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A High Frequency Boundary Element Method for Scattering by Convex Polygons with Impedance Boundary Conditions

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Abstract. We consider scattering of a time harmonic incident plane wave by a convex polygon with piecewise constant impedance boundary conditions. Standard finite or boundary element methods require the number of degrees of freedom to grow at least linearly with respect to the frequency of the incident wave in order to maintain accuracy. Extending earlier work by Chandler-Wilde and Langdon for the sound soft problem, we propose a novel Galerkin boundary element method, with the approximation space consisting of the products of plane waves with piecewise polynomials supported on a graded mesh with smaller elements closer to the corners of the polygon. Theoretical analysis and numerical results suggest that the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency of the incident wave.

AMS subject classifications: 35J05, 65N38, 65R20

Key words: Boundary integral equation method, high frequency scattering, convex polygons, impedance boundary conditions.

1 Introduction

In this paper we consider two-dimensional scattering of a time-harmonic incident plane wave $u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$, $\mathbf{x} \in \mathbb{R}^2$, where the unit vector $\mathbf{d}$ is the direction of propagation and $k > 0$ is the wavenumber of the incident wave, by a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, with impedance boundary conditions holding on $\Gamma := \partial\Omega$. Define $D := \mathbb{R}^2 \backslash \overline{\Omega}$ to be the unbounded domain exterior to Ω, let $\gamma^+ : H^1(D) \to H^{1/2}(\Gamma)$ and $\gamma^- : H^1(\Omega) \to H^{1/2}(\Gamma)$ denote the exterior and interior trace operators, respectively, and, where $H^1(G,\Delta) := \{v \in$

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$H^1(G):\Delta v\in L^2(G)\}$, let $\partial_{\mathbf{n}}^+ : H^1(D,\Delta)\to H^{-1/2}(\Gamma)$ and $\partial_{\mathbf{n}}^- : H^1(\Omega,\Delta)\to H^{-1/2}(\Gamma)$ denote the exterior and interior normal derivative operators, respectively. (All of $\gamma^\pm$ and $\partial_{\mathbf{n}}^\pm$ are well-defined as bounded linear operators, see [15], where also our various function space notations are defined.) Then the scattering problem we consider is: given $\beta\in L^\infty(\Gamma)$, find the total field $u^t\in C^2(D)\cap H^1_{\rm loc}(D)$ such that

$$\Delta u^t+k^2u^t=0, \qquad \text{in } D, \tag{1.1}$$

$$\partial_{\mathbf{n}}^+ u^t+\mathrm{i}k\beta\gamma^+u^t=0, \quad \text{on } \Gamma, \tag{1.2}$$

and such that the scattered field $u^s:=u^t-u^i$ satisfies the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r}(\mathbf{x})-\mathrm{i}ku^s(\mathbf{x})=o\left(r^{-1/2}\right), \tag{1.3}$$

as $r:=|\mathbf{x}|\to\infty$, uniformly with respect to $\mathbf{x}/|\mathbf{x}|$.

It is a standard result that this boundary value problem is uniquely solvable if $\mathrm{Re}\beta\geq 0$, which physically is a condition that the impedance boundary does not emit energy. (See [9] for a proof in the case that Γ is C^2, and [15, Lemma 9.9, Exercise 9.5] for the main ideas for the extension to the case of Lipschitz Γ.) Our concern in this paper is to develop a novel and very effective high frequency boundary element method for the particular case when Ω is a convex polygon and β is constant on each side of Γ, corresponding to an obstacle made up from several, homogeneous, materials, each with a different relative surface admittance.

The problem (1.1)–(1.3) has received significant recent attention in the literature [1, 19,22]. Standard boundary or finite element approximations suffer from the requirement that the number of degrees of freedom must increase at least linearly with respect to k in order to maintain accuracy. Although asymptotic schemes can provide reasonable approximations when k is very large [12], there thus exists a wide range of frequencies for which numerical schemes are prohibitively expensive whilst asymptotic approaches are insufficiently accurate. This difficulty for scattering problems has been well documented in the literature in recent years, and numerous novel approaches to reducing the computational cost for moderate to large k have been proposed. The scattering problem that has received the most attention in the literature is the sound soft problem, i.e. (1.1) and (1.3) with the boundary condition $\gamma^+u^t=0$ on Γ replacing (1.2). Using a boundary element approach, with a hybrid approximation space consisting of the product of plane waves with piecewise polynomials, very efficient schemes have been developed for scattering by smooth obstacles [10] and by convex polygons [6], with in each case the number of degrees of freedom required to achieve a prescribed level of accuracy depending only very mildly on k. In contrast, the only numerical scheme for (1.1)–(1.3) that we are aware of that has been developed specifically for the purpose of efficiency at high frequencies is that in [19], where a circle of piecewise constant impedance is considered. There the approximation space is enriched with plane waves traveling in multiple directions; this reduces the number of degrees of freedom required per wavelength from ten to three (for engineering accuracy), but the cost still grows linearly with k. For a full review of recent developments in integral equation methods for high frequency scattering we refer to [3].

The approach we will take in this paper combines ideas from our earlier work for sound soft convex polygons [6] with ideas developed for solving a two-dimensional problem of high frequency scattering by an inhomogeneous half-plane of piecewise constant impedance [7, 14]. In [7] a method in the spirit of the geometrical theory of diffraction was applied to obtain a representation of the solution, with the known leading order behaviour being subtracted off, leaving only the remaining scattered field due to the discontinuities in the impedance boundary conditions to be approximated. This diffracted field was expressed as a product of oscillatory and non-oscillatory functions, with a rigorous error analysis, supported by numerical experiments, demonstrating that the number of degrees of freedom required to maintain accuracy as $k\to\infty$ grows only logarithmically with respect to k. This approach was improved in [14], where derivation of sharper regularity estimates regarding the rate of decay of the scattered field away from impedance discontinuities led to error estimates independent of k.

The plan of this paper is as follows. In Section 2 we derive regularity results, demonstrating in particular that γ^+u^t can be written as the known leading order physical optics solution plus the products of plane waves with unknown functions that are non-oscillatory, highly peaked near the corners of the polygon and rapidly decaying away from the corners. In Section 3, we discuss the boundary integral equation formulation of (1.1)–(1.3). We describe our approximation space and Galerkin boundary element method in Section 4, and present numerical results demonstrating the efficiency of our scheme at high frequencies in Section 5. Finally, in Section 6 we present some conclusions.

2 Regularity results

Our aim in this section is to investigate the regularity of u^t, deriving bounds on derivatives which are sufficiently explicit, in particular in their dependence on the wavenumber, so that we can prove the effectiveness of our novel boundary element approximation space. In this endeavour we will, as part of our arguments, relate all bounds on derivatives to

$$M:=\sup_{x\in D}|u^t(x)|. \tag{2.1}$$

We note first of all that $\gamma^+u^t\in H^{1/2}(\Gamma)\subset L^2(\Gamma)$ so that the impedance boundary condition (1.2) implies that $\partial_{\mathbf{n}}^+u^t\in L^2(\Gamma)$. It follows from standard regularity results for elliptic problems in Lipschitz domains [15, Theorem 4.24] that $\gamma^+u^t\in H^1(\Gamma)$ and thus, from Theorem 6.12 and the accompanying discussion in [15], that $u^t\in H^{3/2}_{\mathrm{loc}}(D)$, so that, by standard Sobolev imbedding theorems [15], $u^t\in C(\overline{D})$ (as a consequence of which $M<\infty$).

From this point on in the paper we restrict attention to the case shown in Fig. 1 where Ω is a convex polygon. We write the boundary of the polygon as $\Gamma=\cup_{j=1}^n\Gamma_j$, where Γ_j, $j=1,\cdots,n$, are the n sides of the polygon, with j increasing anticlockwise as shown in

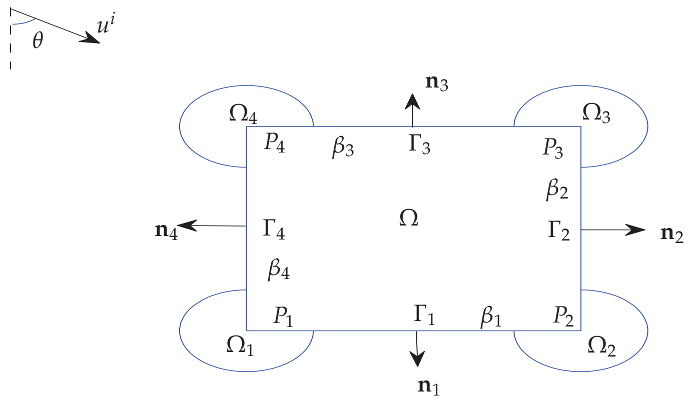

Figure 1: Notation for scattering by an impedance polygon.

