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A Distributed Control Approach for the Boundary Optimal Control of the Steady MHD Equations

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Abstract. A new approach is presented for the boundary optimal control of the MHD equations in which the boundary control problem is transformed into an extended distributed control problem. This can be achieved by considering boundary controls in the form of lifting functions which extend from the boundary into the interior. The optimal solution is then sought by exploring all possible extended functions. This approach gives robustness to the boundary control algorithm which can be solved by standard distributed control techniques over the interior of the domain.

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1 Introduction

Fluid flows interacting with magnetic fields is a feature in many science and engineering settings such as fusion technology, fission nuclear reactors cooled using liquid metals, and submarine propulsion devices [5, 18]. Such flows are described by the magnetohydrodynamic (MHD) system of equations for which numerous formulations have been proposed and analyzed in the literature, based on differing physical assumptions about the MHD model; see, e.g., [9, 15, 20, 21, 23]. For example, for the description of electromagnetic phenomena, the Maxwell equations or some related simplifications employing different sets of state variables have been used, with the state variables consisting of one combination or another of quantities such as the magnetic field, the current density, the

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electric field, and the electric potential [9]. The mechanical behavior of the fluid flow is often described by the Navier-Stokes equations. It is well known that, whereas the Navier-Stokes equations are posed over the region occupied by the fluid, the Maxwell equations extend to all of three-dimensional space [20]. In addition, initial, boundary, and interface conditions are imposed; the specific choices for these constraints help define specific physical models and affect aspects of the mathematical models such as weak formulations along with the choice of the associated function spaces.

Several approaches have been proposed for optimal control problems constrained by the MHD equations; see, e.g., [9, 12, 17]. Compared to the case of distributed controls, standard approaches for treating boundary control problems are not entirely straightforward to implement numerically. In fact, boundary controls involve normal or tangential components of the magnetic field so that the direct implementation of such controls causes substantial difficulties on general domains. Furthermore, such implementations often lead to unnecessarily smooth controls [11, 20] or involve overpenalization that can adversely affect the accuracy of approximations and the conditioning of discretized systems.

We instead introduce a novel approach in which the boundary control problem is transformed, through lifting functions, into a distributed control problem from which an *optimal distributed magnetic field control* may be determined. By appropriately restricting the optimal distributed control to the boundary, any and all possible boundary controls can be determined.

The paper is organized as follows. In Section 2, we introduce the MHD optimal boundary control problem. We also introduce our modification to that problem that allows us to instead determine an optimal distributed control from which the boundary control may be defined by restriction. Then, in Section 3, we provide a weak formulation of the MHD state equations suitable for our purposes and prove the existence of a solution of those equations. This result is needed to obtain the results of Section 4 in which a precise, functional analytic definition of the optimal control is given, followed by a proof of the existence of an optimal solution. The technique of Lagrange multipliers, the first-order necessary condition, and the optimality system are discussed in Sections 4.2 and 4.3. Section 5 contains the results of some numerical experiments.

2 Description of the optimal control problem

The optimal control problem we consider consists of a cost or objective functional, a set of control functions, and a set of state equations that act as constraints.

Let $\Omega \subset \mathbb{R}^3$ denote an open bounded connected domain with $C^{1,1}$ boundary Γ. We denote by Γ_1 a subset of Γ with positive surface measure. For the constraint or state equations, we have the steady-state MHD model given by [6,19,22]

$$\begin{cases} -\dfrac{1}{Re}\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p-S_1(\nabla\times\boldsymbol{B})\times\boldsymbol{B}-\boldsymbol{f}=\boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u}=0, \\ \dfrac{1}{Re_m}\nabla\times(\nabla\times\boldsymbol{B})-\nabla\times(\boldsymbol{u}\times\boldsymbol{B})+\nabla s=\boldsymbol{0}, \\ \nabla\cdot\boldsymbol{B}=0, \end{cases} \quad \text{on } \Omega. \tag{2.1}$$

Here, $\boldsymbol{u}$ and p denote the fluid velocity and pressure, respectively, $\boldsymbol{B}$ the magnetic field, s the Lagrange multiplier associated to the divergence-free constraint on the magnetic field, $Re=UL/\nu$ the viscous Reynolds number, $Re_m=\mu_0\sigma UL$ the magnetic Reynolds number, $H_m=BL\sqrt{\sigma/\mu}$ the Hartmann number, and $S_1=H_m^2/ReRe_m$, where U, B, L, ν, μ_0, and σ denote reference values for the velocity, magnetic field, length, kinematic viscosity, magnetic permeability, and electrical conductivity of the fluid, respectively. The MHD system (2.1) is completed with appropriate boundary conditions for the velocity and magnetic field which are discussed below.

