,q\rangle+k\delta}\overline{\mathcal{P}}_{\mu}^{q,t}(E).$$

Let next, $(E_i)_i$ be a covering of $\overline{X}_m^\alpha$. Thus,

$$\begin{aligned}\mathcal{P}^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}_m^\alpha)&=\mathcal{P}^{\langle\alpha,q\rangle+t+k\delta}\Big(\bigcup_i(\overline{X}_m^\alpha\cap E_i)\Big)=\sum_i\mathcal{P}^{\langle\alpha,q\rangle+t+k\delta}\Big(\overline{X}_m^\alpha\cap E_i\Big)\\&\le\sum_i\overline{\mathcal{P}}^{\langle\alpha,q\rangle+t+k\delta}\Big(\overline{X}_m^\alpha\cap E_i\Big)\le 2^{\langle\alpha,q\rangle+k\delta}\sum_i\overline{\mathcal{P}}_\mu^{q,t}\Big(\overline{X}_m^\alpha\cap E_i\Big)\\&\le 2^{\langle\alpha,q\rangle+k\delta}\sum_i\overline{\mathcal{P}}_\mu^{q,t}(E_i).\end{aligned}$$

Hence, $\forall m$,

$$\mathcal{P}^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}_m^\alpha)\leq 2^{\langle\alpha,q\rangle+k\delta}\mathcal{P}_\mu^{q,t}(\overline{X}_m^\alpha).$$

Consequently,

$$\mathcal{P}^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}^\alpha)\leq 2^{\langle\alpha,q\rangle+k\delta}\mathcal{P}_\mu^{q,t}(\overline{X}^\alpha).$$

2). i). and ii). follow similar arguments and techniques as previously. □

Proposition 6.1. Let $\alpha\in\mathbb{R}_+^k$ and $q\in\mathbb{R}^k$. The following assertions hold:

a). Whenever $\langle\alpha,q\rangle+b_\mu(q)\geq 0$, we have

i). $\dim\overline{X}^\alpha\leq\langle\alpha,q\rangle+b_\mu(q),\quad\forall q\mathbb{R}_+^k$.

ii). $\dim\underline{X}_\alpha\leq\langle\alpha,q\rangle+b_\mu(q),\quad\forall q\mathbb{R}_-^k$.

b). Whenever $\langle\alpha,q\rangle+B_\mu(q)\geq 0$, we have

i). $\operatorname{Dim}\overline{X}^\alpha\leq\langle\alpha,q\rangle+B_\mu(q),\quad\forall q\mathbb{R}_+^k$.

ii). $\operatorname{Dim}\underline{X}_\alpha\leq\langle\alpha,q\rangle+B_\mu(q),\quad\forall q\mathbb{R}_-^k$.

Proof. a). i). It follows from Lemma 6.1, assertion 1) i),

$$\mathcal{H}^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}^\alpha)=0,\quad\forall t>b_\mu(q),\quad\delta>0.$$

Consequently,

$$\dim\overline{X}^\alpha\leq\langle\alpha,q\rangle+t+k\delta,\quad\forall t>b_\mu(q),\quad\delta>0.$$

Hence,

$$\dim\overline{X}^\alpha\leq\langle\alpha,q\rangle+b_\mu(q).$$

a). ii). It follows from Lemma 6.1, assertion 2) i), as previously, that

$$\mathcal{H}^{\langle\alpha,q\rangle+t+k\delta}(\underline{X}^\alpha)=0,\quad\forall t>b_\mu(q),\quad\delta>0.$$

Hence,

$$\dim\underline{X}_\alpha\leq\langle\alpha,q\rangle+t+k\delta,\quad\forall t>b_\mu(q),\quad\delta>0,$$

and finally,

$$\dim\underline{X}_\alpha\leq\langle\alpha,q\rangle+b_\mu(q).$$

b). i). Observing Lemma 6.1, assertion 1) ii), we obtain

$$\mathcal{P}^{\langle\alpha,q\rangle+t+k\delta}(\overline{X}^\alpha),\quad\forall t>B_\mu(q),\quad\delta>0.$$

Consequently,

$$\operatorname{Dim}\overline{X}^\alpha\leq\langle\alpha,q\rangle+t+k\delta,\quad\forall t>B_\mu(q),\quad\delta>0.$$