Fig. 1. We denote the corners of the polygon by P_j, $j=1,\cdots,n$, and we set $P_{n+1}=P_1$, so that, for $j=1,\cdots,n$, Γ_j is the line joining P_j with P_{j+1}. We denote the length of Γ_j by $L_j=|P_{j+1}-P_j|$, the external angle at vertex P_j by $\Omega_j\in(\pi,2\pi)$, and the outwards unit normal vector to Γ_j by $\mathbf{n}_j$. We let $\theta\in[0,2\pi)$ denote the angle of the incident plane wave direction $\mathbf{d}$, as measured anticlockwise from the downward vertical $(0,-1)$. We also assume from this point on that β takes a constant value on each side of the polygon Γ_j; that is, $\beta(\mathbf{x})=\beta_j$, $\mathbf{x}\in\Gamma_j$, $j=1,\cdots,n$, where $\beta_j\in\mathbb{C}$, $\mathrm{Re}(\beta_j)>0$, $j=1,\cdots,n$.

The following lemma is our first tool to derive explicit regularity estimates. In this lemma we use the notations $B_\delta:=\{\mathbf{x}\in\mathbb{R}^2:|\mathbf{x}|<\delta\}$, $S_\delta:=\{\mathbf{x}\in B_\delta:x_2>0\}$, and $\gamma_\delta:=\{\mathbf{x}=(x_1,0):|x_1|<\delta\}$, for $\delta>0$.

Lemma 2.1. *Suppose that $\delta>0$ and $N>0$, and that $u\in H^1(S_\delta)\cap C^2(S_\delta)\cap C(\overline{S_\delta})$, $m\in C(\overline{S_\delta})$, and $\alpha\in C^{0,\varepsilon}(\gamma_\delta)$, for some $\varepsilon\in(0,1)$, with $\|m\|_\infty\le N$ and $\|\alpha\|_{C^{0,\varepsilon}(\gamma_\delta)}\le N$. Suppose also that $\Delta u+mu=0$ in S_δ and $\partial u/\partial x_2+\mathrm{i}\alpha u=0$ on γ_δ. Then $u\in C^1(S_\delta\cup\gamma_\delta)$ and, for some constant $C>0$ depending only on δ, N, and ε,*

$$|\nabla u(\mathbf{x})|\le C\sup_{\mathbf{y}\in S_\delta}|u(\mathbf{y})|,\quad \mathbf{x}\in S_{\delta/2}. \tag{2.2}$$

Proof. Let $G(\mathbf{x},\mathbf{y})$ be the Dirichlet Green's function for the Laplace operator for the ball B_δ, given explicitly for example in [20, p.107], and let $G^*(\mathbf{x},\mathbf{y}):=G(\mathbf{x},\mathbf{y})+G(\mathbf{x}',\mathbf{y})$, where, for $\mathbf{x}=(x_1,x_2)\in B_\delta$, $\mathbf{x}':=(x_1,-x_2)$. Then, by applications of Green's second theorem to u and $G^*(\mathbf{x},\cdot)$, and noting that $G^*(\mathbf{x},\cdot)=0$ on ∂B_δ and that $\partial G^*(\mathbf{x},\mathbf{y})/\partial y_2=0$ and $\partial u/\partial x_2+\mathrm{i}\alpha u=0$ on γ_δ, one obtains that, for $\mathbf{x}\in S_\delta$, where $\Gamma_\delta:=\partial B_\delta\cap\partial S_\delta$,

$$u(\mathbf{x})=\mathrm{i}\int_{\gamma_\delta}G^*(\mathbf{x},\mathbf{y})\alpha(\mathbf{y})u(\mathbf{y})\mathrm{d}s(\mathbf{y})-\int_{\Gamma_\delta}\frac{\partial G^*(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})}u(\mathbf{y})\mathrm{d}s(\mathbf{y})+\int_{S_\delta}G^*(\mathbf{x},\mathbf{y})m(\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}. \tag{2.3}$$

Taking the limit as $\mathbf{x}\to\gamma_\delta$ we see that this equation holds also for $\mathbf{x}\in\gamma_\delta$. Now the second and third integrals in (2.3) are continuously differentiable in $S_\delta\cup\gamma_\delta$, with gradient whose magnitude is $\leq CM^*$ in $\overline{S_{3\delta/4}}$, where $M^* = \sup_{\mathbf{y}\in S_\delta}|u(\mathbf{y})|$ and C depends only on N and δ. It follows from (2.3) (with $\mathbf{x}\in\gamma_\delta$) and mapping properties of the single-layer potential (e.g. [9]) that $u\in C^{0,\varepsilon}(\gamma_{p\delta})$, for every $p\in(0,1)$, with

$$\|u\|_{C^{0,\varepsilon}(\gamma_{3\delta/4})} \leq CM^*,$$

where C here depends on all of ε, N, and δ. Again using (2.3) and standard mapping properties of the single-layer potential [9], we see that $u\in C^1(S_\delta\cup\gamma_\delta)$ and that the bound (2.2) holds. □

Our first bounds on derivatives of u^t will be bounds on ∇u^t in D. We note first of all that it follows from standard interior elliptic regularity estimates [11, Theorem 3.9, Lemma 4.1] that there exists an absolute constant $C>0$ such that, for every $\epsilon>0$,

$$|\nabla u^t(\mathbf{x})| \leq C\epsilon^{-1}(1+(k\epsilon)^2)M \tag{2.4}$$

if $\mathbf{x}\in D$ and the distance of $\mathbf{x}$ from Γ, $\mathrm{dist}(\mathbf{x},\Gamma)>\epsilon$. Using Lemma 2.1 applied to u^t in domains $\{\mathbf{x}\in D: |\mathbf{x}-\mathbf{x}^*|<2c/k\}$ with $\mathbf{x}^*\in\Gamma$, $c>0$, and $\mathrm{dist}(\mathbf{x}^*,\{P_1,\cdots,P_n\})>2c/k$, we can extend this bound up to Γ, excluding the corner points P_j. Precisely, if $|\beta_j|\leq B$, for $j=1,\cdots,n$, then it follows from (2.4) and Lemma 2.1 that there exists $C>0$, depending only on B and c, such that, if $\mathbf{x}\in\overline{D}$ and $\mathrm{dist}(\mathbf{x},\{P_1,\cdots,P_n\})>c/k$, then

$$|\nabla u^t(\mathbf{x})| \leq CkM. \tag{2.5}$$

To obtain bounds which apply throughout D we make a further application of Lemma 2.1. Suppose $\omega>0$ is such that $\pi+\omega<\Omega_j<2\pi$, $j=1,\cdots,n$. Let us use that $\mathbb{R}^2$ is isomorphic to the complex plane $\mathbb{C}$; for $\mathbf{x}=(x_1,x_2)\in D$ let $\tilde{\mathbf{x}}:=x_1+\mathrm{i}x_2\in\mathbb{C}$ and, conversely, if $z\in\mathbb{C}$, let $\hat{z}=(\mathrm{Re}z,\mathrm{Im}z)\in\mathbb{R}^2$. In terms of this notation, set $P_0:=P_n$ and let $z_j:=\tilde{P}_j$, $j=0,1,\cdots,n+1$. Given a typical corner P_j, with $1\leq j\leq n$, let $c>0$ be such that

$$\mathrm{dist}(P_j,\{P_{j+1},P_{j-1}\})>2c/k.$$

Let $\theta_i:=\arg(z_{i-1}-z_i)$, for $i=1,\cdots,n+1$, so that $\theta_j+\Omega_j=\arg(z_{j+1}-z_j)$. Our application of Lemma 2.1 is to $u\in H^1(S_{2c})\cap C^2(S_{2c})\cap C(\overline{S_{2c}})$ given by

$$u(\mathbf{y})=u^t(\widehat{g(\tilde{\mathbf{y}})}), \quad \mathbf{y}\in S_{2c},$$

where the conformal mapping g is given by $g(w)=z_j+(\mathrm{e}^{\mathrm{i}\theta_j}2c/k)(w/(2c))^{\Omega_j/\pi}$, with $\arg(g(w)-z_j)\in[\theta_j,\theta_j+\Omega_j]$. This function u satisfies the conditions of Lemma 2.1 with $\delta=2c$ and $\varepsilon=\omega/\pi$, and with

$$\begin{aligned} m(\mathbf{y}) &= k^2|g'(\tilde{\mathbf{y}})|^2=(\Omega_j/\pi)^2|\mathbf{y}/(2c)|^{2(\Omega_j/\pi-1)}, && \mathbf{y}\in S_{2c},\\ \alpha(\mathbf{y}) &= k|g'(\tilde{\mathbf{y}})|\beta_j^*(y_1)=(\Omega_j/\pi)|y_1/(2c)|^{\Omega_j/\pi-1}\beta_j^*(y_1), && \mathbf{y}\in\gamma_{2c}, \end{aligned}$$

where $\beta_j^*(y_1):=\beta_{j-1}$, for $y_1>0$, $:=\beta_j$, for $y_1<0$. Thus, applying Lemma 2.1, it follows that there exists $C>0$ depending only on B, c, and ω, such that $|\nabla u(\mathbf{y})|\le CM$, for $\mathbf{y}\in S_c$. Since, within distance $2c/k$ of P_j,

$$|\nabla u^t(\mathbf{x})|=|f'(\tilde{\mathbf{x}})||\nabla u(\widehat{f(\tilde{\mathbf{x}})})|=k(k|\mathbf{x}-P_j|/(2c))^{\pi/\Omega_j-1}|\nabla u(\widehat{f(\tilde{\mathbf{x}})})|,$$

where $f(z):=g^{-1}(z)=2c[(z-z_j)k\mathrm{e}^{-\mathrm{i}\theta_j}/(2c)]^{\pi/\Omega_j}$, we see that, if $\mathbf{x}\in\overline{D}$ and $|\mathbf{x}-P_j|<c/k$, then

$$|\nabla u^t(\mathbf{x})|\le Ck\left(k|\mathbf{x}-P_j|\right)^{\pi/\Omega_j-1}M, \tag{2.6}$$

where C depends only on B, c, and ω.

