Analysis in Theory and Applications
Anal. Theory Appl., Vol. **32**, No. 4 (2016), pp. 373-386

DOI: 10.4208/ata.2016.v32.n4.5

Algorithms and Identities for (q,h)-Bernstein Polynomials and (q,h)-Bézier Curves–A Non-Blossoming Approach

Ilija Jegdić[1,*], Jungsook Larson[2] and Plamen Simeonov[2]

[1] *Department of Mathematics, Texas Southern University, Houston, TX 77004, USA.*
[2] *Department of Mathematics and Statistics, University of Houston–Downtown, Houston, TX 77002, USA.*

Received 17 March 2015; Accepted (in revised version) 23 September 2016

Abstract. We establish several fundamental identities, including recurrence relations, degree elevation formulas, partition of unity and Marsden identity, for quantum Bernstein bases and quantum Bézier curves. We also develop two term recurrence relations for quantum Bernstein bases and recursive evaluation algorithms for quantum Bézier curves. Our proofs use standard mathematical induction and other elementary techniques.

Key Words: Bernstein polynomials, Bézier curves, Marsden's identity, recursive evaluation.

AMS Subject Classifications: 11C08, 65DXX, 65D15, 65D17

1 Introduction and definitions

Bernstein bases are polynomial bases used as blending functions for the construction of Bézier curves and surfaces. These bases have been used extensively over the last half century in geometric modeling, computer aided geometric design (CAGD), and approximation theory. The main application of Bézier curves and surfaces is in mathematical modeling of curves and surfaces that are used in various real life problems. One essential property of a Bézier curve or a Bézier surface is that it can be computed very efficiently using affine recursive evaluation algorithms. This is due to certain structural properties of the Bernstein basis functions that other polynomial bases do not possess.

The classical Bernstein polynomials were introduced by Bernstein in 1912 and have found many applications in applied and computational mathematics since then. The classical Bézier curves and surfaces were introduced in 1962 by the French engineer Pierre

*Corresponding author. *Email addresses:* `ilija.jegdic@tsu.edu` (I. Jegdić), `jung_sook@yahoo.com` (J. Larson), `simeonovp@uhd.edu` (P. Simeonov)

Bézier who worked for the French car manufacturer Renault. He used Bézier curves and surfaces to design and model aerodynamic car bodies [1]. The q-Bernstein polynomials were introduced and studied only recently by G. Phillips and his collaborators [7]. The theory of quantum q- and h-Bézier curves in the context of the quantum q- and h-blossoming was developed very recently by Goldman, Simeonov, and Zafiris [4,9,10].

In this paper, our main goal is to state and prove several of the most important properties of the (q,h)-Bernstein polynomials and (q,h)-Bézier curves such as recurrence relations, degree elevation algorithms, the partition of unity property, linear independence (polynomial basis), recursive evaluation algorithms, and a (q,h)-Marsden identity. This work extends and generalizes some analogous results of Goldman, Simeonov, and Zafiris [2,4,9,10] for q- and h-Bernstein polynomials and q- and h-Bézier curves. Most of our proofs will use the method of mathematical induction (with respect to the polynomial degree), instead of the blossoming techniques used by Goldman, Simeonov, and Zafiris [2,4,9,10], since we lack the machinery of the (q,h)-blossoming theory. The advantage of our approach is that we can establish all these important properties almost from scratch using only the very popular and well-understood induction argument, instead of the much less familiar theory of quantum blossoming.

We begin with some notation and terminology. Let $g(t)=qt+h$ be a linear function, $q\neq 0,-1$. The j-th composition of the function g is defined by

$$g^{[j]}(t)=(\underbrace{g\circ g\circ\cdots\circ g}_{j\text{ times}})(t),\quad j\geq 1.$$

We set $g^{[0]}(t)=t$. For example

$$\begin{aligned}
g^{[2]}(t)&=(g\circ g)(t)=g(g(t))=q^2t+qh+h,\\
g^{[3]}(t)&=g(g^{[2]}(t))=q\cdot g^{[2]}(t)+h=q^3t+(q^2+q+1)h.
\end{aligned}$$

Notice that

$$q+1=\frac{1-q^2}{1-q}=[2]_q\quad\text{and}\quad q^2+q+1=\frac{1-q^3}{1-q}=[3]_q,$$

where $[n]_q=\frac{1-q^n}{1-q}$ if $q\neq 1$, $[n]_q=n$ if $q=1$ and $[n]_0=1$ are the so-called q-integers [6]. By induction it is easy to show that

$$g^{[n]}(t)=q^n\cdot t+\left(\frac{1-q^n}{1-q}\right)h=q^nt+[n]_qh. \tag{1.1}$$

The (q,h)-Bernstein polynomials of degree n on the interval $[a,b]$ are defined by [3]

$$B_k^n(t;[a,b];q,h)=\begin{bmatrix}n\\k\end{bmatrix}_q\frac{\prod_{j=0}^{k-1}(t-g^{[j]}(a))\cdot\prod_{j=0}^{n-k-1}(b-g^{[j]}(t))}{\prod_{j=0}^{n-1}(b-g^{[j]}(a))}, \tag{1.2}$$