Hence,

$$\mathrm{Dim}\overline{X}^{\alpha} \leq \langle \alpha,q\rangle + B_{\mu}(q).$$

b). ii). observing Lemma 6.1, assertion 2) ii), we obtain

$$\mathcal{P}^{\langle \alpha,q\rangle+t+k\delta}(\underline{X}_{\alpha}) = 0, \quad \forall t > B_{\mu}(q), \quad \delta > 0.$$

Hence,

$$\mathrm{Dim}\underline{X}_{\alpha} \leq \langle \alpha,q\rangle + t + k\delta, \quad \forall t > B_{\mu}(q), \quad \delta > 0,$$

and finally,

$$\mathrm{Dim}\underline{X}_{\alpha} \leq \langle \alpha,q\rangle + B_{\mu}(q).$$

Thus, we complete the proof. □

Lemma 6.2. *$\forall q \in \mathbb{R}^k$ such that*

$$\langle \alpha,q\rangle + b_{\mu}(q) < 0 \quad \textit{or} \quad \langle \alpha,q\rangle + B_{\mu}(q) < 0,$$

we have

$$X(\alpha) = \emptyset.$$

Proof. It is based on

1. For $q \in \mathbb{R}^k_-$ with $\langle \alpha,q\rangle + b_{\mu}(q) < 0$ or $\langle \alpha,q\rangle + B_{\mu}(q) < 0$, $\underline{X}_{\alpha} = \emptyset$.
2. For $q \in \mathbb{R}^k_+$ with $\langle \alpha,q\rangle + b_{\mu}(q) < 0$ or $\langle \alpha,q\rangle + B_{\mu}(q) < 0$, $\overline{X}^{\alpha} = \emptyset$.

Indeed, let $q \in \mathbb{R}^k_-$ and assume that $\underline{X}_{\alpha} \neq \emptyset$. This means that there exists at least one point $x \in \mathrm{supp}(\mu)$ for which $\underline{\alpha}_{\mu_j}(x) \geq \alpha_j$, for $1 \leq j \leq k$. Consequently, for all $\varepsilon > 0$, there is a sequence $(r_n)_n \downarrow 0$ and satisfying

$$0 < r_n < \frac{1}{n} \quad \text{and} \quad \mu_j(B(x,r_n)) < r_n^{\alpha_j - \varepsilon}, \quad 1 \leq j \leq k.$$

Hence,

$$\Big(\mu(B(x,r_n))\Big)^q (2r_n)^t > 2^t r_n^{\langle(\alpha-\varepsilon\mathbb{I}),q\rangle+t}.$$

Choosing $t = \langle(\varepsilon\mathbb{I} - \alpha),q\rangle$, this induces that $\mathcal{H}^{q,t}_{\mu}(\{x\}) > 2^t$ and consequently,

$$b_{\mu}(q) \geq \dim^q_{\mu}(\{x\}) \geq t, \quad \forall \varepsilon > 0.$$

Letting $\varepsilon \downarrow 0$, it results that $b_{\mu}(q) \geq -\langle \alpha,q\rangle$ which is impossible. So as the first part of 1. The remaining part as well as 2 can be checked by similar techniques. □

Theorem 6.1. *Let $\mu=(\mu_1,\mu_2,\cdots,\mu_k)$ be a vector-valued Borel probability measure on $\mathbb{R}^d$ and $q\in\mathbb{R}^k$ fixed. Let further $t_q\in\mathbb{R}$, $r_q>0$, $\underline{K}_q$, $\overline{K}_q>0$, ν_q a Borel probability measure supported by supp(μ), $\varphi_q:\mathbb{R}_+\to\mathbb{R}$ be such that $\varphi_q(r)=o(\log r)$, as $r\to 0$. Let finally $(r_{q,n})_n\subset[0,1]\downarrow 0$ and satisfying*

$$\frac{\log r_{q,n+1}}{\log r_{q,n}}\to 1 \quad \textit{and} \quad \sum_n r_{q,n}^{\varepsilon}<\infty, \quad \forall\varepsilon>0.$$

Assume next the following assumptions:

A1). $\forall x\in supp(\mu) and r\in[0,r_q]$,

$$\underline{K}_q\le\frac{\nu_q(B(x,r))}{\Big(\mu(B(x,r))\Big)^q(2r)^{t_q}\exp(\varphi_q(r))}\le\overline{K}_q.$$