This analysis, leading to (2.6), applies in particular in the special case $\beta=0$ of Neumann boundary conditions on Γ when the behaviour (2.6) agrees with that predicted by a separation of variables solution in polar coordinates local to the corner P_j (cf. [6, Theorem 2.3]). The bound (2.6) is also consistent with the well-known Malyuzhinets solution (see [18] and the references therein) for scattering by a wedge with impedance boundary conditions.

In order to deduce more detailed regularity estimates we combine ideas from [6] (for the related sound soft problem) and [7,14] (for an impedance half-plane problem). These more detailed estimates are bounds on derivatives of all orders related to the trace of the total field γ^+u^t, relevant to the analysis of boundary element methods based on a direct integral equation formulation obtained from Green's theorem (see Section 3 below).

For $j=1,\cdots,n$, let D_j denote the half-plane to one side of Γ_j given by $D_j:=\{\mathbf{x}\in\mathbb{R}^2:(\mathbf{x}-P_j)\cdot\mathbf{n}_j>0\}\subset D$. The boundary ∂D_j of this half-plane consists of the line segment Γ_j and two semi-infinite line segments Γ_j^+ and Γ_j^- to either side of Γ_j, as shown in Fig. 2. Let

$$\Phi(\mathbf{x},\mathbf{y}):=\frac{\mathrm{i}}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|),\quad \mathbf{x},\mathbf{y}\in\mathbb{R}^2,\quad \mathbf{x}\neq\mathbf{y}, \tag{2.7}$$

with $H_0^{(1)}$ the Hankel function of the first kind of order zero, so that Φ is the standard fundamental solution of the Helmholtz equation. Let $G_j(\mathbf{x},\mathbf{y})$ denote the impedance Green's function for the half plane D_j for impedance β_j, given explicitly by

$$G_j(\mathbf{x},\mathbf{y})=\Phi(\mathbf{x},\mathbf{y})+\Phi(\mathbf{x},\mathbf{y}')+P_{\beta_j}(k(\mathbf{x}-\mathbf{y}')\cdot\mathbf{s}_j,k(\mathbf{x}-\mathbf{y}')\cdot\mathbf{n}_j),\quad \text{for } \mathbf{x},\mathbf{y}\in\overline{D_j}.$$

Here $\mathbf{y}'$ denotes the image of $\mathbf{y}$ on reflection in the line ∂D_j, $\mathbf{s}_j$ is a unit vector in the direction $P_{j+1}-P_j$, parallel to Γ_j, and, where $U:=\mathbb{R}\times[0,\infty)$ is the upper half-plane, $P_{\beta^*}\in C(\overline{U})\cap C^\infty(\overline{U}\setminus\{(0,0)\})$ is defined (see e.g. [4,5]), for $\mathrm{Re}\beta^*>0$, by

$$P_{\beta^*}(\xi,\eta):=-\frac{\mathrm{i}\beta^*}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}(\eta(1-s^2)^{1/2}-\xi s))}{(1-s^2)^{1/2}((1-s^2)^{1/2}+\beta^*)}\mathrm{d}s,\quad \text{for } \xi\in\mathbb{R},\quad \eta\ge 0.$$

By applications of Green's theorems, utilizing that both u^s and $G_j(\mathbf{x},\cdot)$ satisfy the Sommerfeld radiation conditions (see (1.3) and [4, (7)], respectively), and using the impedance

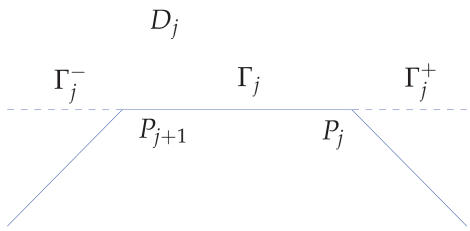

Figure 2: The half-plane D_j and its boundary.

boundary condition satisfied by G_j, that

$$\frac{\partial G_j(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(y)} + \mathrm{i}k\beta_j G_j(\mathbf{x},\mathbf{y}) = 0, \quad \text{on } \partial D_j,$$

it can be deduced (see [16, Theorem 4.3] for details) that

$$u^s(\mathbf{x}) = -\int_{\partial D_j} G_j(\mathbf{x},\mathbf{y})\left(\frac{\partial u^s}{\partial \mathbf{n}}(\mathbf{y}) + \mathrm{i}k\beta_j u^s(\mathbf{y})\right)\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in D_j. \tag{2.8}$$

We note that the integral on the right-hand side is well-defined since u^t is continuous and bounded in $\overline{D}$ and ∇u^t satisfies the bounds (2.5) and (2.6), while $G_j(\mathbf{x},\cdot)$ decreases sufficiently rapidly at infinity [7, (2.10)] so that it is absolutely integrable on ∂D_j. We note also that, since the left and right hand sides of (2.8) are both continuous in $\overline{D_j}$, (2.8) holds in fact for all $\mathbf{x}\in\overline{D_j}$.

In the case that Γ_j is not illuminated by the incident wave (by this we mean the case that $\mathbf{d}\cdot\mathbf{n}_j \ge 0$), it can be shown that (2.8) holds for $\mathbf{x}\in\overline{D_j}$ also with u^s replaced by u^i. (The point is that (see [8, Remark 2.15, Theorem 2.19(ii)]), in the case $\mathbf{d}\cdot\mathbf{n}_j\ge 0$, u^i can be approximated in $\overline{D_j}$ by a bounded sequence of solutions of the Helmholtz equation which satisfy the Sommerfeld radiation condition and which converge uniformly on compact subsets of $\overline{D_j}$ to u^i, so that (2.8) holds first for each member of this sequence and then, in the limit, also for u^i.) Adding the equations (2.8) satisfied by u^s and u^i, we see that

$$u^t(\mathbf{x}) = -\int_{\Gamma_j^-\cup\Gamma_j^+} G_j(\mathbf{x},\mathbf{y})\left(\frac{\partial u^t}{\partial \mathbf{n}}(\mathbf{y}) + \mathrm{i}k\beta_j u^t(\mathbf{y})\right)\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\overline{D_j}, \tag{2.9}$$

if Γ_j is a shadow side, since

$$u^t = u^i + u^s \quad \text{and} \quad \frac{\partial u^t}{\partial \mathbf{n}} + \mathrm{i}k\beta_j u^t = 0, \quad \text{on } \Gamma_j.$$

On illuminated sides (where $\mathbf{d}\cdot\mathbf{n}_j < 0$), (2.8) still holds for u^s and we can follow the argument of [7, p.653] to deduce that

$$u^t(\mathbf{x}) = u_j^t(\mathbf{x}) - \int_{\Gamma_j^-\cup\Gamma_j^+} G_j(\mathbf{x},\mathbf{y})\left(\frac{\partial u^t}{\partial \mathbf{n}}(\mathbf{y}) + \mathrm{i}k\beta_j u^t(\mathbf{y})\right)\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\overline{D_j}, \tag{2.10}$$

where $u_j^t(\mathbf{x}) := u^i(\mathbf{x}) + R_{\beta_j}(\theta - \theta_{j+1})u^r(\mathbf{x})$, u^r is the plane wave

$$u^r(\mathbf{x}) := \exp(\mathrm{i}k(c_j + \mathbf{x}\cdot\mathbf{d}_j')),$$

with $\mathbf{d}_j' := \mathbf{d}\cdot\mathbf{s}_j\mathbf{s}_j - \mathbf{d}\cdot\mathbf{n}_j\mathbf{n}_j$ and $c_j := P_j\cdot(\mathbf{d}-\mathbf{d}_j')$, and R_{β_j} is the plane wave reflection coefficient for a plane of impedance β_j, defined by $R_{\beta_j}(\theta^*) := (\cos\theta^* - \beta_j)/(\cos\theta^* + \beta_j)$.