$k=0,\cdots,n$. Here $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are the q-binomial coefficients defined by [6]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad k=0,\cdots,n,$$

and $(a;q)_n$ denotes the q-shifted factorial defined by [8]

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n\geq 1, \quad (a;q)_0 = 1.$$

Therefore, we have the following limiting relation

$$\lim_{(q,h)\to(1,0)} B_k^n(t;[a,b];q,h) = B_k^n(t;[a,b]), \tag{1.3}$$

where

$$B_k^n(t;[a,b]) = \binom{n}{k}(t-a)^k(b-t)^{n-k}, \quad k=0,\cdots,n,$$

are the classical Bernstein polynomials of degree n on the interval $[a,b]$, [2].

The paper is organized as follows. In Section 2 we establish recurrence relations for the (q,h)-Bernstein polynomials. The degree elevation formula for the (q,h)-Bernstein polynomials is established in Section 3. The partition of unity property is given and proved in Section 4. We show that the (q,h)-Bernstein polynomials form a basis for the space of degree n polynomials in Section 5. Using the recurrence relations from Section 2, we derive two recursive evaluation algorithms for (q,h)-Bézier curves in Section 6. Degree elevation for (q,h)-Bézier surfaces is discussed in Section 7, and the proof of the (q,h)-Marsden identity is given in Section 8. We conclude the paper by discussing future work in Section 9.

2 Recurrence relations for the (q,h)-Bernstein polynomials

In this section we derive two recurrence relations for the (q,h)-Bernstein polynomials using the two recurrence relations for the q-binomial coefficients [6]:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q, \tag{2.1a}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q, \tag{2.1b}$$

$k=0,\cdots,n$. We set $\begin{bmatrix} n \\ -1 \end{bmatrix}_q=0$ and $\begin{bmatrix} n \\ n+1 \end{bmatrix}_q=0$. Substituting (2.1a) into (1.2), we get

$$\begin{aligned}
B_k^n(t;[a,b];q,h)=&\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{\left(b-g^{[n-k-1]}(t)\right)\prod_{j=0}^{k-1}\left(t-g^{[j]}(a)\right)\prod_{j=0}^{n-k-2}\left(b-g^{[j]}(t)\right)}{\left(b-g^{[n-1]}(a)\right)\prod_{j=0}^{n-2}\left(b-g^{[j]}(a)\right)}\\
&+q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{\left(t-g^{[k-1]}(a)\right)\prod_{j=0}^{k-2}\left(t-g^{[j]}(a)\right)\prod_{j=0}^{n-k-1}(b-g^{[j]}(t))}{\left(b-g^{[n-1]}(a)\right)\prod_{j=0}^{n-2}\left(b-g^{[j]}(a)\right)}\\
=&\left(\frac{b-g^{[n-k-1]}(t)}{b-g^{[n-1]}(a)}\right)B_k^{n-1}(t;[a,b];q,h)\\
&+q^{n-k}\left(\frac{t-g^{[k-1]}(a)}{b-g^{[n-1]}(a)}\right)B_{k-1}^{n-1}(t;[a,b];q,h). \qquad (2.2)
\end{aligned}$$

Similarly, substituting (2.1b) into (1.2), we derive

$$\begin{aligned}
B_k^n(t;[a,b];q,h)=&q^k\left(\frac{b-g^{[n-k-1]}(t)}{b-g^{[n-1]}(a)}\right)B_k^{n-1}(t;[a,b];q,h)\\
&+\left(\frac{t-g^{[k-1]}(a)}{b-g^{[n-1]}(a)}\right)B_{k-1}^{n-1}(t;[a,b];q,h). \qquad (2.3)
\end{aligned}$$

3 Degree elevation formula for the (q,h)-Bernstein polynomials

First we need to find coefficients $c(n,k)$ and $d(n,k)$ such that

$$\frac{c(n,k)[n+1]_q}{[k+1]_q}\cdot\frac{\left(t-g^{[k]}(a)\right)}{\left(b-g^{[n]}(a)\right)}+\frac{d(n,k)[n+1]_q}{[n+1-k]_q}\cdot\frac{\left(b-g^{[n-k]}(t)\right)}{\left(b-g^{[n]}(a)\right)}=1. \qquad (3.1)$$

