Commun. Comput. Phys.
doi: 10.4208/cicp.181212.220113a

Vol. **14**, No. 4, pp. 1094-1102
October 2013

A Constructive Method for Computing Generalized Manley-Rowe Constants of Motion

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Received 18 December 2012; Accepted (in revised version) 22 January 2013

Communicated by Michel A. Van Hove

Available online 10 May 2013

Abstract. The Manley-Rowe constants of motion (MRC) are conservation laws written out for a dynamical system describing the time evolution of the amplitudes in resonant triad. In this paper we extend the concept of MRC to resonance clusters of any form yielding generalized Manley-Rowe constants (gMRC) and give a constructive method how to compute them. We also give details of a *Mathematica* implementation of this method. While MRC provide integrability of the underlying dynamical system, gMRC generally do not but may be used for qualitative and numerical study of dynamical systems describing generic resonance clusters.

PACS: 47.10.Df, 47.10.Fg, 02.70.Dh

Key words: Resonance clusters, Hamiltonian formulation, discrete wave turbulence, NR-diagram, invariants of motion.

1 Introduction

Weakly nonlinear wave systems showing resonances can be found in a great variety of physical systems, among them surface water waves, atmospheric planetary waves, plasma drift waves, etc. In Hamiltonian formulation the equation of motion in Fourier space can be written out as

$$i\dot{B}_{\mathbf{k}} = \partial\mathcal{H}/\partial B^*_{\mathbf{k}}, \tag{1.1}$$

where $B_{\mathbf{k}}$ is the amplitude of the Fourier mode corresponding to wavevector $\mathbf{k}$ and the Hamiltonian $\mathcal{H}$ is represented as an expansion in powers of terms $\mathcal{H}_j$, each representing all products of j amplitudes $B_{\mathbf{k}}$:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{int}, \quad \mathcal{H}_{int} = \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \cdots. \tag{1.2}$$

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The quadratic Hamiltonian $\mathcal{H}_2$ describes linear motion and the interaction Hamiltonian $\mathcal{H}_{int}$ describes nonlinear interaction of waves.

If the cubic Hamiltonian $\mathcal{H}_3 \neq 0$, $\mathcal{H}_{int} \approx \mathcal{H}_3$, three-wave interaction is dominant and the main contribution to the nonlinear evolution comes from triads of waves each satisfying the resonance conditions

$$\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)=0, \quad \mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3=0. \tag{1.3}$$

The corresponding dynamical system describing time evolution of the amplitudes B_j of a triad reads

$$\dot{B}_1 = ZB_2^*B_3, \quad \dot{B}_2 = ZB_1^*B_3, \quad \dot{B}_3 = -ZB_1B_2, \tag{1.4}$$

the interaction coefficient $Z \neq 0$ being determined from a known function over the solutions of (1.3).

System (1.4) possesses three conservation laws, first found in [1] called Manley-Rowe constants (MRC) of motion. For a resonant triad, the MRC may be written as:

$$I_{13} = |B_1|^2+|B_3|^2, \quad I_{23} = |B_2|^2+|B_3|^2, \quad I_{12} = |B_1|^2-|B_2|^2. \tag{1.5}$$

They are linearly dependent, but any two of them form a linearly independent subset, enough to provide the integrability of (1.4), see [2].

In a 3-wave system, a resonant triad is called a primary resonance cluster, and clusters consisting of more than 3 modes (common clusters) may be decomposed into triads having joint modes, [3]. Common clusters describe many real physical phenomena: in nonlinear water wave systems [4], in piezoelectric semiconductors [5], in geophysics [6,7], in optics [8] etc.

In this paper we present a simple constructive method for deducing generalized MRC (gMRC) for a common resonance cluster. The gMRC are polynomials on $|B_j|^2$ with integer coefficients. The method has been developed by EK for the course on nonlinear resonance analysis held at the J. Kepler University, Linz, since 2005. A *Mathematica* code based on this method has been written by Loredana Tec and can be downloaded from [9]. The method has been used afterwards by the author, collaborators and students but its description has not been published. Recent publication [10] where gMRC are studied by far more complicated methods indicates that a simpler algorithm and available program code may be of interest for physicists working in the area of discrete wave turbulence.