We now define

$$\tilde{g}(\mathbf{x}) := \frac{\partial u^t}{\partial \mathbf{n}}(\mathbf{x}) + \mathrm{i}k\beta_j u^t(\mathbf{x}), \quad \mathbf{x}\in\partial D_j,$$

noting that $\tilde{g}(\mathbf{x}) = 0$ for $\mathbf{x}\in\Gamma_j$. To write (2.10) and (2.9) more explicitly we introduce new dimensionless functions whose arguments can be interpreted as arc-length along ∂D_j. These functions are defined by

$$\phi_j(s) := u^t(P_j + s\mathbf{s}_j) \quad \text{and} \quad g_j(s) := \tilde{g}(P_j + s\mathbf{s}_j), \quad \text{for } s\in\mathbb{R},$$

and we also define $\psi_j(s) := u_j^t(P_j + s\mathbf{s}_j)$, if Γ_j is a side illuminated by the incident field ($\mathbf{d}\cdot\mathbf{n}_j < 0$), and set $\psi_j(s) := 0$ otherwise. Also, let

$$\kappa_j(t) := \frac{\mathrm{i}}{2}H_0^{(1)}(|t|) + P_{\beta_j}(|t|,0), \quad \text{for } t\in\mathbb{R}\setminus\{0\},$$

noting that $G_j(\mathbf{x},\mathbf{y}) = \kappa_j(k|\mathbf{x}-\mathbf{y}|)$ if $\mathbf{x},\mathbf{y}\in\partial D_j$ with $\mathbf{x}\neq\mathbf{y}$. Then equations (2.10) and (2.9), restricted to Γ_j, can be written as the statement that, for $0 < s < L_j$,

$$\begin{aligned}\phi_j(s) &= \psi_j(s) - \int_{-\infty}^0 \kappa_j(k(s-t))g_j(t)\,\mathrm{d}t - \int_{L_j}^\infty \kappa_j(k(s-t))g_j(t)\,\mathrm{d}t \\ &= \psi_j(s) + \mathrm{e}^{\mathrm{i}ks}v_j^+(s) + \mathrm{e}^{-\mathrm{i}ks}v_j^-(L_j - s),\end{aligned} \tag{2.11}$$

where, defining $\check{\kappa}_j(s) := \mathrm{e}^{-\mathrm{i}|s|}\kappa_j(s)$, $s\in\mathbb{R}\setminus\{0\}$, the functions v_j^+ and v_j^- are defined by

$$v_j^+(s) := -\int_{-\infty}^0 \check{\kappa}_j(k(s-t))\mathrm{e}^{-\mathrm{i}kt}g_j(t)\,\mathrm{d}t, \qquad s\geq 0,$$

and

$$\begin{aligned}v_j^-(s) &:= -\int_{L_j}^\infty \check{\kappa}_j(k(t-L_j+s))\mathrm{e}^{\mathrm{i}kt}g_j(t)\,\mathrm{d}t \\ &= -\mathrm{e}^{\mathrm{i}kL_j}\int_{-\infty}^0 \check{\kappa}_j(k(s-t))\mathrm{e}^{-\mathrm{i}kt}g_j(L_j-t)\,\mathrm{d}t, \qquad s\geq 0.\end{aligned}$$

In the next theorem we establish that the functions $v_j^\pm$ are at most slowly oscillatory, in that their derivatives are rapidly decaying at infinity.

Theorem 2.1. *Suppose that, for some $B>0$, $c>0$, and $\omega\in(0,\pi)$, it holds that*

$$\mathrm{Re}\beta_j\geq B^{-1},\quad |\beta_j|\leq B,\quad \pi+\omega\leq\Omega_j<2\pi,\quad kL_j\geq 2c,\quad \textit{for } j=1,\cdots,n, \tag{2.12}$$

and set $\alpha_j^+:=\pi/\Omega_j$, for $j=1,\cdots,n$, and $\alpha_j^-:=\alpha_{j+1}^+$, $j=1,\cdots,n-1$, $\alpha_n^-:=\alpha_1^+$. Then there exist constants C_m, $m=0,1,\cdots$, dependent only on m, B, ω, and c, such that, for $j=1,\cdots,n$ and $m\in\mathbb{N}$,

$$|v_j^\pm(s)|\leq\begin{cases} C_0M, & 0<ks\leq 1,\\ C_0M(ks)^{-1/2}, & ks\geq 1,\end{cases} \tag{2.13a}$$

$$|v_j^{\pm^{(m)}}(s)|\leq\begin{cases} C_mk^mM(ks)^{\alpha_j^\pm-m}, & 0<ks\leq 1,\\ C_mk^mM(ks)^{-1/2-m}, & ks\geq 1.\end{cases} \tag{2.13b}$$

Proof. We shall prove only the bounds on $v_j^{+^{(m)}}$; the bounds on $v_j^{-^{(m)}}$ follow analogously. Throughout the proof let C_m denote a positive constant, depending only on m, B, ω, and c, not necessarily the same at each occurrence.

For $s\geq 0$ and $m=0,1,\cdots$,

$$v_j^{+^{(m)}}(s)=-k^m\int_{-\infty}^0\check{\kappa}_j^{(m)}(k(s-t))\mathrm{e}^{-\mathrm{i}kt}g_j(t)\,\mathrm{d}t.$$

From (2.5) and (2.6), respectively, it follows that $|g_j(t)|\leq CkM$ if $t\leq -c/k$, while

$$|g_j(t)|\leq Ck(k|t|)^{\alpha_j^+-1}M,\quad \text{if } -c/k<t<0,$$

where the constant $C>0$ depends only on B, c, and ω. Thus it is easily seen that

$$|v_j^{+^{(m)}}(s)|\leq Ck^mM[A_1(ks)+A_2(ks)],\qquad s\geq 0,$$

where, for $\sigma\geq 0$,

$$A_1(\sigma):=\int_{-\infty}^{-c}\left|\check{\kappa}_j^{(m)}(\sigma-\tau)\right|\mathrm{d}\tau,\qquad A_2(\sigma):=\int_{-c}^{0}\left|\check{\kappa}_j^{(m)}(\sigma-\tau)\right||\tau|^{\alpha_j^+-1}\mathrm{d}\tau.$$

It is shown in [14, lemma 2.2] that

$$|\check{\kappa}_j(t)|\leq\begin{cases} C_0(1+|\log t|), & \text{for } 0<t\leq 1,\\ C_0t^{-3/2}, & \text{for } t>1.\end{cases}$$

Using these bounds it is easily seen that, in the case $m=0$, $A_1(\sigma)+A_2(\sigma)\leq C_0$, for $0\leq\sigma\leq 1$, $\leq C_0\sigma^{-1/2}$ for $\sigma>1$. From this bound the bound (2.13) on $|v_j^+|$ follows.

To bound A_1 and A_2 when $m\in\mathbb{N}$ we use the bounds from [14, lemma 2.2] that

$$|\check{\kappa}_j^{(m)}(t)|\leq\begin{cases} C_mt^{-m}, & \text{for } 0<t\leq 1,\\ C_mt^{-3/2-m}, & \text{for } t>1.\end{cases}$$

Using these bounds it is easily seen that $A_1(\sigma) \le C_m(c+\sigma)^{-1/2-m}$, for $\sigma \ge 0$, and that $A_2(\sigma) \le C_m \sigma^{-1/2-m}$ for $\sigma \ge c$, while, for $0<\sigma<c$,

$$\begin{aligned} A_2(\sigma) &\le C_m \int_{-c}^{0} (\sigma-\tau)^{-m} |\tau|^{\alpha_j^+ -1} \mathrm{d}\tau \\ &= C_m \left(\int_{\sigma}^{c} (\sigma+\tau)^{-m} \tau^{\alpha_j^+ -1} \mathrm{d}\tau + \int_{0}^{\sigma} (\sigma+\tau)^{-m} \tau^{\alpha_j^+ -1} \mathrm{d}\tau \right) \\ &\le C_m \left(\int_{\sigma}^{c} \tau^{\alpha_j^+ -1-m} \mathrm{d}\tau + \sigma^{-m} \int_{0}^{\sigma} \tau^{\alpha_j^+ -1} \mathrm{d}\tau \right) \le C_m \sigma^{\alpha_j^+ -m}. \end{aligned}$$

From these bounds the bound (2.13) on $|v_j^{+(m)}|$ for $m \in \mathbb{N}$ follows. □

3 Boundary integral equation formulation

Where Φ is defined by (2.7), applying Green's representation theorem [15] to u^s gives

$$u^s(\mathbf{x}) = \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \gamma^+ u^s(\mathbf{y}) - \Phi(\mathbf{x},\mathbf{y}) \partial_{\mathbf{n}}^+ u^s(\mathbf{y}) \right) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D. \tag{3.1}$$

Applying Green's second theorem [15] to $\Phi(\mathbf{x},\cdot)$ and u^i in Ω we see that

$$0 = \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} u^i(\mathbf{y}) - \Phi(\mathbf{x},\mathbf{y}) \frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{y}) \right) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D. \tag{3.2}$$

Then adding (3.1) and (3.2) and using the boundary condition (1.2), we find that

$$u^t(\mathbf{x}) = u^i(\mathbf{x}) + \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} + \mathrm{i}k\beta(\mathbf{y})\Phi(\mathbf{x},\mathbf{y}) \right) \gamma^+ u^t(\mathbf{y}) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D. \tag{3.3}$$

Applying the trace operator γ^+ and using the jump relations [15, Theorem 6.11], we obtain a standard boundary integral equation (cf. [9, Section 3.9]) for $\gamma^+ u^t$, that

$$\begin{aligned} \frac{1}{2}\gamma^+ u^t(\mathbf{x}) = u^i(\mathbf{x}) + \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} + \mathrm{i}k\beta(\mathbf{y})\Phi(\mathbf{x},\mathbf{y}) \right) \gamma^+ u^t(\mathbf{y}) \mathrm{d}s(\mathbf{y}),& \\ \mathbf{x} \in \Gamma \setminus \{P_1,\cdots,P_n\}.& \end{aligned} \tag{3.4}$$

It is well known [9,21] that, while (3.4) is uniquely solvable for all but a countable set of positive wavenumbers k, with the associated linear operator bounded and invertible on $H^s(\Gamma)$, for $0 \le s \le 1$, in particular on $L^2(\Gamma)$, (3.4) is not uniquely solvable for all wavenumbers. Precisely, if k is such that the Dirichlet problem for the Helmholtz equation in the interior region Ω has a non-trivial solution u_D (k is a so-called *irregular frequency*), then (3.4) has infinitely many solutions.To avoid this problem, the standard solution is to use a combined-layer formulation [2,9], taking a linear combination of (3.4) with the equation that we get by applying the normal derivative operator $\partial_{\mathbf{n}}^+$ to (3.3), which gives a more elaborate integral equation formulation, involving hypersingular integral operators. This is the approach used in the boundary element method of [22], for example. Another option is to use the so-called CHIEF scheme, the basic idea of which is to overdetermine the solution of (3.4) with equations arising from enforcing a version of (3.3) at interior points $\mathbf{x}\in\Omega$ called the CHIEF points. This scheme, proposed in [21], is used, for example, in [19].