Suppose we have found these coefficients. Then we can write

$$\begin{aligned}
B_k^n(t;[a,b];q,h)=&\frac{c(n,k)[n+1]_q\left(t-g^{[k]}(a)\right)}{[k+1]_q\left(b-g^{[n]}(a)\right)}B_k^n(t;[a,b];q,h)\\
&+\frac{d(n,k)[n+1]_q\left(b-g^{[n-k]}(a)\right)}{[n+1-k]_q\left(b-g^{[n]}(a)\right)}B_k^n(t;[a,b];q,h).
\end{aligned}$$

From the last equation and (1.2) it follows that

$$B_k^n(t;[a,b];q,h)=c(n,k)B_{k+1}^{n+1}(t;[a,b];q,h)+d(n,k)B_k^{n+1}(t;[a,b];q,h), \qquad (3.2)$$

$k=0,\cdots,n$. Eq. (3.2) expresses a (q,h)-Bernstein polynomial of degree n as a linear combination of two (q,h)-Bernstein polynomials of degree $n+1$. Now, we return to finding the coefficients $c(n,k)$ and $d(n,k)$. Comparing the like terms in (3.1), we obtain the linear system

$$c(n,k)\frac{[n+1]_q}{[k+1]_q}-d(n,k)\frac{[n+1]_q}{[n+1-k]_q}q^{n-k}=0, \tag{3.3a}$$

$$-c(n,k)\frac{[n+1]_q}{[k+1]_q}\frac{g^{[k]}(a)}{\left(b-g^{[n]}(a)\right)}+d(n,k)\frac{[n+1]_q}{[n+1-k]_q}\frac{\left(b-[n-k]_qh\right)}{\left(b-g^{[n]}(a)\right)}=1. \tag{3.3b}$$

Multiplying (3.3a) by $\frac{g^{[k]}(a)}{\left(b-g^{[n]}(a)\right)}$ and adding to (3.3b) we get

$$d(n,k)=\frac{[n+1-k]_q\left(b-g^{[n]}(a)\right)}{[n+1]_q\left(b-q^{n-k}g^{[k]}(a)-[n-k]_qh\right)}=\frac{[n+1-k]_q}{[n+1]_q}, \tag{3.4}$$

since

$$\begin{aligned}
&b-q^{n-k}g^{[k]}(a)-[n-k]_qh\\
=&b-q^{n-k}(q^ka+[k]_qh)-[n-k]_qh\\
=&b-q^na-\frac{h}{1-q}\left(q^{n-k}(1-q^k)+(1-q^{n-k})\right)\\
=&b-q^na-[n]_qh=b-g^{[n]}(a).
\end{aligned} \tag{3.5}$$

From (3.3a) and (3.4) it follows that

$$c(n,k)=\frac{[k+1]_q}{[n+1-k]_q}q^{n-k}d(n,k)=\frac{[k+1]_q}{[n+1]_q}q^{n-k}. \tag{3.6}$$

From (3.2), (3.4), and (3.6) we obtain the degree elevation formula

$$B_k^n(t;[a,b];q,k)=q^{n-k}\frac{[k+1]_q}{[n+1]_q}B_{k+1}^{n+1}(t;[a,b];q,h)+\frac{[n+1-k]_q}{[n+1]_q}B_k^{n+1}(t;[a,b];q,h). \tag{3.7}$$

4 Partition of unity

Proposition 4.1. The (q,h)-Bernstein polynomials satisfy the partition of unity property

$$\sum_{k=0}^{n}B_k^n(t;[a,b];q,h)=1. \tag{4.1}$$

Proof. To prove (4.1), we use induction with respect to $n\geq 0$.

When $n=0$, by (1.2) we have $B_0^0(t;[a,b];q,h)=1$. Therefore (4.1) holds in this case. Assume that (4.1) is true for some $n\geq 0$. Now we prove (4.1) for the case $n+1$. Using recurrence relation (2.2), we can write

$$\begin{aligned}&\sum_{k=0}^{n+1} B_k^{n+1}(t;[a,b];q,h)\\ =&\sum_{k=0}^{n+1}\Big\{\frac{(b-g^{[n-k]}(t))}{(b-g^{[n]}(a))}B_k^n(t;[a,b];q,h)+q^{n+1-k}\frac{(t-g^{[k-1]}(a))}{(b-g^{[n]}(a))}B_{k-1}^n(t;[a,b];q,h)\Big\},\end{aligned}$$