A2). $C_q(p)=\lim\limits_{n\to+\infty}C_{q,n}(p)$ exists and finite for all $p\in\mathbb{R}$, where

$$C_{q,n}(p)=\frac{1}{-\log r_{q,n}}\log\Big(\int_{supp(\mu)}\Big(\mu(B(x,r_{q,n}))\Big)^p d\nu_q(x)\Big).$$

Then, the following assertions hold.

i).

$$\begin{aligned}&dim(\underline{X}_{-\nabla_+C_q(0)}\cap\overline{X}^{-\nabla_-C_q(0)})\\ &\ge\begin{cases}-\nabla_-C_q(0)q+\Lambda_\mu(q)\ge-\nabla_-C_q(0)q+B_\mu(q)\ge-\nabla_-C_q(0)q+b_\mu(q), & q\in\mathbb{R}_-^k,\\ -\nabla_+C_q(0)q+\Lambda_\mu(q)\ge-\nabla_+C_q(0)q+B_\mu(q)\ge-\nabla_+C_q(0)q+b_\mu(q), & q\in\mathbb{R}_+^k.\end{cases}\end{aligned}$$

ii). Whenever C_q is differentiable at 0, we have

$$f_\mu(-\nabla C_q(0))=b_\mu^*(-\nabla C_q(0))=B_\mu^*(-\nabla C_q(0))=\Lambda_\mu^*(-\nabla C_q(0)).$$

Theorem 6.2. *Assume that the hypotheses of Theorem 6.1 are satisfied for all $q\in\mathbb{R}^k$. Then, the following assertions hold:*

i). $\alpha_\mu=-B_\mu$, ν_q, a.s., whenever B_μ is differentiable at q.

ii). $Dom(B)\subseteq\alpha_\mu(supp(\mu))$ and $f_\mu=B_\mu^*$ on $Dom(B)$.

The proof of this result is based on the application of a large deviation formalism. This will permit to obtain a measure ν supported by $\underline{X}_{-\nabla_+C(0)}\cap\overline{X}^{-\nabla_-C(0)}$. To do this, we re-formulate a mixed large deviation formalism to be adapted to the mixed multifractal formalism raised in our work.

Theorem 6.3 (The Mixed Large Deviation Formalism). *Consider a sequence of vector-valued random variables* $(W_n=(W_{n,1},W_{n,2},\cdots,W_{n,k}))_n$ *on a probability space* $(\Omega,\mathcal{A},\mathbb{P})$ *and* $(a_n)_n\subset[0,+\infty]$ *with* $\lim_{n\to+\infty}a_n=+\infty$. *Let next the function*

$$\begin{aligned}&C_n:\ \mathbb{R}^k\to\overline{\mathbb{R}},\\&t\mapsto C_n(t)=\frac{1}{a_n}\log\Big(E(\exp(\langle t,W_n\rangle))\Big).\end{aligned}$$

Assume that

A1). $C_n(t)$ *is finite for all* n *and* t.

A2). $C(t)=\lim_{n\to+\infty}C_n(t)$ *exists and is finite for all* t.

There holds that

i). The function C *is convex.*

ii). If $\nabla_-C(t)\le\nabla_+C(t)<\alpha$, *for some* $t\in\mathbb{R}^k$, *then*

$$\limsup_{n\to+\infty}\frac{1}{a_n}\log\Big(e^{-a_nC(t)}E\Big(\exp(\langle t,W_n\rangle)1_{\{\frac{W_n}{a_n}\ge\alpha\}}\Big)\Big)<0.$$

iii). If $\sum_n e^{-\varepsilon a_n}<\infty$ *for all* $\varepsilon>0$, *then*

$$\limsup_{n\to+\infty}\frac{W_n}{a_n}\le\nabla_+C(0)\qquad\mathbb{P}\ a.s.$$

iv). If $\alpha<\nabla_-C(t)\le\nabla_+C(t)$, *for some* $t\in\mathbb{R}^k$, *then*

$$\limsup_{n\to+\infty}\frac{1}{a_n}\log\Big(e^{-a_nC(t)}E\Big(\exp(\langle t,W_n\rangle)1_{\{\frac{W_n}{a_n}\le\alpha\}}\Big)\Big)<0.$$

v). If $\sum_n e^{-\varepsilon a_n}$ *is finite for all* $\varepsilon>0$, *then*

$$\nabla_-C(0)\le\limsup_{n\to+\infty}\frac{W_n}{a_n}\qquad\mathbb{P}\ a.s.$$

Proof. i). It follows from Holder's inequality.