In the next section we will describe a Galerkin boundary element method for solving (3.4) numerically, and will see in Section 5 that this method seems to work well even when k is an irregular frequency. That we do not see problems at irregular frequencies seems to be associated with our novel approximation space, specifically designed to accurately approximate γ^+u^t. Rather than requiring a fixed number of degrees of freedom per wavelength, which is the case for conventional solvers, our scheme allows us to resolve many wavelengths with each degree of freedom. As a result, our approximation space will not approximate arbitrary oscillatory functions accurately. In particular, we expect that the solutions to the homogeneous version of (3.4), which are oscillatory but with a different phase to γ^+u^t, will be poorly approximated with our scheme, and so anticipate that, provided k is large enough compared to the total number of degrees of freedom, solving (3.4) with the boundary element scheme we propose should approximate γ^+u^t well, avoiding problems of nonuniqueness. For numerical results supporting this claim see Section 5, and for an elaboration of this argument for the related sound-soft problem, supported by extensive numerical results, see [13].

It is important to note, in any case, that, while we show numerical results for (3.4) in Section 5, the novel boundary element approximation space we design is just as relevant to any other direct integral equation formulation, in particular to the combined-layer integral equation formulation, in which γ^+u^t is also the unknown.

4 Galerkin boundary element method

Our aim now is to use the regularity results in Theorem 2.1 to design an optimal approximation space for γ^+u^t, which can be used for the numerical solution of (3.4). We represent $\mathbf{x}\in\Gamma$ parametrically by

$$\mathbf{x}(s)=P_j+\left(s-\tilde{L}_{j-1}\right)\left(\frac{P_{j+1}-P_j}{L_j}\right), \qquad \text{for } s\in\left(\tilde{L}_{j-1},\tilde{L}_j\right), \quad j=1,\cdots,n,$$

where $\tilde{L}_j:=\sum_{m=1}^{j}L_m$, and then write (3.4) in parametric form as

$$\phi(s)-2\int_0^L K(s,t)\phi(t)\mathrm{d}t=2f(s), \tag{4.1}$$

where $\phi(s)=u^t(\mathbf{x}(s))$, $L=\sum_{j=1}^{n}L_j$,

$$K(s,t):=\left(\frac{\partial\Phi(\mathbf{x}(s),\mathbf{x}(t))}{\partial\mathbf{n}(\mathbf{x}(t))}+\mathrm{i}k\beta(\mathbf{x}(t))\Phi(\mathbf{x}(s),\mathbf{x}(t))\right),$$

and $f(s)=u^i(\mathbf{x}(s))$. The first step in our numerical method is to separate off the explicitly known high frequency leading order behaviour which we denote by $\Psi(s)$. From (2.11) and Theorem 2.1 it is clear that this leading order behaviour is

$$\Psi(s):=\begin{cases} u_j^t(\mathbf{x}(s)) & \text{on illuminated sides,} \\ 0 & \text{on shadow sides.} \end{cases}$$

Introducing the new unknown $\varphi=\phi-\Psi$, and substituting into (4.1), we have

$$\varphi(s)-2\int_0^L K(s,t)\varphi(t)\,\mathrm{d}t=2f(s)-\Psi(s)+2\int_0^L K(s,t)\Psi(t)\,\mathrm{d}t, \tag{4.2}$$

which we can write in operator form as

$$(I-\mathcal{K})\varphi=F, \tag{4.3}$$

where

$$\mathcal{K}v(s):=2\int_0^L K(s,t)v(t)\,\mathrm{d}t, \quad F(s):=2f(s)-\Psi(s)+2\int_0^L K(s,t)\Psi(t)\,\mathrm{d}t,$$

and I is the identity operator. Thinking of (4.3) as an operator equation on $L^2(0,L)$, this is the equation that we are going to solve for the unknown φ by a Galerkin boundary element method.

We now design our Galerkin approximation space $V_{N,\nu}\subset L^2(0,L)$ in such a way as to efficiently represent φ, based on the representation (2.11) and the bounds in Theorem 2.1. Note that the notations in (2.11) and (4.2) are related by

$$\varphi(\tilde{L}_{j-1}+s)=\phi_j(s)-\psi_j(s), \quad \text{for } 0\le s\le L_j, \quad j=1,\cdots,n. \tag{4.4}$$

Our estimates in Theorem 2.1 are similar to those for the same scattering problem but with sound-soft boundary conditions [6, Theorem 3.3, Corollary 3.4], but with different exponents for $0<ks\le 1$. Hence our approximation space is similar to (although not the same as) that defined in [6]. To describe this approximation space we begin by defining a composite graded mesh on a finite interval $[0,A]$, which comprises a polynomial grading near 0 and a geometric grading on the rest of the interval $[0,A]$. This mesh will be a component in the boundary element mesh that we will use on each side of the polygon.

Definition 4.1. For $A>\lambda>0$, $q\ge 1$, $N=2,3,\cdots$, we define

$$N_1:=\lceil Nq\rceil \quad \text{and} \quad N_2:=\left\lceil \frac{-\log(A/\lambda)}{q\log(1-1/N_1)}\right\rceil,$$

where, for $s\in\mathbb{R}$, $\lceil s\rceil$ denotes the smallest integer greater than or equal to s. The mesh $\Lambda_{N,A,\lambda,q}:=(y_0,\cdots,y_{N_1+N_2})$ then consists of the points

$$y_i:=\lambda\left(\frac{i}{N_1}\right)^q, \quad i=0,\cdots,N_1, \quad \text{and} \quad y_{N_1+j}:=\lambda\left(\frac{A}{\lambda}\right)^{j/N_2}, \quad j=1,\cdots,N_2. \tag{4.5}$$

For large N, N_1 and N_2 are both proportional to N, and the definitions of N_1 and N_2 ensure a smooth transition between the two parts of the mesh; in particular $y_{N_1}-y_{N_1-1}$ and $y_{N_1+1}-y_{N_1}$ are both asymptotically equal to λ/N as $N\to\infty$.

We now show that piecewise polynomials supported on the graded mesh defined in (4.5) are well-suited to the approximation of $v_j^{\pm}$. For $a<b$ let $\|\cdot\|_{2,(a,b)}$ denote the norm on $L^2(a,b)$,

$$\|g\|_{2,(a,b)}:=\left\{\int_a^b |g(s)|^2\mathrm{d}s\right\}^{1/2}.$$

For $A>\lambda>0$, $\nu\in\mathbb{N}\cup\{0\}$, $q\geq 1$, where y_i, $i=0,1,\cdots,N_1+N_2$, are the points of the mesh in Definition 4.1, let $\Pi_{N,\nu}\subset L^2(0,A)$ denote the set of piecewise polynomials

$$\Pi_{N,\nu}:=\Big\{\sigma\in L^2(0,A):\sigma|_{(y_{j-1},y_j)}\text{ is a polynomial of degree }\leq\nu\text{ for }j=1,\cdots,N_1+N_2\Big\},$$

and let P_N be the orthogonal projection operator from $L^2(0,A)$ to $\Pi_{N,\nu}$, so that setting $p=P_Ng$ minimizes $\|g-p\|_{2,(0,A)}$ over all $p\in\Pi_{N,\nu}$. Our error estimates for our boundary element method approximation space are based on the following theorem (cf. [6, Theorem 4.2]). We omit the proof which is a minor variant of the proof of [6, Theorem 4.2], referring the reader to [16] for details. Note that the relevance of this result is that, by Theorem 2.1, $v_j^{\pm}$ satisfies the conditions of this theorem with $\alpha=\alpha_j^{\pm}$.