where $B_{n-1}^n=0$ and $B_{n+1}^n=0$. Then

$$\begin{aligned}&\sum_{k=0}^{n+1} B_k^{n+1}(t;[a,b];q,h)\\ =&\sum_{k=0}^{n}\frac{(b-g^{[n-k]}(t))}{(b-g^{[n]}(a))}B_k^n(t;[a,b];q,h)+\sum_{k=0}^{n}q^{n-k}\frac{(t-g^{[k]}(a))}{(b-g^{[n]}(a))}B_k^n(t;[a,b];q,h)\\ =&\sum_{k=0}^{n}\frac{b-q^{n-k}t-\left(\frac{1-q^{n-k}}{1-q}\right)h+q^{n-k}t-q^{n-k}\left(q^k a+\left(\frac{1-q^k}{1-q}\right)h\right)}{(b-g^{[n]}(a))}B_k^n(t;[a,b];q,h)\\ =&\sum_{k=0}^{n}\frac{\left\{b-q^n a-\left(\frac{1-q^n}{1-q}\right)h\right\}}{(b-g^{[n]}(a))}B_k^n(t;[a,b];q,h)\\ =&\sum_{k=0}^{n}B_k^n(t;[a,b];q,h)=1,\end{aligned}$$

by the induction hypothesis. Therefore (4.1) is true for every n. □

5 Polynomial basis

Proposition 5.1. The (q,h)-Bernstein polynomials of degree n form a basis for the space of polynomials of degree at most n.

Proof. It suffices to show that for every $n\geq 0$ there exist coefficients $\{C_{n,m,k}\}_{k=0}^n$ such that

$$t^m=\sum_{k=0}^{n}C_{n,m,k}B_k^n(t;[a,b];q,h),\quad m=0,\cdots,n. \tag{5.1}$$

We use induction with respect to n. When $n=0$ we have $B_0^0(t;[a,b];q,h)=1$. Assume that (5.1) is true for some degree $n\geq 0$. We now prove (5.1) for degree $n+1$.

First, let $0 \le m \le n$. By the induction hypothesis and the degree elevation equation (3.2) we have

$$\begin{aligned} t^m &= \sum_{k=0}^{n} C_{n,m,k} B_k^n(t;[a,b];q,h) \\ &= \sum_{k=0}^{n} C_{n,m,k} \Big\{ c(n,k) B_{k+1}^{n+1}(t;[a,b];q,h) + d(n,k) B_k^{n+1}(t;[a,b];q,h) \Big\} \\ &= \sum_{k=0}^{n+1} C_{n+1,m,k} B_k^{n+1}(t;[a,b];q,h), \end{aligned}$$

where

$$C_{n+1,m,k} = C_{n,m,k-1} c(n,k-1) + C_{n,m,k} d(n,k), \tag{5.2}$$

$k=0,\cdots,n+1$, and $c(n,k)$ and $d(n,k)$ are given by (3.6) and (3.4).

Now consider the monomial t^{n+1}. By the induction hypothesis

$$\begin{aligned} t^{n+1} = t\cdot t^n &= \sum_{k=0}^{n} C_{n,n,k} t B_k^n(t;[a,b];q,h) \\ &= \sum_{k=0}^{n} C_{n,n,k} \Big\{ \tilde{c}(n,k) B_{k+1}^{n+1}(t;[a,b];q,h) + \tilde{d}(n,k) B_k^{n+1}(t;[a,b];q,h) \Big\}, \end{aligned}$$

where by (1.2) the coefficients $\tilde{c}(n,k)$ and $\tilde{d}(n,k)$ must satisfy the equation

$$\left\{ \tilde{c}(n,k) \frac{[n+1]_q}{[k+1]_q} (t - g^{[k]}(a)) + \tilde{d}(n,k) \frac{[n+1]_q}{[n+1-k]_q} (b - g^{[n-k]}(t)) \right\} \times \frac{1}{b - g^{[n]}(a)} = t. \tag{5.3}$$

Equating the coefficients of t on the both sides of (5.3), we obtain

$$\frac{[n+1]_q}{[k+1]_q} \tilde{c}(n,k) - \frac{[n+1]_q}{[n+1-k]_q} q^{n-k} \tilde{d}(n,k) = b - g^{[n]}(a). \tag{5.4}$$

Equating the constant coefficients on both sides of (5.3) we obtain

$$-\frac{[n+1]_q g^{[k]}(a)}{[k+1]_q} \tilde{c}(n,k) + \frac{[n+1]_q}{[n+1-k]_q} \big(b - [n-k]_q h\big) \tilde{d}(n,k) = 0. \tag{5.5}$$

Multiplying (5.4) by $g^{[k]}(a)$ and adding to (5.5) yields

$$\frac{\tilde{d}(n,k)[n+1]_q}{[n+1-k]_q} \Big(-q^{n-k} g^{[k]}(a) + b - [n-k]_q h \Big) = (b - g^{[n]}(a)) g^{[k]}(a).$$

Solving the last equation for $\tilde{d}(n,k)$ and applying (3.5), we get

$$\tilde{d}(n,k) = \frac{\Big(b - g^{[n]}(a)\Big) g^{[k]}(a) [n+1-k]_q}{\big(-q^{n-k} g^{[k]}(a) + b - [n-k]_q h\big) [n+1]_q} = \frac{[n+1-k]_q}{[n+1]_q} g^{[k]}(a). \tag{5.6}$$