ii). Let $h\in\mathbb{R}_+^{*,k}$ be such that $C(t)+\langle\alpha,h\rangle-C(t+h)>0$. We have

$$\begin{aligned}
&\frac{1}{a_n}\log\Big[e^{-a_nC(t)}\mathbb{E}\Big(\exp(\langle t,W_n\rangle)1_{\{\frac{W_n}{a_n}\geq\alpha\}}\Big)\Big]\\
=&\frac{1}{a_n}\log\Big[e^{-a_nC(t)}\int_{\{\frac{W_n}{a_n}\geq\alpha\}}e^{\langle t,W_n\rangle}d\mathbb{P}\Big]\\
=&\frac{1}{a_n}\log\Big[e^{-a_n(C(t)+\langle\alpha,h\rangle)}\int_{\{\frac{W_n}{a_n}\geq\alpha\}}e^{\langle t,W_n\rangle+a_n\langle\alpha,h\rangle}d\mathbb{P}\Big]\\
\leq&\frac{1}{a_n}\log\Big[e^{-a_n(C(t)+\langle\alpha,h\rangle)}\int_{\{\frac{W_n}{a_n}\geq\alpha\}}e^{\langle t+h,W_n\rangle}d\mathbb{P}\Big]\\
\leq&\frac{1}{a_n}\log\Big[e^{-a_n(C(t)+\langle\alpha,h\rangle)}\mathbb{E}(\exp(\langle t+h,W_n\rangle))\Big]\\
=&\frac{1}{a_n}\log\Big[e^{-a_n(C(t)+\langle\alpha,h\rangle-C_n(t+h))}\Big]\\
=&-(C(t)+\langle\alpha,h\rangle-C_n(t+h)).
\end{aligned}$$

Next, by taking the limsup as $n\longrightarrow+\infty$, the result follows immediately.

iii). Denote for $n,m\in\mathbb{N}$,

$$T_{n,m}=\Big\{\frac{W_n}{a_n}\geq\nabla_+C(0)+\frac{1}{m}\Big\}.$$

By choosing $t=0$ and $\alpha=\nabla_+C(0)+1/m$ in item ii), and observing that $C(0)=0$, we obtain

$$\limsup_{n\to+\infty}\frac{1}{a_n}\log\Big(E\Big(1_{\{\frac{W_n}{a_n}\geq\nabla_+C(0)+\frac{1}{m}\}}\Big)\Big)<0,$$

which means that $\limsup_n\frac{1}{a_n}\log\mathbb{P}(T_{n,m})<0$. Consequently, for some $\varepsilon>0$ and n large enough, there holds that $\limsup_n\frac{1}{a_n}\log\mathbb{P}(T_{n,m})<-\varepsilon$. Thus, $\mathbb{P}(T_{n,m})<e^{-\varepsilon a_n}$ which implies the convergence of the series $\sum_n\mathbb{P}(T_{n,m})$. Hence, using Borel-Cantelli theorem, we obtain

$$\mathbb{P}(\limsup_n T_{n,m})=0$$

for all m. Therefore,

$$\mathbb{P}\Big(\limsup_n\frac{W_n}{a_n}>\nabla_+C(0)\Big)=\mathbb{P}\Big(\bigcup_m\limsup_n T_{n,m}\Big)=0$$

and finally,

$$\limsup_n\frac{W_n}{a_n}\leq\nabla_+C(0),\quad\mathbb{P}\ a.s..$$

Thus, we complete the proof. ☐

Proof of Theorem 6.1. For simplicity we denote $t=t_q$, $\underline{K}=\underline{K}_q$, $\overline{K}=\overline{K}_q$, $\varphi=\varphi_q$, $\nu=\nu_q$ and $r_n=r_{q,n}$. Next, for $x\in\mathrm{supp}(\mu)$, let

$$\underline{\alpha}_{\mu_j}(x,r_n)=\liminf_n\frac{\log\big[\mu_j(B(x,r_n)\big]}{\log r_n}\quad\text{and}\quad\overline{\alpha}_{\mu_j}(x,r_n)=\limsup_n\frac{\log\big[\mu_j(B(x,r_n)\big]}{\log r_n}.$$

i). Using the hypothesis A1). and Lemma 4.3 we obtain $b_\mu(q)=B_\mu(q)=\Lambda_\mu(q)=t$. Next, it is straightforward that the set