Theorem 4.1. *Suppose that $g\in C^\infty(0,\infty)$, $k>0$, $A>\lambda:=2\pi/k$ and $\alpha\in[1/2,1]$, and that there exist constants $c_m>0$, $m=0,1,2,\cdots$, such that, for $m\in\mathbb{N}$,*

$$|g(s)|\leq\begin{cases} c_0, & 0<ks\leq 1,\\ c_0(ks)^{-1/2}, & ks\geq 1,\end{cases}\qquad |g^{(m)}(s)|\leq\begin{cases} c_mk^m(ks)^{\alpha-m}, & 0<ks\leq 1,\\ c_mk^m(ks)^{-1/2-m}, & ks\geq 1.\end{cases}$$

Then, where $q:=(2\nu+3)/(1+2\alpha)$, there exists a constant C_ν^*, dependent only on ν, such that for $N=2,3,\cdots$,

$$\|g-P_Ng\|_{2,(0,A)}\leq\frac{C_\nu^*\overline{c}_\nu(1+\log(A/\lambda))^{1/2}}{k^{1/2}N^{\nu+1}},$$

where $\overline{c}_\nu:=\max(c_1,c_{\nu+1})$.

From this point on we assume that $L_j\geq\lambda$, for $j=1,\cdots,n$, where $\lambda:=2\pi/k$ is the wavelength. Setting $\alpha_j=\pi/\Omega_j$, $j=1,\cdots,n$, and $\alpha_{n+1}:=\alpha_1$, we define $q_j:=(2\nu+3)/(1+2\alpha_j)\in(2\nu/3+1,\nu+3/2)$, for $j=1,\cdots,n+1$, and the two meshes

$$\Gamma_j^+:=\tilde{L}_{j-1}+\Lambda_{N,L_j,\lambda,q_j},\quad \Gamma_j^-:=\tilde{L}_j-\Lambda_{N,L_j,\lambda,q_{j+1}},\quad\text{for } j=1,\cdots,n.$$

Letting $\mathrm{e}_\pm(s):=\mathrm{e}^{\pm\mathrm{i}ks}$, $s\in[0,L]$, we then define the approximation space associated with each mesh as

$$V_{\Gamma_j^+,\nu}:=\{\sigma\mathrm{e}_+:\sigma\in\Pi_{\Gamma_j^+,\nu}\},\qquad V_{\Gamma_j^-,\nu}:=\{\sigma\mathrm{e}_-:\sigma\in\Pi_{\Gamma_j^-,\nu}\},$$

for $j=1,\cdots,n$, where

$$\begin{aligned}\Pi_{\Gamma_j^+,\nu}:=\Big\{&\sigma\in L^2(0,L):\sigma|_{(\tilde{L}_{j-1}+y_{m-1},\tilde{L}_{j-1}+y_m)} \text{ is a polynomial of degree } \le\nu,\\ &\text{for } m=1,\cdots,N_1^++N_2^+, \text{ and } \sigma|_{(0,\tilde{L}_{j-1})\cup(\tilde{L}_j,L)}=0\Big\},\\ \Pi_{\Gamma_j^-,\nu}:=\Big\{&\sigma\in L^2(0,L):\sigma|_{(\tilde{L}_j-\tilde{y}_m,\tilde{L}_j-\tilde{y}_{m-1})} \text{ is a polynomial of degree } \le\nu,\\ &\text{for } m=1,\cdots,N_1^-+N_2^-, \text{ and } \sigma|_{(0,\tilde{L}_{j-1})\cup(\tilde{L}_j,L)}=0\Big\},\end{aligned}$$

with $0=y_0<y_1<\cdots<y_{N_1^++N_2^+}=L_j$ the points of the mesh $\Lambda_{N,L_j,\lambda,q_j}$ and $0=\tilde{y}_0<\tilde{y}_1<\cdots<\tilde{y}_{N_1^-+N_2^-}=L_j$ the points of the mesh $\Lambda_{N,L_j,\lambda,q_{j+1}}$, and (N_1^+,N_2^+) and (N_1^-,N_2^-) the values of (N_1,N_2) for the meshes $\Lambda_{N,L_j,\lambda,q_j}$ and $\Lambda_{N,L_j,\lambda,q_{j+1}}$, respectively. Our approximation space $V_{N,\nu}$ is then the linear span of $\bigcup_{j=1,\cdots,n}\{V_{\Gamma_j^+,\nu}\cup V_{\Gamma_j^-,\nu}\}$. The total number of the degrees of freedom is $\mathsf{M}_N=(\nu+1)\sum_{j=1}^n N_j^*$, where N_j^* is the sum of the values of N_1+N_2 (the number of subintervals) for the meshes $\Lambda_{N,L_j,\lambda,q_j}$ and $\Lambda_{N,L_j,\lambda,q_{j+1}}$. Since $-1/\log(1-1/N_1)<N_1$, for $N_1\in\mathbb{N}$, and $1<q_j<\nu+3/2$, we see that

$$\begin{aligned}N_j^*&<\left(Nq_j+1+(Nq_j+1)\log\left(\frac{L_j}{\lambda}\right)/q_j+1\right)+\left(Nq_{j+1}+1+(Nq_{j+1}+1)\log\left(\frac{L_j}{\lambda}\right)/q_{j+1}+1\right)\\ &<(2\nu+3)N+4+2(N+1)\log\left(\frac{L_j}{\lambda}\right),\end{aligned}$$

so that

$$\mathsf{M}_N<(\nu+1)n\left((2\nu+3)N+4+2(N+1)\log\left(\frac{\bar{L}}{\lambda}\right)\right)<(\nu+1)nN\left(2\nu+5+3\log\left(\frac{k\bar{L}}{2\pi}\right)\right),\tag{4.6}$$

where $\bar{L}:=(L_1\cdots L_n)^{1/n}$.

It follows from equations (2.11) and (4.4), Theorem 2.1 (applied with $c=\pi$), and Theorem 4.1 that φ can be approximated very well by an element of the approximation space $V_{N,\nu}$. Precisely, these equations and theorems imply that, if the conditions of Theorem 2.1 are satisfied, then on each interval $(\tilde{L}_{j-1},\tilde{L}_j)$ (corresponding to side Γ_j), there exist elements σ_j^+ and σ_j^- of $\Pi_{\Gamma_j^+,\nu}$ and $\Pi_{\Gamma_j^-,\nu}$, respectively, such that, for some constant $C_\nu>0$, depending only on ν, B, and ω,

$$\|(\sigma_j^+\mathrm{e}_++\sigma_j^-\mathrm{e}_-)-\varphi\|_{2,(\tilde{L}_{j-1},\tilde{L}_j)}\le\frac{C_\nu M(1+\log(L_j/\lambda))^{1/2}}{k^{1/2}N^{\nu+1}}.$$

It follows that, where $\phi_N:=\sum_{j=1}^n(\sigma_j^+\mathrm{e}_++\sigma_j^-\mathrm{e}_-)\in V_{N,\nu}$,

$$\|\phi_N-\varphi\|_{2,(0,L)}^2=\sum_{j=1}^n\|(\sigma_j^+\mathrm{e}_++\sigma_j^-\mathrm{e}_-)-\varphi\|_{2,(\tilde{L}_{j-1},\tilde{L}_j)}^2\le\frac{C_\nu^2M^2n(1+\log(\bar{L}/\lambda))}{kN^{2(\nu+1)}}.$$

Combining this bound with (4.6) we see that we have shown the following theorem.

Theorem 4.2. *Suppose that, for $j=1,\cdots,n$, $kL_j\geq 2\pi$ and that β_j and Ω_j satisfy* (2.12). *Then, where ϕ_N denotes the best approximation to φ from the approximation space $V_{N,\nu}\subset L^2(0,L)$, it holds that*

$$\begin{aligned}\|\phi_N-\varphi\|_{2,(0,L)} &\leq C_\nu M\frac{n^{1/2}\left(1+\log\left(\frac{k\bar{L}}{2\pi}\right)\right)^{1/2}}{k^{1/2}N^{\nu+1}}\\ &\leq C_\nu' M\frac{n^{\nu+3/2}\left(1+\log\left(\frac{k\bar{L}}{2\pi}\right)\right)^{\nu+3/2}}{k^{1/2}\mathsf{M}_N^{\nu+1}},\end{aligned}$$

where the constants C_ν and C_ν' depend only on ν, B, and ω.

We note that this bound makes clear that, assuming M defined by (2.1) is bounded as $k\to\infty$, to achieve any desired L^2 error in the best approximation from $V_{N,\nu}$, it is enough to keep the degrees of freedom M_N fixed as $k\to\infty$. Further, keeping N fixed (which means increasing M_N logarithmically as k increases) will come close to keeping $k^{1/2}\|\phi_N-\varphi\|_{2,(0,L)}$ bounded as k increases.