Then from (5.5) and (5.6) it follows that

$$\tilde{c}(n,k)=\frac{[k+1]_q}{[n+1-k]_q}\frac{\left(b-[n-k]_q h\right)}{g^{[k]}(a)}\tilde{d}(n,k)=\frac{[k+1]_q}{[n+1]_q}\left(b-[n-k]_q h\right). \tag{5.7}$$

Therefore $\left\{B_k^{n+1}(t;[a,b];q,h)\right\}_{k=0}^{n+1}$ is a basis for the space of polynomials of degree at most $n+1$, and this completes the proof of Proposition 5.1. □

6 Recursive evaluation algorithms for (q,h)-Bézier curves

A (q,h)-Bézier curve of degree n on the interval $[a,b]$ is a polynomial curve of the form

$$P(t)=\sum_{k=0}^{n}P_k B_k^n(t;[a,b];q,h). \tag{6.1}$$

Given a degree n polynomial $P(t)$, the coefficients $\{P_k\}_{k=0}^n$ in Eq. (6.1) are unique, because by Proposition 5.1 the (q,h)-Bernstein polynomials form a basis. These coefficients are called the control points of the (q,h)-Bézier curve $P(t)$.

Using recurrence relations (2.2) and (2.3), we can derive two recursive evaluation algorithms for (q,h)-Bézier curves. Below we describe these two algorithms.

We set $P_k^0=P_k$, $k=0,\cdots,n$ to be the control points at level $r=0$. Suppose that for some $0\le r\le n-1$ we have found points $\{P_k^r\}_{k=0}^{n-r}$ such that

$$P(t)=\sum_{k=0}^{n-r}P_k^r B_k^{n-r}(t;[a,b];q,h). \tag{6.2}$$

Here $P_k^r=P_k^r(t)$, $k=0,\cdots,n-r$, are polynomials of degree r.

Substituting (2.2) into (6.2) (with $B_{-1}^{n-r-1}=B_{n-r}^{n-r-1}=0$) we obtain

$$\begin{aligned}
P(t)=&\sum_{k=0}^{n-r}P_k^r\left\{\left(\frac{b-g^{[n-r-k-1]}(t)}{b-g^{[n-r-1]}(a)}\right)B_k^{n-r-1}(t;[a,b];q,h)\right.\\
&\left.+q^{n-r-k}\left(\frac{t-g^{[k-1]}(a)}{b-g^{[n-r-1]}(a)}\right)B_{k-1}^{n-r-1}(t;[a,b];q,h)\right\}\\
=&\sum_{k=0}^{n-r-1}B_k^{n-r-1}(t;[a,b];q,h)\left\{\left(\frac{b-g^{[n-r-k-1]}(t)}{b-g^{[n-r-1]}(a)}\right)P_k^r\right.\\
&\left.+q^{n-r-k-1}\left(\frac{t-g^{[k]}(a)}{b-g^{[n-r-1]}(a)}\right)P_{k+1}^r\right\}\\
=&\sum_{k=0}^{n-r-1}P_k^{r+1}B_k^{n-r-1}(t;[a,b];q,h),
\end{aligned}$$

where the control points at level $r+1$ are

$$P_k^{r+1}=\left(\frac{b-g^{[n-r-k-1]}(t)}{b-g^{[n-r-1]}(a)}\right)P_k^r+q^{n-r-k-1}\left(\frac{t-g^{[k]}(a)}{b-g^{[n-r-1]}(a)}\right)P_{k+1}^r, \tag{6.3}$$

$k=0,\cdots,n-r-1$.

At the last level n of this algorithm we get a single point P_0^n which gives the value of the Bézier curve at t, that is $P_0^n=P(t)$.

Similarly, substituting recurrence relation (2.3) into Eq. (6.2), we derive the second recursive evaluation algorithm for (q,h)-Bézier curves:

$$P_k^{r+1}=q^k\left(\frac{b-g^{[n-r-k-1]}(t)}{b-g^{[n-r-1]}(a)}\right)P_k^r+\left(\frac{t-g^{[k]}(a)}{b-g^{[n-r-1]}(a)}\right)P_{k+1}^r, \tag{6.4}$$

$k=0,\cdots,n-r-1$, $r=0,\cdots,n-1$. Again, at the last level n we get $P_0^n=P(t)$.