2 Computation of gMRC

2.1 Computation of MRC

Let us rewrite (1.4) as

$$\dot{B}_1 = ZB_2^*B_3, \qquad \dot{B}_2 = ZB_1^*B_3, \qquad \dot{B}_3 = -ZB_1B_2, \tag{2.1a}$$

$$\dot{B}_1^* = ZB_2B_3^*, \qquad \dot{B}_2^* = ZB_1B_3^*, \qquad \dot{B}_3^* = -ZB_1^*B_2^*, \tag{2.1b}$$

where the second equation is the complex conjugate of the first. As $B_jB_j^*=|B_j|^2$ and

$$\mathbf{d}(B_jB_j^*)/\mathbf{d}t=B_j\mathbf{d}B_j^*/\mathbf{d}t+B_j^*\mathbf{d}B_j/\mathbf{d}t, \tag{2.2}$$

multiplication of each equation for B_j and B_j^* by B_j^* and B_j, respectively yields

$$\begin{cases} \mathbf{d}|B_1|^2/\mathbf{d}t=ZB_1^*B_2^*B_3+ZB_1B_2B_3^*, \\ \mathbf{d}|B_2|^2/\mathbf{d}t=ZB_1^*B_2^*B_3+ZB_1B_2B_3^*, \\ \mathbf{d}|B_3|^2/\mathbf{d}t=-ZB_1B_2B_3^*-ZB_1^*B_2^*B_3, \end{cases} \Rightarrow$$
$$\begin{cases} \mathbf{d}|B_1|^2/\mathbf{d}t-\mathbf{d}|B_2|^2/\mathbf{d}t=0, \\ \mathbf{d}|B_1|^2/\mathbf{d}t+\mathbf{d}|B_3|^2/\mathbf{d}t=0, \\ \mathbf{d}|B_2|^2/\mathbf{d}t+\mathbf{d}|B_3|^2/\mathbf{d}t=0, \end{cases} \Rightarrow \begin{cases} I_{12}=|B_1|^2-|B_2|^2=\text{const}, \\ I_{13}=|B_1|^2+|B_3|^2=\text{const}, \\ I_{23}=|B_2|^2+|B_3|^2=\text{const}, \end{cases} \tag{2.3}$$

which is the set of MRC found originally in [1].

2.2 Computation of gMRC for a two triad cluster

Regard a simple common cluster in a three-wave system formed by two triads a and b connected by the low frequency modes $B_{1a}=B_{1b}$. The corresponding dynamical system reads

$$\begin{cases} \dot{B}_{1a}=Z_aB_{2a}^*B_{3a}+Z_bB_{2b}^*B_{3b}, \\ \dot{B}_{1a}^*=Z_aB_{2a}B_{3a}^*+Z_bB_{2b}B_{3b}^*, \\ \dot{B}_{2a}=Z_aB_{1a}^*B_{3a}, \qquad \dot{B}_{2a}^*=Z_aB_{1a}B_{3a}^*, \\ \dot{B}_{3a}=-Z_aB_{1a}B_{2a}, \qquad \dot{B}_{3a}^*=-Z_aB_{1a}^*B_{2a}^*, \\ \dot{B}_{2b}=Z_bB_{1a}^*B_{3b}, \qquad \dot{B}_{2b}^*=Z_bB_{1a}^*B_{3b}, \\ \dot{B}_{3b}=-Z_bB_{1a}B_{2b}, \qquad \dot{B}_{3b}^*=-Z_bB_{1a}^*B_{2b}^*. \end{cases} \tag{2.4}$$

Considerations similar to those above yield three gMRC:

$$\begin{cases} \mathbf{d}|B_{1a}|^2/\mathbf{d}t=Z_aB_{1a}^*B_{2a}^*B_{3a}+Z_bB_{1a}^*B_{2b}^*B_{3b} \\ \qquad\qquad +Z_aB_{1a}B_{2a}B_{3a}^*+Z_bB_{1a}B_{2b}B_{3b}^*, \\ \mathbf{d}|B_{2a}|^2/\mathbf{d}t=Z_aB_{1a}^*B_{2a}^*B_{3a}+Z_aB_{1a}B_{2a}B_{3a}^*, \\ \mathbf{d}|B_{3a}|^2/\mathbf{d}t=-Z_aB_{1a}B_{2a}B_{3a}^*-Z_aB_{1a}^*B_{2a}^*B_{3a}, \\ \mathbf{d}|B_{2b}|^2/\mathbf{d}t=Z_bB_{1a}^*B_{2b}^*B_{3b}+Z_bB_{1a}B_{2b}B_{3b}^*, \\ \mathbf{d}|B_{3b}|^2/\mathbf{d}t=-Z_bB_{1a}B_{2b}B_{3b}^*-Z_bB_{1a}^*B_{2b}^*B_{3b}, \end{cases} \Rightarrow$$
$$\begin{cases} \mathbf{d}(|B_{1a}|^2+|B_{3a}|^2+|B_{3b}|^2)/\mathbf{d}t=0, \\ \mathbf{d}(|B_{2a}|^2+|B_{3a}|^2)/\mathbf{d}t=0, \\ \mathbf{d}(|B_{2b}|^2+|B_{3b}|^2)/\mathbf{d}t=0, \end{cases} \Rightarrow \begin{cases} |B_{1a}|^2+|B_{3a}|^2+|B_{3b}|^2=\text{const}, \\ |B_{2a}|^2+|B_{3a}|^2=\text{const}, \\ |B_{2b}|^2+|B_{3b}|^2=\text{const}. \end{cases} \tag{2.5}$$

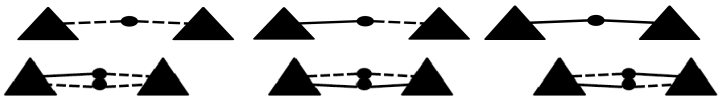

Figure 1: NR-diagrams for all 2-triad clusters.

It is important to note that the choice of the connecting mode influences the form of the generalized MRCs. Indeed, for the connection $B_{1a}=B_{1b}$ we got the set of constants (2.5) whereas the connection $B_{1a}=B_{3b}$ yields a different set of MRCs:

$$\begin{cases} I_{23|a}=|B_{2|a}|^2+|B_{3|a}|^2, \\ I_{12|b}=|B_{1|b}|^2-|B_{2|b}|^2, \\ I_{|a,b}=|B_{3|a}|^2+|B_{1|b}|^2+|B_{3|b}|^2. \end{cases} \tag{2.6}$$

The sets (2.5) and (2.6) are linearly independent and the corresponding dynamical systems are not equivalent.

As it was first shown in [11] the connections within a cluster of triads may be classified into *three connection types* depending on whether a connection involves both or one or none of the high frequency modes B_{3a}, B_{3b}. A high frequency mode is unstable due to the criterion of decay instability and is called *active* or A-mode while low frequency modes B_{1a}, B_{2a}, B_{1b}, B_{2b} are called passive or P-modes. Each connection within a common cluster can be either of AA-, AP- or PP-type (see [2] for more details). In the NR-diagram representation they are shown as two bold, bold-dashed and dashed-dashed half edges correspondingly and each triad is shown as a triangle

Examples of all possible NR-diagrams for 2-triad clusters are shown in Fig. 1. For the NR-diagrams shown in the upper panel on the left and in the middle the sets of MRCs are given in (2.5) and (2.6) respectively. The variety of 3-triad clusters is substantially richer (a few dozen). Examples of NR-diagrams for some possible clusters are shown in Fig. 2. In real physical applications, clusters of 5 to 10 triads and more are common, e.g., [7,11–13]. The sheer voluminous of computations for deducing MRCs makes the use of a computer algebra system mandatory.

Another important issue is that as any linear combination of gMRCs is a gMRC, gMRCs are not unique; any of the constants in (2.5) can be replaced by some of their linear combinations:

$$I_1=|B_{1a}|^2-|B_{2a}|^2+|B_{3b}|^2, \quad \text{or} \quad I_2=|B_{1a}|^2+|B_{2a}|^2+2|B_{3a}|^2+|B_{3b}|^2, \quad \text{or else.} \tag{2.7}$$

However, for qualitative and numerical studies we are naturally interested in computing "a minimal set" of gMRC, i.e., a set of the shortest possible form which contains specific variables. For instance, for the dynamical system (2.4), the set

$$\begin{cases} I_{23|a}=|B_{2a}|^2+|B_{3a}|^2, \quad I_{23|b}=|B_{2b}|^2+|B_{3b}|^2, \\ I_{|ab}=|B_{1a}|^2+|B_{3a}|^2+|B_{3b}|^2, \end{cases} \tag{2.8}$$