Of course we have no way of computing the best approximation ϕ_N; to select what we hope is something close to the best approximation from $V_{N,\nu}$ we use the Galerkin method. Let $(\cdot,\cdot)$ denote the usual inner product on $L^2(0,L)$, defined by $(\chi_1,\chi_2):=\int_0^L\chi_1(s)\overline{\chi}_2(s)\,\mathrm{d}s$. Then our Galerkin method approximation $\varphi_N\in V_{\Gamma,\nu}$ is defined by

$$(\varphi_N,\rho)-(\mathcal{K}\varphi_N,\rho)=(F,\rho),\qquad \text{for all } \rho\in V_{\Gamma,\nu}. \tag{4.7}$$

For brevity we give further detail only for the case $\nu=0$. Writing φ_N as a linear combination of the basis functions of $V_{\Gamma,0}$, we have

$$\varphi(s)\approx\varphi_N(s)=\sum_{j=1}^{\mathsf{M}_N}v_j\rho_j(s),\qquad 0\leq s\leq L, \tag{4.8}$$

where ρ_j is the jth basis function, M_N is the dimension of $V_{\Gamma,0}$, and the constants v_j, $j=1,\cdots,\mathsf{M}_N$, satisfy a linear system equivalent to (4.7). To make explicit this linear system, for $p=1,\cdots,n$, let $n_p^\pm$ be the number of points in the mesh $\Gamma_p^\pm$. Denote the points of the mesh $\Gamma_p^\pm$ by $s_{p,l}^\pm$, for $l=1,\cdots,n_p^\pm$, $p=1,\cdots,n$, with $s_{p,l}^\pm<\cdots<s_{p,n_p^\pm}^\pm$. Setting

$$n_1:=0,\qquad n_p:=\sum_{j=1}^{p-1}(n_j^++n_j^-),\ \text{ for } p=2,\cdots,n-1,$$

define, for $p=1,\cdots,n$,

$$\begin{aligned}\rho_{n_p+j}(s)&:=\mathrm{e}^{\mathrm{i}ks}\chi_{(s_{p,j-1}^+,s_{p,j}^+)}(s)\Big/\sqrt{s_{p,j}^+-s_{p,j-1}^+}, && j=1,\cdots,n_p^+,\\ \rho_{n_p+n_p^++j}(s)&:=\mathrm{e}^{-\mathrm{i}ks}\chi_{(s_{p,j-1}^-,s_{p,j}^-)}(s)\Big/\sqrt{s_{p,j}^--s_{p,j-1}^-}, && j=1,\cdots,n_p^-,\end{aligned}$$

where $\chi_{(a,b)}$ denotes the characteristic function of the interval (a,b). Substituting (4.8) into (4.7) gives the linear system

$$\sum_{j=1}^{M_N} \left[(\rho_j,\rho_m) - (\mathcal{K}\rho_j,\rho_m) \right] v_j = (F,\rho_m), \quad m = 1,\cdots,\mathsf{M}_N. \tag{4.9}$$

For implementation details, namely discussion of how to compute the integrals in (4.9), we refer to [16].

5 Numerical results

For our numerical examples we take the scattering object, Ω, to be a square, with vertices $(0,0)$, $(2\pi,0)$, $(2\pi,2\pi)$ and $(0,2\pi)$, so that $L=8\pi$. We take $\beta_j=1$ on each side Γ_j, $j=1,\cdots,n$, and in our first example we take the incident angle to be $\theta=\pi/4$. We solve (4.9) to find the coefficients v_j, $j=1,\cdots,\mathsf{M}_N$, and then use (4.8) to compute φ_N. In Table 1 we show both absolute and relative estimated L^2 errors for $k=5$ and 320 (all L^2 norms approximated by discrete L^2 norms, sampling at 1000000 evenly spaced points on the boundary), tabulated as a function of N. (Recall that N is the local number of degrees of freedom per wavelength in each of the graded meshes we use at distance λ from the corner where the mesh is refined.) We also show the number of degrees of freedom, M_N, and $\mathsf{M}_N/(L/\lambda)$, the average number of degrees of freedom per wavelength (which needs to be in the range 5-10 for standard low order boundary element methods to achieve engineering accuracy). Note that, for the same values of N, greater accuracy is achieved for $k=320$ than for $k=5$, even though the average numbers of degrees of freedom per wavelength are very small for $k=320$. We also tabulate the estimated order of convergence (EOC), given by

$$\text{EOC} := \log_2 \left(\|\varphi - \varphi_N\|_2 / \|\varphi - \varphi_{2N}\|_2 \right).$$

Table 1: Errors in the Galerkin method approximation, scattering by square with non-grazing incidence, $k=5$ and $k=320$.

k	N	M_N	$\mathsf{M}_N\lambda/L$	$\|\varphi_{128}-\varphi_N\|_2$	$\|\varphi_{128}-\varphi_N\|_2/\|\varphi_{128}\|_2$	EOC	$\mathrm{cond}_2 A$
5	4	56	2.8	1.3053×10^{-1}	3.0383×10^{-1}	0.9	1.82×10^{1}
	8	104	5.2	7.0707×10^{-2}	1.6458×10^{-1}	0.8	8.50×10^{1}
	16	200	10.0	3.9573×10^{-2}	9.2110×10^{-2}	0.9	4.66×10^{2}
	32	376	18.8	2.1229×10^{-2}	4.9412×10^{-2}	1.0	2.13×10^{3}
	64	752	37.6	1.0731×10^{-2}	2.4978×10^{-2}	-	1.45×10^{4}
320	4	120	0.094	1.6285×10^{-2}	3.0284×10^{-1}	0.9	2.97×10^{1}
	8	248	0.194	8.7732×10^{-3}	1.6315×10^{-1}	0.9	4.04×10^{1}
	16	472	0.369	4.8664×10^{-3}	9.0494×10^{-2}	1.0	4.62×10^{1}
	32	904	0.706	2.4775×10^{-3}	4.6072×10^{-2}	1.1	5.14×10^{1}
	64	1832	1.431	1.1782×10^{-3}	2.1910×10^{-2}	-	1.34×10^{2}

Table 2: Errors in the Galerkin method approximation, scattering by square with non-grazing incidence, $N=32$.

k	M_N	$\mathsf{M}_N\lambda/L$	$k^{1/2}\|\varphi_{128}-\varphi_{32}\|_2$	$\|\varphi_{128}-\varphi_{32}\|_2/\|\varphi_{128}\|_2$	$\mathrm{cond}_2 A$
5	376	18.80	4.7469×10^{-2}	4.9412×10^{-2}	2.13×10^{3}
10	464	11.60	4.7033×10^{-2}	4.9022×10^{-2}	6.16×10^{2}
20	552	6.90	4.7047×10^{-2}	4.9006×10^{-2}	5.06×10^{2}
40	640	4.00	4.6715×10^{-2}	4.8627×10^{-2}	2.19×10^{2}
80	728	2.28	4.7897×10^{-2}	4.9871×10^{-2}	7.35×10^{1}
160	816	1.28	4.6208×10^{-2}	4.8177×10^{-2}	3.81×10^{1}
320	904	0.71	4.4319×10^{-2}	4.6072×10^{-2}	5.14×10^{1}

We expect, given that $\nu=0$, EOC ≈ 1 from Theorem 4.2. This is clearly seen in the table. Note also that the 2-norm condition number, $\mathrm{cond}_2 A$, of the matrix $A=[(\rho_j,\rho_m)-(\mathcal{K}\rho_j,\rho_m)]$ only increases very moderately as N increases for the larger value of k.

Table 2 shows the errors for fixed $N=32$ and increasing k. The 2-norm condition number is small and decreasing as k increases. As predicted by Theorem 4.2, $k^{1/2}\|\varphi_{128}-\varphi_N\|_2$ remains approximately fixed as k increases. The relative L^2 error $\|\varphi_{128}-\varphi_N\|_2/\|\varphi_{128}\|_2$ also remains roughly constant as k increases. At the same time, the number of degrees of freedom M_N, bounded above by (4.6), grows only in proportion to $\log k$ as k increases, and the average degrees of freedom per wavelength reduces rapidly as k increases. These observations indicate the effectiveness and robustness of our new boundary element scheme, particularly at high frequencies.

As a second numerical example, we repeat the above experiment with incident angle $\theta=\pi/2$, and everything else unchanged, so that the direction of incidence is now parallel to two sides of the (square) polygon, i.e. there is grazing incidence on two of the polygon sides. Our theoretical results are robust with respect to variations in the direction of incidence, and this can be clearly seen in the results of Tables 3 and 4 which mimic closely the results of Tables 1 and 2 respectively. This is the case even though the solution behaviour on the sides with grazing incidence is qualitatively different. In Fig. 3, we plot $|\varphi(s)|$ for $s\in[0,L]$, for each of the two cases $\theta=\pi/4$ (non-grazing incidence) and $\theta=\pi/2$ (grazing incidence), both for $k=40$. In each case, the decay of $|\varphi(s)|$ away from the corners (at $s=0$, 2π, 4π, 6π and 8π, corresponding to the corners $(0,0)$, $(2\pi,0)$, $(2\pi,2\pi)$, $(0,2\pi)$ and $(0,0)$ respectively) can clearly be seen, but, on the first side Γ_1 $(0\le s\le 2\pi)$, between $(0,0)$ and $(2\pi,0)$, $\varphi(s)$ decreases more slowly as s increases for $\theta=\pi/2$. This is to be expected from the analysis of [17] of the far field for scattering by an impedance wedge. In the case when the wedge has surface admittance β and occupies the quarter plane $x_1\ge 0$, $x_2\le 0$, the results of [17] predict that $\varphi(s)=\mathcal{O}(s^{-3/2})$ as $s\to\infty$ for non-grazing incidence, while, precisely, they predict that

$$\varphi(s)\sim\frac{\sqrt{2}}{\beta\sqrt{\pi ks}}\mathrm{e}^{\mathrm{i}(ks+\pi/4)},\quad \text{as } s\to\infty, \tag{5.1}$$