7 Degree elevation for (q,h)-Bézier curves

Let

$$P(t)=\sum_{k=0}^{n}P_kB_k^n(t;[a,b];q,h) \tag{7.1}$$

be a (q,h)-Bézier curve of degree n on the interval $[a,b]$. We want to write $P(t)$ as a (q,h)-Bézier curve of degree $n+1$, that is,

$$P(t)=\sum_{k=0}^{n+1}\tilde{P}_kB_k^{n+1}(t;[a,b];q,h). \tag{7.2}$$

Substituting the degree-elevation formula (3.2) into (7.1), we get

$$\begin{aligned}P(t)&=\sum_{k=0}^{n}P_k\left\{c(n,k)B_{k+1}^{n+1}(t;[a,b];q,h)+d(n,k)B_k^{n+1}(t;[a,b];q,h)\right\}\\&=\sum_{k=0}^{n+1}\left(d(n,k)P_k+c(n,k-1)P_{k-1}\right)B_k^{n+1}(t;[a,b];q,h),\end{aligned}$$

where $c(n,-1)=d(n,n+1)=0$ and the coefficients $c(n,k)$ and $d(n,k)$ are given by (3.6) and (3.4). Therefore, the degree elevated control points $\{\tilde{P}_k\}_{k=0}^{n+1}$ in (7.2) are given by

$$\tilde{P}_k=c(n,k-1)P_{k-1}+d(n,k)P_k, \tag{7.3}$$

$k=0,\cdots,n+1$, where $P_{-1}=0$ and $P_{n+1}=0$.

8 The (q,h)-Marsden identity

Theorem 8.1 ((q,h)-Marsden Identity)**.** *The (q,h)-Bernstein polynomials on the interval $[a,b]$ satisfy*

$$\prod_{j=0}^{n-1}(x-g^{[j]}(t))=\sum_{k=0}^{n}\left\{\prod_{\nu=k}^{n-1}(x-g^{[\nu]}(a))\right\}\left\{\prod_{\nu=0}^{k-1}(x-g^{[\nu]}(b))\right\}B_k^n(t,[a,b];q,h), \tag{8.1}$$

where $g(t)=qt+h$ and $n\geq 1$.

Proof. To prove (8.1) we use induction with respect to n.

First consider the case $n=1$. By (1.2) and (1.1) we have

$$B_0^0=1,\quad B_0^1(t;[a,b];q,h)=\frac{b-t}{b-a},\quad B_1^1(t;[a,b];q,h)=\frac{t-a}{b-a}. \tag{8.2}$$

In this case the left-hand side of (8.1) is $(x-t)$, while the right-hand side of (8.1) is

$$(x-a)B_0^1+(x-b)B_1^1=(x-a)\frac{(b-t)}{(b-a)}+(x-b)\frac{(t-a)}{(b-a)}=(x-t).$$

Therefore, (8.1) is true when $n=1$.

Next assume that (8.1) holds for some $n\geq 1$. Set

$$P_{n,k}(x)=\prod_{\nu=k}^{n-1}(x-g^{[\nu]}(a))\cdot\prod_{\nu=0}^{k-1}(x-g^{[\nu]}(b)),\quad k=0,\cdots,n. \tag{8.3}$$

Then (8.1) takes the form

$$\prod_{j=0}^{n-1}(x-g^{[j]}(t))=\sum_{k=0}^{n}P_{n,k}(x)B_k^n(t;[a,b];q,h). \tag{8.4}$$

Now we prove (8.4) for $n+1$. The left-hand side of (8.4) for $n+1$ is

$$\begin{aligned}\prod_{j=0}^{n}(x-g^{[j]}(t))&=(x-g^{[n]}(t))\prod_{j=0}^{n-1}(x-g^{[j]}(t))\\&=\sum_{k=0}^{n}P_{n,k}(x)(x-g^{[n]}(t))B_k^n(t;[a,b];q,h)\\&=\sum_{k=0}^{n}P_{n,k}(x)\Big(e_{n,k}B_k^{n+1}(t;[a,b];q,h)+f_{n,k}B_{k+1}^{n+1}(t;[a,b];q,h)\Big)\\&=\sum_{k=0}^{n+1}(P_{n,k}(x)e_{n,k}+P_{n,k-1}(x)f_{n,k-1})B_k^{n+1}(t;[a,b];q,h),\end{aligned} \tag{8.5}$$

with $P_{n,n+1}=0$ and $P_{n,-1}=0$, provided that we can find $e_{n,k}=e_{n,k}(x)$ and $f_{n,k}=f_{n,k}(x)$ such that

$$(x-g^{[n]}(t))B_k^n = e_{n,k}B_k^{n+1}+f_{n,k}B_{k+1}^{n+1}, \quad k=0,\cdots,n. \tag{8.6}$$

To simplify the notation we omit some of the variables and parameters in (8.6) and in the equations that follow. From (1.2) and (8.6) we get

$$x-g^{[n]}(t)=\tilde{e}_{n,k}(b-g^{[n-k]}(t))+\tilde{f}_{n,k}(t-g^{[k]}(a)), \tag{8.7}$$

where

$$\tilde{e}_{n,k}=\frac{[n+1]_q}{[n+1-k]_q}\cdot\frac{e_{n,k}}{\big(b-g^{[n]}(a)\big)}, \quad \tilde{f}_{n,k}=\frac{[n+1]_q}{[k+1]_q}\cdot\frac{f_{n,k}}{\big(b-g^{[n]}(a)\big)}. \tag{8.8}$$