By using well-known vector identities for divergence-free fields, the system (2.1) takes the form

$$\begin{cases} -\dfrac{1}{Re}\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla r-S_1(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}-\boldsymbol{f}=\boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u}=0, \\ \dfrac{1}{Re_m}\nabla\times(\nabla\times\boldsymbol{B})+(\boldsymbol{u}\cdot\nabla)\boldsymbol{B}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}+\nabla s=\boldsymbol{0}, \\ \nabla\cdot\boldsymbol{B}=0, \end{cases} \quad \text{on } \Omega, \tag{2.2}$$

where the modified pressure r is defined as

$$r=p+\frac{S_1}{2}\nabla|\boldsymbol{B}|^2.$$

To the system (2.2) we append the boundary conditions

$$\boldsymbol{u}=\boldsymbol{g}, \qquad \text{on } \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\tau}(\boldsymbol{u},r)=\boldsymbol{t}, \quad \text{on } \Gamma\backslash\Gamma_1, \tag{2.4}$$

$$\boldsymbol{B}=\boldsymbol{\Phi}_0, \qquad \text{on } \Gamma, \tag{2.5}$$

where

$$\boldsymbol{\tau}(\boldsymbol{u},r)=-\frac{1}{Re}\frac{\partial\boldsymbol{u}}{\partial n}+r\boldsymbol{n}+\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u}. \tag{2.6}$$

The boundary conditions (2.3) and (2.4) are standard velocity and stress boundary conditions, respectively, for the velocity $\boldsymbol{u}$ and modified pressure r. The definition (2.6) can be physically interpreted as a *generalized stress* at the boundary that takes into account both dissipative (due to viscous dissipation) and conservative energy (due to mechanical pressure, magnetic energy, and kinetic energy) contributions at the boundary. The boundary condition (2.5) is not the most commonly used boundary condition for the magnetic field $\boldsymbol{B}$. Instead, one of the pairs of boundary conditions

$$\boldsymbol{B}\cdot\boldsymbol{n}=\Phi_1 \quad \text{and} \quad \boldsymbol{E}\times\boldsymbol{n}=\boldsymbol{\Psi}_1, \qquad \text{on } \Gamma, \tag{2.7}$$

or

$$\boldsymbol{E}\cdot\boldsymbol{n}=\Psi_2 \quad \text{and} \quad \boldsymbol{B}\times\boldsymbol{n}=\boldsymbol{\Phi}_2, \qquad \text{on } \Gamma, \tag{2.8}$$

where

$$\boldsymbol{E}=\frac{1}{Re_m}\nabla\times\boldsymbol{B}-\boldsymbol{u}\times\boldsymbol{B} \tag{2.9}$$

denotes the electric field, is usually applied on Γ. However, (2.5) (or equivalently, boundary conditions on the pair $\boldsymbol{B}\cdot\boldsymbol{n}$ and $\boldsymbol{B}\times\boldsymbol{n}$) is, for us, an expedient boundary condition to use in the boundary optimal control problem we consider; its use does not engender any loss of generality, as will become evident later.

The *controls* in the problem we consider can be chosen to be the data pair $(\Phi_1,\boldsymbol{\Psi}_1)$ in (2.7) or the data pair $(\boldsymbol{\Phi}_2,\Psi_2)$ in (2.8) or even the datum $\boldsymbol{\Phi}_0$ in (2.5).

In standard approaches (see, e.g., [11, 16, 17]) for determining an optimal boundary control function $\boldsymbol{\Phi}_0$, the optimal control problem consists in minimizing the objective functional

$$\mathcal{J}_b(\boldsymbol{w},\boldsymbol{\Phi})=\frac{\alpha}{2}\int_\Omega|\boldsymbol{u}-\boldsymbol{u}_d|^2d\boldsymbol{x}+\frac{\beta}{2}\int_\Gamma|\boldsymbol{\Phi}_0|^2d\boldsymbol{x}+\frac{\gamma}{2}\int_\Gamma|\nabla\boldsymbol{\Phi}_0|^2d\boldsymbol{x}. \tag{2.10}$$

The goal of the optimal control problem is to match, as well as possible, the velocity $\boldsymbol{u}$ to a prescribed velocity $\boldsymbol{u}_d$, so that the α-term in the functional is the real goal of control. The β- and γ-terms are penalty regularization terms that limit the size of the boundary control function $\boldsymbol{\Phi}_0$ and allow this function to have a sufficient smoothness. Analogous functionals may be defined if one instead uses one the boundary control pairs $(\Phi_1,\boldsymbol{\Psi}_1)$ or $(\Phi_2,\boldsymbol{\Psi}_2)$.