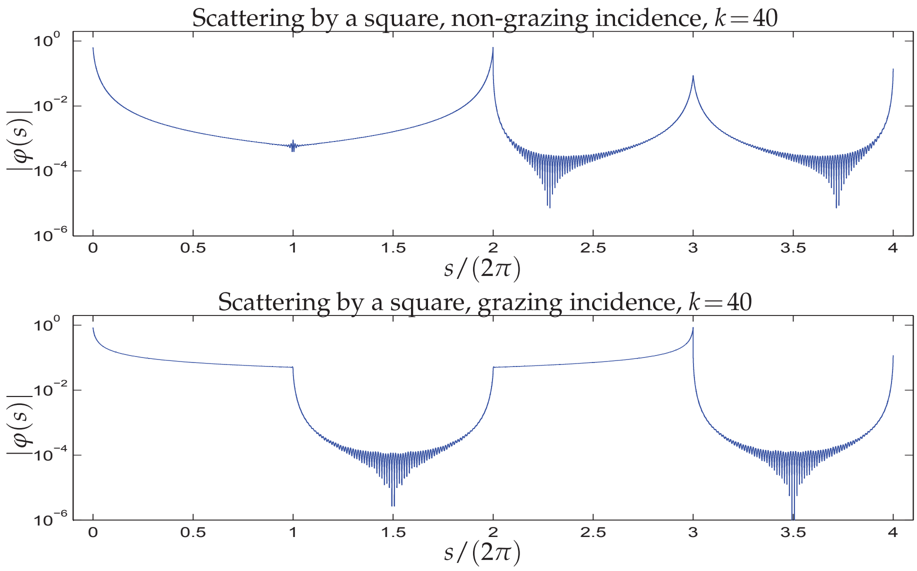

Figure 3: A comparison of $\varphi(s)$ for grazing ($\theta=\pi/2$) and non-grazing ($\theta=\pi/4$) incidence, $k=40$.

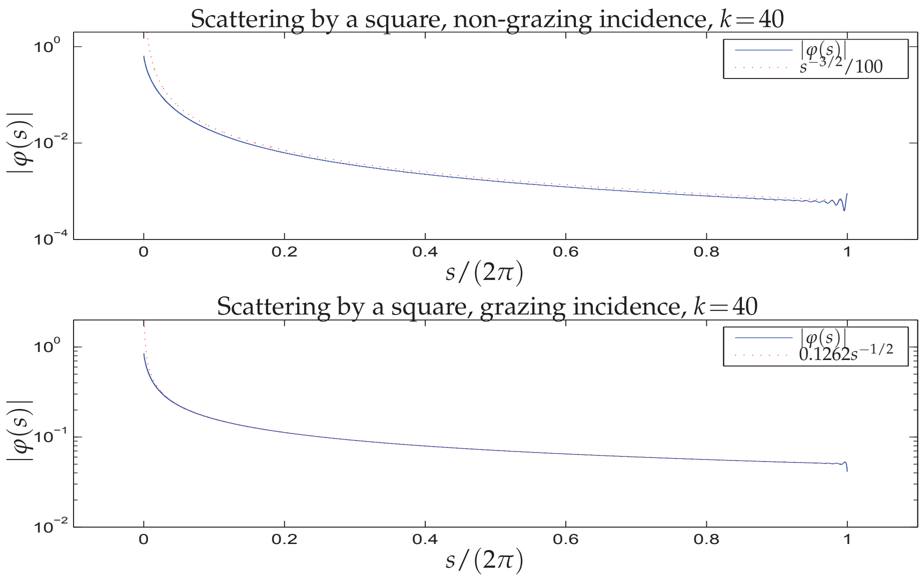

Figure 4: A comparison of $\varphi(s)$ on Γ_1, the side of the square between $(0,0)$ and $(2\pi,0)$, for grazing ($\theta=\pi/2$) and non-grazing ($\theta=\pi/4$) incidence, $k=40$.

for $\theta=\pi/2$. In Fig. 4 we plot $|\varphi(s)|$ for $s\in[0,2\pi]$, and for comparison we also plot $s^{-3/2}/100$ for the case of non-grazing incidence, and $\sqrt{2}/(|\beta|\sqrt{\pi ks})\approx 0.1262s^{-1/2}$ for the case of grazing incidence. For grazing incidence, we see that $|\varphi(s)|$ decays almost exactly like $0.1262s^{-1/2}$, indeed the plots are almost indistinguishable, as we might expect

Table 3: Errors in the Galerkin method approximation, scattering by square with grazing incidence, $k=10$, $k=40$ and $k=160$.

k	N	M_N	$M_N\lambda/L$	$\|\varphi_{128}-\varphi_N\|_2$	$\|\varphi_{128}-\varphi_N\|_2/\|\varphi_{128}\|_2$	EOC	$\mathrm{cond}_2 A$
10	4	64	1.6	8.7735×10^{-2}	1.1611×10^{-1}	0.9	7.57×10^{0}
	8	128	3.2	4.5634×10^{-2}	6.0391×10^{-2}	0.9	2.12×10^{1}
	16	240	6.0	2.4679×10^{-2}	3.2659×10^{-2}	0.9	1.30×10^{2}
	32	464	11.6	1.2982×10^{-2}	1.7179×10^{-2}	0.9	6.16×10^{2}
	64	936	23.4	6.7312×10^{-3}	8.9079×10^{-3}	-	3.52×10^{3}
40	4	88	0.55	4.5011×10^{-2}	1.0418×10^{-1}	0.9	1.17×10^{1}
	8	176	1.10	2.3649×10^{-2}	5.4737×10^{-2}	0.9	1.65×10^{1}
	16	336	2.10	1.3108×10^{-2}	3.0339×10^{-2}	0.9	2.57×10^{1}
	32	640	4.00	6.9543×10^{-3}	1.6096×10^{-2}	1.0	2.19×10^{2}
	64	1296	8.10	3.5132×10^{-3}	8.1314×10^{-3}	-	1.94×10^{3}
160	4	104	0.163	2.4487×10^{-2}	1.0196×10^{-1}	1.0	2.14×10^{1}
	8	224	0.350	1.2325×10^{-2}	5.1315×10^{-2}	0.9	2.93×10^{1}
	16	424	0.663	6.7984×10^{-3}	2.8306×10^{-2}	0.9	3.38×10^{1}
	32	816	1.275	3.5982×10^{-3}	1.4982×10^{-2}	0.9	3.81×10^{1}
	64	1648	2.575	1.8654×10^{-3}	7.7666×10^{-3}	-	4.02×10^{2}

Table 4: Errors in the Galerkin method approximation, scattering by square with grazing incidence, $N=32$.

k	M_N	$M_N\lambda/L$	$k^{1/2}\|\varphi_{128}-\varphi_{32}\|_2$	$\|\varphi_{128}-\varphi_{32}\|_2/\|\varphi_{128}\|_2$	$\mathrm{cond}_2 A$
5	376	18.80	4.0399×10^{-2}	1.8364×10^{-2}	2.13×10^{3}
10	464	11.60	4.1053×10^{-2}	1.7179×10^{-2}	6.16×10^{2}
20	552	6.90	4.2087×10^{-2}	1.6399×10^{-2}	5.06×10^{2}
40	640	4.00	4.3983×10^{-2}	1.6096×10^{-2}	2.19×10^{2}
80	728	2.28	4.4923×10^{-2}	1.5548×10^{-2}	7.35×10^{1}
160	816	1.28	4.5514×10^{-2}	1.4982×10^{-2}	3.81×10^{1}
320	904	0.71	4.6313×10^{-2}	1.4565×10^{-2}	5.14×10^{1}

from (5.1). This also suggests that our estimates in Theorem 2.1 appear to be sharp in this case. For non-grazing incidence we see that $|\varphi(s)|$ decays like $Cs^{-3/2}$, indicating that our estimates are not sharp in this case.

We remark, finally, that all the values of k for which we show results in the tables above are irregular frequencies in the sense of Section 3, i.e. they are values of k for which the Dirichlet problem for the Helmholtz equation in the interior Ω has a non-trivial solution u_D and so the integral equation (3.4) has infinitely many solutions. (For our numerical examples these irregular frequencies are $k=\sqrt{m^2+n^2}$, for $m,n\in\mathbb{N}$, with the corresponding non-trivial solutions u_D given by $u_D(\mathbf{x}) = \sin mx_1 \sin nx_2$.) This lack of uniqueness at a continuous level does not appear to translate to the discrete level; the Galerkin method with our approximation space, carefully tailored to φ, seems to select the right solution – see the discussion in Section 3 and [13].

6 Conclusions

In this paper we have derived regularity estimates for scattering by a convex polygon with piecewise constant impedance boundary conditions, and we have used these to derive a novel approximation space for $\gamma^+ u^t$ consisting of the products of plane waves with piecewise polynomials supported on a graded mesh, with larger elements further from the corners of the polygon. We have implemented a Galerkin boundary element method; numerical results suggest that the number of degrees of freedom required to achieve a prescribed level of accuracy need grow only logarithmically with respect to the frequency of the incident wave. These numerical experiments are backed up by rigorous error bounds on the error in the best approximation from the approximation space.

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