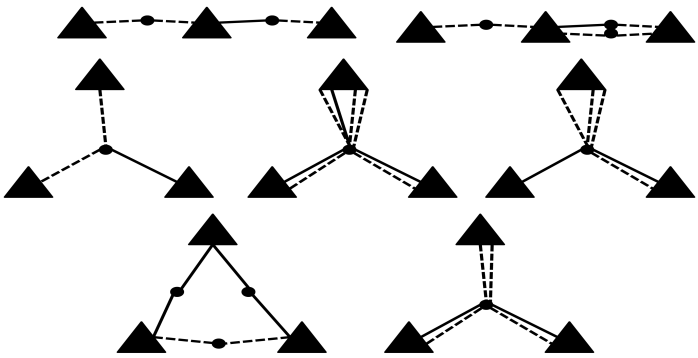

Figure 2: NR-diagrams for some 3-triad clusters.

is minimal while the set

$$\begin{cases} I_{23|a} = |B_{2a}|^2 + |B_{3a}|^2, \quad I_{23|b} = |B_{2b}|^2 + |B_{3b}|^2, \\ I_2 = |B_{1a}|^2 + |B_{2a}|^2 + 2|B_{3a}|^2 + |B_{3b}|^2, \end{cases} \tag{2.9}$$

is not. Minimal sets of gMRC can be computed using Gröbner bases.

3 Implementation details

The computation of gMRC has been implemented in *Mathematica* as a package `Nonlinear Resonance` including the following procedures (algorithms are given below):

- `DynamicalSystem` for computing dynamical systems for common clusters in a three wave system;
- `ConservationLawsLinComb` for computing the MRCs for a given cluster by considering all linear combinations of the variables from the corresponding dynamical system having coefficients from $\{0, \pm 1\}$. The set of conservation laws is obtained by computing the null space of the set of these vertices;
- `ConservationLawsGb` for computing a Gröbner basis of the ideal of the MRCs for a common cluster.

Each of those takes as input a list of triads, where every triad is represented by a list of its vertices. The procedure `DynamicalSystem` returns the dynamical system of the cluster specified by the input. The procedure `ConservationLawsLinComb` returns a "minimal" basis of the set of conservation laws – i.e., a set of MRCs – corresponding to the input cluster. The procedure `ConservationLawsGb` returns a normalized reduced Gröbner basis of the ideal of conservation laws with respect to the lexicographic term ordering on the $|B_{ij}|^2$s such that

$$|B_{ij}|^2 > |B_{kl}|^2 \begin{cases} \text{if } i < k \text{ or} \\ \text{if } i = k \text{ and } j < l. \end{cases}$$

Algorithm 3.1: DynamicalSystem

Require: $L=((L_{ij})_{j=a}^{c})_{i=1}^{s}$ a list of triads representing a cluster.
Ensure: The dynamical system R of the cluster.
$M:=(Z_1B_{1b}^*,B_{1c},Z_1B_{1a}^*B_{1c},-Z_1B_{1a}B_{1b},\cdots,Z_sB_{sb}^*,B_{sc},Z_sB_{sa}^*B_{sc},-Z_sB_{sa}B_{sb})$;
$N:=(B_{1a},B_{1b},B_{1c},\cdots,B_{sa},B_{sb},B_{sc})$;
$\tilde{L}:=(L_{1a},L_{1b},L_{1c},\cdots,L_{sa},L_{sb},L_{sc})$;
for i from $3s$ by -1 to 2 **do**
  for j from $i-1$ by -1 to 1 **do**
    if $\tilde{L}_i=\tilde{L}_j$ **then**
      $M_j:=M_i+M_j$;
      $M_i:=NULL$;
      In M substitute N_i by N_j;
      $N_i:=NULL$;
      $j:=1$;
    end if
  end for
end for
for i from 1 to $|M|$ **do**
  $R_i:=\left(\frac{d}{dt}N_i=M_i\right)$;
  $R_{i+|M|}:=\left(\frac{d}{dt}N_i^*=M_i^*\right)$;
end for
return R.