However, unlike for standard boundary control formulations, we do not deal with the control choices $\boldsymbol{\Phi}_0$ or $(\Phi_1,\boldsymbol{\Psi}_1)$ or $(\Phi_2,\boldsymbol{\Psi}_2)$ directly or with the cost functional (2.10). Instead, we choose to determine an *optimal distributed magnetic field control* from which any and all of the possible boundary controls can be determined by restriction; naturally, the cost functional we use changes so that the penalization term involves the distributed control.

Remark 2.1. A seemingly negative aspect of our approach, once discretization is effected, is the increase in the number of the degrees of freedom to be controlled, i.e., instead of being determined by the number of grid points on the boundary, the number of degrees of freedom for the approximation of the distributed control is determined by the number of grid points in the interior of the domain. However, one does not have to apply a distributed control on the whole domain; one need apply it only on any subset of that domain whose closure contains the boundary. In so doing, the number of degrees of freedom needed to approximate the distributed control can be made small and thus not significantly contribute to the total computational cost. Furthermore, boundary control is often applied on only part of the boundary, so that in this case the closure of the interior subdomain need only contain that part, further reducing the number of degrees of freedom employed. However, for simplicity, we treat the case where the distributed control acts on the whole interior domain; all our results hold for the case of distributed controls acting on subsets of that domain.

To this end, we define the decomposition of the magnetic field given by

$$\boldsymbol{B}=\boldsymbol{Q}+\boldsymbol{\Phi}, \qquad \text{on } \Omega, \tag{2.11}$$

with

$$\nabla\cdot\boldsymbol{\Phi}=0, \quad \text{in } \Omega \qquad \text{and} \qquad \boldsymbol{\Phi}=\boldsymbol{B}, \quad \text{on } \Gamma \tag{2.12}$$

so that $\boldsymbol{Q}$ is solenoidal and satisfies homogeneous boundary conditions on Γ, i.e., we have $\nabla\cdot\boldsymbol{Q}=0$ in Ω and $\boldsymbol{Q}=\boldsymbol{0}$ on Γ. Note that the fourth equation in (2.1) or (2.2) implies that $\int_\Gamma \boldsymbol{B}\cdot\boldsymbol{n}d\Gamma=0$ so that, from (2.11) and (2.12), the compatibility conditions $\int_\Gamma \boldsymbol{\Phi}\cdot\boldsymbol{n}d\Gamma=\int_\Omega \nabla\cdot\boldsymbol{\Phi}d\boldsymbol{x}=0$ and $\int_\Gamma \boldsymbol{Q}\cdot\boldsymbol{n}d\Gamma=\int_\Omega \nabla\cdot\boldsymbol{Q}d\boldsymbol{x}=0$ are automatically satisfied.

Although we do not treat $\boldsymbol{g}$ in (2.3) as a control, it is convenient from the analysis point of view to also treat homogeneous velocity boundary conditions so that we also decompose the velocity field as

$$\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{\mu}, \qquad \text{on } \Omega, \tag{2.13}$$

with

$$\nabla\cdot\boldsymbol{\mu}=0, \quad \text{in } \Omega \qquad \text{and} \qquad \boldsymbol{\mu}=\boldsymbol{g}, \quad \text{on } \Gamma_1 \tag{2.14}$$

so that $\boldsymbol{w}$ is solenoidal and satisfies homogeneous boundary conditions on Γ_1, i.e., we have $\nabla\cdot\boldsymbol{w}=0$ in Ω and $\boldsymbol{w}=\boldsymbol{0}$ on Γ_1. Then, $\boldsymbol{w}$, r, $\boldsymbol{Q}$ and s are the *state functions* in our control problem, $\boldsymbol{\Phi}$ is the *distributed magnetic field control function*, and $\boldsymbol{\mu}$ is an auxiliary function used to satisfy the velocity boundary condition. Substitution of (2.11) and (2.13) into (2.2) results in the final form of the *constraint* or *state equations*