Equating the constant terms and the t-terms in (8.7), we obtain the system

$$x-[n]_qh=\big(b-[n-k]_qh\big)\,\tilde{e}_{n,k}-g^{[k]}(a)\tilde{f}_{n,k}, \tag{8.9a}$$

$$-q^n=-q^{n-k}\tilde{e}_{n,k}+\tilde{f}_{n,k}, \tag{8.9b}$$

where we used (1.1). Adding the second equation in (8.9) times $g^{[k]}(a)$ to the first equation in (8.9) yields

$$x-[n]_qh-q^ng^{[k]}(a)=\Big(b-[n-k]_qh-q^{n-k}g^{[k]}(a)\Big)\,\tilde{e}_{n,k}.$$

Therefore

$$\tilde{e}_{n,k}=\frac{x-[n]_qh-q^ng^{[k]}(a)}{b-[n-k]_qh-q^{n-k}g^{[k]}(a)}=\frac{x-q^ng^{[k]}(a)-[n]_qh}{b-g^{[n]}(a)}, \tag{8.10}$$

where we used (3.5). Then the second equation in (8.9) and (1.1) yield

$$\begin{aligned}
\tilde{f}_{n,k}=&q^{n-k}\tilde{e}_{n,k}-q^n\\
=&\frac{q^{n-k}x-q^{2n-k}g^{[k]}(a)-q^{n-k}[n]_qh-q^nb+q^ng^{[n]}(a)}{b-g^{[n]}(a)}\\
=&\frac{q^{n-k}x-q^{2n}a-q^{2n-k}[k]_qh-q^{n-k}[n]_qh-q^nb+q^{2n}a+q^n[n]_qh}{b-g^{[n]}(a)}\\
=&\frac{q^{n-k}x-q^nb-(q^{2n-k}-q^{2n}+q^{n-k}-q^{2n-k}-q^n+q^{2n})h/(1-q)}{b-g^{[n]}(a)}\\
=&\frac{q^{n-k}\big(x-q^kb-[k]_qh\big)}{b-g^{[n]}(a)}=\frac{q^{n-k}\Big(x-g^{[k]}(b)\Big)}{b-g^{[n]}(a)}.
\end{aligned} \tag{8.11}$$

From (8.8), (8.10), and (8.11) it follows that

$$e_{n,k}=\frac{[n+1-k]_q}{[n+1]_q}\Big(x-[n]_qh-q^ng^{[k]}(a)\Big) \tag{8.12}$$

and

$$f_{n,k}=\frac{[k+1]_q}{[n+1]_q}q^{n-k}\left(x-g^{[k]}(b)\right). \tag{8.13}$$

By (8.3) the coefficient of B_k^{n+1} in the last line of (8.5) equals

$$\begin{aligned} e_{n,k}P_{n,k}(x)+f_{n,k-1}P_{n,k-1}(x)=&\prod_{\nu=k}^{n-1}(x-g^{[\nu]}(a))\prod_{\nu=0}^{k-2}(x-g^{[\nu]}(b))\\ &\times\left\{(x-g^{[k-1]}(b))e_{n,k}+(x-g^{[k-1]}(a))f_{n,k-1}\right\}. \end{aligned} \tag{8.14}$$

Next, by (8.12) and (8.13), we get

$$\begin{aligned} &[n+1]_q\left\{(x-g^{[k-1]}(b))e_{n,k}+(x-g^{[k-1]}(a))f_{n,k-1}\right\}\\ =&(x-g^{[k-1]}(b))[n+1-k]_q(x-[n]_qh-q^ng^{[k]}(a))\\ &+q^{n+1-k}[k]_q(x-g^{[k-1]}(a))(x-g^{[k-1]}(b))\\ =&(x-g^{[k-1]}(b))\left\{[n+1-k]_q\left(x-[n]_qh-q^ng^{[k]}(a)\right)\right.\\ &\left.+q^{n+1-k}[k]_q(x-g^{[k-1]}(a))\right\}. \end{aligned} \tag{8.15}$$

We now simplify the last expression in (8.15). We can write

$$[n+1-k]_q(x-[n]_qh-q^ng^{[k]}(a))+q^{n+1-k}[k]_q(x-g^{[k-1]}(a))=Ax-Ba-Ch. \tag{8.16}$$