Algorithm 3.2: ConservationLawsLinComb

Require: L a list of triads representing a cluster.
Ensure: A minimal set M of MRCs for the cluster.
$\tilde{L}$:=DynamicalSystem(L);
Let n be the number of nodes appearing in the dynamical system $\tilde{L}$, say $B_{\lambda_1},\cdots,B_{\lambda_n}$;
for $1\le i\le n$ **do**
  $\frac{d}{dt}(|B_{\lambda_i}|^2):=B_{\lambda_i}\frac{d}{dt}(B_{\lambda_i}^*)+\frac{d}{dt}(B_{\lambda_i})B_{\lambda_i}^*$, where $\frac{d}{dt}(B_{\lambda_i})$, $\frac{d}{dt}(B_{\lambda_i}^*)$ are determined by the dynamical system;
end for
Determine a basis M of

$$\left\{(v_1,\cdots,v_n)\in\{0,\pm1\}^n \;\middle|\; \frac{d}{dt}(v_1|B_{\lambda_1}|^2+\cdots+v_n|B_{\lambda_n}|^2)=0\right\};$$

return M.

This is usually computed faster than the basis returned by `conservationlaws` but its elements need not only have coefficients ±1 and 0. Computation time for arbitrary clusters consisting of 20-30 triads is about a few seconds.

Algorithm 3.3: ConservationLawsGb

Require: L a list of triads representing a cluster.
Ensure: A Gröbner basis G of the ideal of conservation laws and the corresponding term order $\prec$.
1: $\tilde{L}$:=DynamicalSystem(L);
2: Let n be the number of nodes appearing in the dynamical system $\tilde{L}$, say $B_{\lambda_1},\cdots,B_{\lambda_n}$;
3: **for** $1\leq i\leq n$ **do**
4: $\frac{d}{dt}(|B_{\lambda_i}|^2):=B_{\lambda_i}\frac{d}{dt}(B^*_{\lambda_i})+\frac{d}{dt}(B_{\lambda_i})B^*_{\lambda_i}$, where $\frac{d}{dt}(B_{\lambda_i})$, $\frac{d}{dt}(B^*_{\lambda_i})$ are determined by the dynamical system;
5: **end for**
6: Choose a term order $\prec$ such that for $t_1\in\left[\frac{d}{dt}(|B_{\lambda_1}|^2),\cdots,\frac{d}{dt}(|B_{\lambda_n}|^2)\right]$, $1\neq t_2\in\left[B_{\lambda_1},\cdots,B_{\lambda_n},B^*_{\lambda_1},\cdots,B^*_{\lambda_n}\right]$, $t_3\in\left[\frac{d}{dt}(|B_{\lambda_1}|^2),\cdots,\frac{d}{dt}(|B_{\lambda_n}|^2),B_{\lambda_1},\cdots,B_{\lambda_n},B^*_{\lambda_1},\cdots,B^*_{\lambda_n}\right]$ we have $t_1\prec t_2t_3$ and such that $\prec$ restricted to $\left[\frac{d}{dt}(|B_{\lambda_1}|^2),\cdots,\frac{d}{dt}(|B_{\lambda_n}|^2)\right]$ is degree lexicographic;
7: Compute a Gröbner basis $\tilde{G}$ of the ideal $\left\langle\frac{d}{dt}(|B_{\lambda_i}|^2)-B_{\lambda_i}\frac{d}{dt}(B^*_{\lambda_i})+\frac{d}{dt}(B_{\lambda_i})B^*_{\lambda_i}|i=1,\cdots,n\right\rangle\trianglelefteq\mathbb{Q}\left[\frac{d}{dt}(|B_{\lambda_1}|^2),\cdots,\frac{d}{dt}(|B_{\lambda_n}|^2),B_{\lambda_1},\cdots,B_{\lambda_n},B^*_{\lambda_1},\cdots,B^*_{\lambda_n}\right]$ with respect to $\prec$;
8: $G=\tilde{G}\cap\mathbb{Q}\left[\frac{d}{dt}(|B_{\lambda_1}|^2),\cdots,\frac{d}{dt}(|B_{\lambda_n}|^2)\right]$;
9: Remove all elements of G with degree ≥ 2;
10: In G substitute $\frac{d}{dt}(|B_{\lambda_i}|^2)$ by $|B_{\lambda_i}|^2$ for $i=1,\cdots,n$;
11: **return** G and $\prec$.