$$\begin{cases} -\dfrac{1}{Re}\Delta(\boldsymbol{w}+\boldsymbol{\mu})+\big((\boldsymbol{w}+\boldsymbol{\mu})\cdot\nabla\big)(\boldsymbol{w}+\boldsymbol{\mu})+\nabla r \\ \qquad\qquad -S_1\big((\boldsymbol{Q}+\boldsymbol{\Phi})\cdot\nabla\big)(\boldsymbol{Q}+\boldsymbol{\Phi})=\boldsymbol{f}, \\ \nabla\cdot\boldsymbol{w}=0, \\ \dfrac{1}{Re_m}\nabla\times\big(\nabla\times(\boldsymbol{Q}+\boldsymbol{\Phi})\big)+\big((\boldsymbol{w}+\boldsymbol{\mu})\cdot\nabla\big)(\boldsymbol{Q}+\boldsymbol{\Phi}) \\ \qquad\qquad -\big((\boldsymbol{Q}+\boldsymbol{\Phi})\cdot\nabla\big)(\boldsymbol{w}+\boldsymbol{\mu})+\nabla s=\boldsymbol{0}, \\ \nabla\cdot\boldsymbol{Q}=0, \end{cases} \qquad \text{on } \Omega, \tag{2.15}$$

along with

$$\boldsymbol{w}=\boldsymbol{0}, \qquad \text{on } \Gamma_1, \tag{2.16}$$

$$\boldsymbol{\tau}(\boldsymbol{w}+\boldsymbol{\mu},r)=\boldsymbol{t}, \qquad \text{on } \Gamma\backslash\Gamma_1, \tag{2.17}$$

$$\boldsymbol{Q}=\boldsymbol{0}, \qquad \text{on } \Gamma, \tag{2.18}$$

which candidate optimal states $\boldsymbol{w}$, r, $\boldsymbol{Q}$ and s and optimal controls $\boldsymbol{\Phi}$ are required to satisfy. Here, $\boldsymbol{f}$ and $\boldsymbol{t}$ are given data functions, although, using well-known approaches [13], these could be used as control functions as well.

Clearly, for every solution $(\boldsymbol{u},r,\boldsymbol{B},s)$ of the MHD state problem (2.2), one can find lifting functions $\boldsymbol{\Phi}$ and $\boldsymbol{\mu}$ as above and a quadruple $(\boldsymbol{w},r,\boldsymbol{Q},s)$ that solves the problem (2.15). Of course, the lifting functions are not uniquely defined, so that different lifting functions will produce different solutions $(\boldsymbol{w},r,\boldsymbol{Q},s)$ of (2.15). However, the sums $\boldsymbol{B}=\boldsymbol{Q}+\boldsymbol{\Phi}$ and $\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{\mu}$ are uniquely determined and, along with r and s, solve (2.2).

To this end, with $\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{\mu}$, we define the *cost* or *objective functional*

$$\mathcal{J}(\boldsymbol{w},\boldsymbol{\Phi})=\frac{\alpha}{2}\int_\Omega|\boldsymbol{w}+\boldsymbol{\mu}-\boldsymbol{u}_d|^2 d\boldsymbol{x}+\frac{\beta}{2}\int_\Omega|\boldsymbol{\Phi}|^2 d\boldsymbol{x}+\frac{\gamma}{2}\int_\Omega|\nabla\boldsymbol{\Phi}|^2 d\boldsymbol{x}, \tag{2.19}$$

where α, β, and γ denote positive constants. The first term in (2.19) embodies the objective of the control problem. The β-term is added to limit the cost of control. The form of the γ-term is motivated by the need to have a sufficiently smooth control function so that the optimal control problem is well-posed. The values of these three constants can be adjusted in order to set the relative importance of the terms in the functional. Large values of α compared to β and γ allow for larger controls and better velocity matching. Observe the difference between the regularization terms in (2.10) and (2.19); the former involves the $H^1(\Gamma)$ norm whereas the latter involves the $H^1(\Omega)$ norm; see Remark 3.2 for a further discussion. Now that we have defined all its ingredients, we can state the optimal control problem we consider: *find an optimal control function* $\boldsymbol{\Phi}^*$ *and an optimal state* $\{\boldsymbol{w}^*,r^*,\boldsymbol{Q}^*,s^*\}$ *such that the functional* $\mathcal{J}(\cdot,\cdot)$ *given in* (2.19) *is minimized and the state system* (2.15)-(2.18) *is satisfied.*