$$M=\Big\{\,x\in\mathrm{supp}(\mu);\ -\nabla_+C(0)\leq\underline{\alpha}_\mu(x,r_n)\leq\overline{\alpha}_\mu(x,r_n)\leq-\nabla_-C(0)\Big\}$$

coincides with $\underline{X}_{-\nabla_+C(0)}\cap\overline{X}^{-\nabla_-C(0)}$. Hence, by setting in the mixed large deviation formalism Theorem 6.3, $\Omega=\mathrm{supp}(\mu)$, $\mathcal{A}=\mathcal{B}(\mathrm{supp}(\mu))$, $IP=\mu$, $W_n(x)=\log(\mu(B(x,r_n)))=(\log(\mu_1(B(x,r_n)),\log(\mu_2(B(x,r_n)),\cdots,\log(\mu_k(B(x,r_n)))$ and $a_n=-\log r_n$, it holds that

$$\underline{\alpha}_\mu(x)\geq\begin{cases}-\nabla_-C(0)q+t, & \text{for } q\leq 0,\\ -\nabla_+C(0)q+t, & \text{for } q\geq 0.\end{cases}$$

Finally, applying the famous Billingsley's Theorem [7], we obtain

$$\dim M\geq\begin{cases}-\nabla_-C(0)q+t, & \text{for } q\leq 0,\\ -\nabla_+C(0)q+t, & \text{for } q\geq 0.\end{cases}$$

ii). Remark that if C is differentiable at 0, item i). states that

$$\dim M\geq-\nabla C(0)q+t\geq\Lambda_\mu^*(-\nabla C(0))\geq B_\mu^*(-\nabla C(0))\geq b_\mu^*(-\nabla C(0)).$$

In the other hand, since the set M is not empty, Lemma 6.2 implies that $-\nabla C(0)q+t\geq 0$. Hence, Proposition 6.1 yields that $\dim M\leq-\nabla C(0)q+t$ for any $q\in\mathbb{R}^k$. Thus, taking the inf on q, we obtain

$$\dim M\leq b_\mu^*(-\nabla C(0))\leq B_\mu^*(-\nabla C(0))\leq\Lambda_\mu^*(-\nabla C(0)).$$

iii). We firstly claim that, there exists $\beta>0$ such that, for all $x\in\mathrm{supp}(\mu)$ and $0<r\ll 1$, we have

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))}<\beta.$$

So let $(B(x_{ij},r_n))_{1\leq i\xi,j}$ the ξ sets relatively to Besicovitch theorem extracted from the set $(B(x_i,r_n))_i$. A careful computation yields that

$$|p+q-\mathbb{I}|I_\mu^{p+q-\mathbb{I}}=C_q(p)+t_q,\quad\forall p,q\in\mathbb{R}^k,\tag{6.1}$$

where

$$|p+q-\mathbb{I}|=\sum_{i=1}^k(p_i+q_i-1).$$

Theorem 5.1 and Proposition 5.1 guarantee that

$$|p+q-\mathbb{I}|I_{\mu}^{p+q-1}=C_{\mu}^{p+q}(\mathrm{supp}(\mu))=\Lambda_{\mu}(p+q).$$

Consequently, $C_q(p)=\Lambda_\mu(p+q)-\Lambda_\mu(p)$. So, if Λ_μ is differentiable at q, C_q will be too at 0 and $\nabla C_q(0)=\nabla\Lambda_\mu(q)$. Thus, using the mixed large deviation formalism, we obtain

$$\alpha_\mu(x)=-\nabla C_q(0);\ \nu_q \text{ for almost all } x\in\mathrm{supp}(\mu),$$

hence, finally, $\alpha_\mu(x)=-\nabla\Lambda_\mu(q)$.

iv). Let q be such that $\nabla\Lambda_\mu(q)$ exists. Then $\nabla C_q(0)$ exists too. So, item ii). states that

$$f_\mu(-\nabla C(0))=\Lambda_\mu^*(-\nabla C(0)).$$

Which completes the proof. □

Proof of Theorem 6.2. i). Using the same notations as previously, we obtain C_q differentiable at 0, B_μ differentiable at q, and $\nabla C_q(0)=\nabla B_\mu(q)$. In the other hand, we obtain also

$$\alpha_\mu(x)=\alpha_\mu(x,r_n)=\lim_n\frac{W_n(x)}{-a_n}=-\nabla C_q(0)=\nabla B_\mu(q), \quad \nu_q,\ a.s.$$

ii). Follows immediately from i). and Theorem 6.1. □

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