For the coefficients A,B, and C in (8.16) using (1.1) we derive

$$A=[n+1-k]_q+q^{n+1-k}[k]_q=\frac{1-q^{n+1-k}+q^{n+1-k}(1-q^k)}{1-q}=[n+1]_q, \tag{8.17a}$$

$$B=[n+1-k]_qq^{n+k}+q^{n+1-k}[k]_qq^{k-1}=\frac{(1-q^{n+1-k})q^{n+k}+q^n(1-q^k)}{1-q}=q^n[n+1]_q, \tag{8.17b}$$

and

$$\begin{aligned} C=&[n+1-k]_q\left([n]_q+q^n[k]_q\right)+q^{n+1-k}[k]_q[k-1]_q\\ =&\frac{(1-q^{n+1-k})(1-q^n+q^n-q^{n+k})+q^{n+1-k}(1-q^k)(1-q^{k-1})}{(1-q)^2}\\ =&\frac{1-q^{n+1-k}-q^{n+k}+q^{2n+1}+q^{n+1-k}-q^{n+1}-q^n+q^{n+k}}{(1-q)^2}\\ =&\frac{(1-q^n)(1-q^{n+1})}{(1-q)^2}=[n]_q[n+1]_q. \end{aligned} \tag{8.18}$$

From (8.16), (8.17a), (8.17b), and (8.18) we get

$$\begin{aligned}&[n+1-k]_q(x-[n]_qh-q^ng^{[k]}(a))+q^{n+1-k}[k]_q(x-g^{[k-1]}(a))\\=&[n+1]_q(x-q^na-[n]_qh)=[n+1]_q(x-g^{[n]}(a)).\end{aligned}\tag{8.19}$$

Then, from (8.15) and (8.19) it follows that

$$(x-g^{[k-1]}(b))e_{n,k}+(x-g^{[k-1]}(a))f_{n,k-1}=(x-g^{[k-1]}(b))(x-g^{[n]}(a)).\tag{8.20}$$

From (8.14), (8.20), and (8.3) we get

$$e_{n,k}P_{n,k}(x)+f_{n,k-1}P_{n,k-1}(x)=\prod_{\nu=k}^{n}(x-g^{[\nu]}(a))\prod_{\nu=0}^{k-1}(x-g^{[\nu]}(b))=P_{n+1,k}(x).\tag{8.21}$$

Finally, combining (8.5) and (8.21) we obtain the right-hand side of equation (8.4) for the case $n+1$. We have shown that (8.4) is true for the case $n+1$. This completes the proof of the (q,h)-Marsden identity. □

9 Future work

We have established several important properties, identities, and algorithms for the (q,h)-Bernstein polynomials and the (q,h)-Bézier curves using only standard mathematical induction. Many of these and other properties have been derived in the recent works [3,4, 8–10].

The most natural next stage of this work is to program and implement the recursive evaluation and degree elevation algorithms for (q,h)-Bézier curves. Then (q,h)-Bernstein polynomials and (q,h)-Bézier surfaces of several variables can be constructed using tensor products of univariate (q,h)-Bernstein polynomials and their analogous properties can be studied. These multivariate (q,h)-Bernstein polynomials can be used as blending functions to define (q,h)-Bézier surfaces, and to study their properties and recursive evaluation algorithms.

This future work is outside of the scope of this paper and will be accomplished in future research projects.

References

[1] G. Farin, Curves and Surfaces for CAGD, 5th ed., Morgan Kaufman, 2001.
[2] R. Goldman, Pyramid Algorithms, A Dynamic Programming Approach to Curves and Surfaces for Geometric Modeling, The Morgan Kaufman Series in Computer Graphics and Geometric Modeling, Elsevier Science, 2003.
[3] R. Goldman and P. Simeonov, Quantum Bernstein bases and quantum Bézier curves, J. Comput. Appl. Math., 288 (2015), 284–303.

[4] R. Goldman and P. Simeonov, Formulas and algorithms for quantum differentiation of quantum Bernstein bases and quantum Bézier curves based on quantum blossoming, Graphical Models, 74 (2012), 326–334.

[5] N. Gregorz, Approximation properties for generalized Bernstein polynomials, J. Math. Anal. Appl., 350 (2009), 50–55.

[6] V. Kac and P. Cheung, Quantum Calculus, Universitext Series, vol. IX, Springer-Verlag, 2002.

[7] G. Phillips, Bernstein polynomials based on the q-integers, Annals Numer. Anal., 4 (1997), 511–518.

[8] P. Simeonov and R. Goldman, Quantum B-splines, BIT, 53(1) (2013), 193–223.

[9] P. Simeonov, V. Zafiris and R. Goldman, h-Blossoming: A new approach to algorithms and identities for h-Bernstein bases and h-Bézier curves, Computer Aided Geometric Design, 28 (2011), 549–565.

[10] P. Simeonov, V. Zafiris and R. Goldman, q-Blossoming: A new approach to algorithms and identities for q-Bernstein bases and q-Bézier curves, J. Approx. Theory, 164(1) (2012), 77–104.