4 Discussion

The construction of any constants of motion aims usually at establishing the integrability of the underlying dynamical system. For instance, integrability of a two triad cluster with $B_{3a}=B_{3b}$ and $Z_a/Z_b=2$ is proven in [14] where an additional constant of motion (a polynomial of sixth order) has been constructed via direct search based on irreducible forms. On the other hand, a two triad cluster with $B_{1a}=B_{1b}$ and $Z_a/Z_b=3/4$ shows chaotic energy exchange among the modes of the cluster (see Poincaré sections for this case in [2], Fig. 4.6).

Examples of integrable dynamical systems are known only for resonance clusters of a very special form and/or for special initial conditions. Moreover, there is no general method to establish integrability of a dynamical system and usually only a small set of constants of motion of any form (other than gMRC) can be found, e.g., [2, 14–18].

The algorithm described in this paper allows to produce in a matter of seconds (making use of a computer algebra system) a set of gMRC which is appropriate for qualitative study of the energy transport in a resonance cluster, for simplifying numerical study by reducing the number of variables in a dynamical system or for checking stability of the chosen numerical scheme.

Generalized MRCs can be analogously constructed for a 4-wave system where the primary clusters are quartets. The explicit form of the gMRCs for a quartet can be found in [2].

If we want to compare the method presented in this paper for computation of gMRCs

to the method presented in [19] let us say the following: MRCs are integrals of motion restricting the solution space of a dynamical system, so it is clear that in any case they are defined by the dynamical system.

If the dynamical system corresponds to a *primary cluster* it is sufficient to know the resonance frequencies in order to write the dynamical system. The method which has been presented in [19] and in which "the derivation yields the integrals deductively and without specifying the interaction details" (see [19], pp. 6095) uses the frequencies alone to compute the MRCs in explicit form.

If the cluster is a common cluster, its set of resonance frequencies does not define uniquely the dynamical system (this subject is discussed in depth in [2], pp. 176–181) and consequently the frequencies alone are not sufficient to compute gMRCs. To have an idea notice that, say, it does not follow from $\omega_3=\omega_4$ that also the wave vectors are equal. For instance, in a 2-dimensional discrete wave system with dispersion function $\omega\sim k^\alpha$ with any real α and $k=|\mathbf{k}|=\sqrt{k_x^2+k_y^2}$ one can easily construct different wave vectors with equal frequencies, e.g., $\mathbf{k}_1=(1,7)$ and $\mathbf{k}_2=(5,5)$. Then $\omega_1=\omega(\mathbf{k}_1)=(1^2+7^2)^{\alpha/2}=50^{\alpha/2}=(5^2+5^2)^{\alpha/2}=\omega(\mathbf{k}_2)=\omega_2$ but $B_1=B(\mathbf{k}_1)\neq B(\mathbf{k}_2)=B_2$.

Below, we give as an example two common clusters having the same set of five frequencies but a different number of degrees of freedom and consequently a different number of gMRCs:

$$\omega_2+\omega_3=\omega_1,\quad \omega_4+\omega_5=\omega_2,\quad \omega_5+\omega_7=\omega_6;\quad \text{with}\quad \omega_3=\omega_4=\omega_6 \tag{4.1}$$

(observed experimentally, two-dimensional gravity-capillary water waves, [4]) and

$$\widetilde{\omega}_1+\widetilde{\omega}_2=\widetilde{\omega}_3,\quad \widetilde{\omega}_4+\widetilde{\omega}_5=\widetilde{\omega}_3 \tag{4.2}$$

(observed in measured data, two-dimensional oceanic planetary waves, [12]).

As a summary we may state that the method developed in [19] – and in fact any method which does not take into account the explicit form of the dynamical system – can be used only for primary clusters but not for a common resonance cluster. The method presented in this paper is more general: starting with a given resonance cluster we define uniquely the corresponding dynamical system and from this we may compute the full set of quadratic invariants of motion both for primary and common clusters.

Acknowledgments

Authors are grateful to the Referee for useful remarks allowing to improve the presentation. E. K. acknowledges the support of the Austrian Science Foundation (FWF) under project P22943-N18 "Nonlinear resonances of water waves" and in part – by the Project of Knowledge Innovation Program (PKIP) of Chinese Academy of Sciences, Grant No. KJCX2. YW. W10. E. K. is very much obliged to the organizing committee of the program "New Directions in Turbulence" (KITPC/ITP-CAS, 2012) and the hospitality of

Kavli ITP, Beijing, where part of this work has been accomplished. L. T. is supported by the Austrian Academy of Sciences as the recipient of a DOC-fFORTE-fellowship.

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