Once the optimal functions $\boldsymbol{Q}^*$ and $\boldsymbol{\Phi}^*$ have been determined, optimal control functions of other types can be determined. For example, if we wish to know the optimal boundary control of type (2.7), we have, from (2.9), (2.11), (2.13), (2.16), and (2.18),

$$\begin{cases} \Phi_1^*=\boldsymbol{B}^*\cdot\boldsymbol{n}=\boldsymbol{\Phi}^*\cdot\boldsymbol{n}, \\ \boldsymbol{\Psi}_1^*=\boldsymbol{E}^*\times\boldsymbol{n}=\Big(\dfrac{1}{Re_m}\nabla\times(\boldsymbol{Q}^*+\boldsymbol{\Phi}^*)-\boldsymbol{\mu}\times\boldsymbol{\Phi}^*\Big)\times\boldsymbol{n}, \end{cases} \quad \text{on } \Gamma, \tag{2.20}$$

whereas optimal boundary controls of the type (2.8) would be determined as

$$\begin{cases} \Psi_2^*=\boldsymbol{B}^*\times\boldsymbol{n}=\boldsymbol{\Phi}^*\times\boldsymbol{n}, \\ \boldsymbol{\Phi}_2^*=\boldsymbol{E}^*\cdot\boldsymbol{n}=\Big(\dfrac{1}{Re_m}\nabla\times(\boldsymbol{Q}^*+\boldsymbol{\Phi}^*)-\boldsymbol{\mu}\times\boldsymbol{\Phi}^*\Big)\cdot\boldsymbol{n}, \end{cases} \quad \text{on } \Gamma. \tag{2.21}$$

Of course, optimal boundary controls of the type (2.5) are simply determined as $\boldsymbol{\Phi}_0^* = \boldsymbol{B}^* = \boldsymbol{\Phi}^*$ on Γ. Thus we see that obtaining the single optimal distributed control $\boldsymbol{\Phi}^*$ along with the optimal state variable $\boldsymbol{Q}^*$ enables one to determine optimal controls of other types, including boundary optimal controls.

Remark 2.2. We observe that, in practical situations, the boundary control given by $\boldsymbol{\Phi}_0$ (or by the pairs $(\Phi_1,\boldsymbol{\Psi}_1)$ or $(\Phi_2,\boldsymbol{\Psi}_2)$) acts only on part of the boundary $\Gamma_c \subset \Gamma$ [13]. In those cases, the corresponding distributed control function $\boldsymbol{\Phi}$ is fixed on the remaining portion $\Gamma_d = \Gamma \backslash \Gamma_c$. Thus, we distinguish the fixed and control parts of the boundary condition for the magnetic field as $\boldsymbol{\Phi}_{0,c}$ on Γ_c and $\boldsymbol{\Phi}_{0,d}$ on Γ_d (and similarly for the other boundary conditions).

3 Weak formulation of the MHD system

3.1 Notations

In order to define a weak formulation of the MHD system (2.2), we introduce several function spaces. Let $\Omega \subset \mathbb{R}^3$ denote an open bounded connected domain with $C^{1,1}$ boundary Γ. We denote by Γ_1 any subset of Γ with positive surface measure. For integers $m \geq 0$, we use the standard notation $H^m(\Omega)$ and $\boldsymbol{H}^m(\Omega)$ to denote Sobolev spaces of scalar- and vector-valued functions, respectively; these spaces are endowed with the standard Sobolev norm $\|\cdot\|_m$. We have that $H^0(\Omega) = L^2(\Omega)$ and $\boldsymbol{H}^0(\Omega) = \boldsymbol{L}^2(\Omega)$ with norm $\|\cdot\|_0$. The scalar product in $\boldsymbol{L}^2(\Omega)$ is denoted by $(\cdot,\cdot)$. Let $\boldsymbol{H}_0^m(\Omega)$ denote the closure of $\boldsymbol{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_m$ and $\boldsymbol{H}^{-m}(\Omega)$ denote the dual space of $\boldsymbol{H}_0^m(\Omega)$. The dual space of $\boldsymbol{H}^1(\Omega)$ is denoted by $\boldsymbol{H}^1(\Omega)^*$. We also define the spaces [1,8]

$$
\begin{aligned}
&\boldsymbol{V}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \mid \nabla \cdot \boldsymbol{u} = 0\},\\
&L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q \, d\boldsymbol{x} = 0\},\\
&L_1^2(\Omega) = \begin{cases} L^2(\Omega), & \text{if } \Gamma_1 \neq \emptyset,\\ L_0^2(\Omega), & \text{if } \Gamma_1 = \emptyset. \end{cases}
\end{aligned}
$$

For $m \geq 1$, the trace operator acting on functions belonging to $H^m(\Omega)$ is denoted by γ_0, e.g., $\gamma_0 f = f|_\Gamma$ and similarly for vector-valued functions; for smooth functions the trace operator simply restricts a function to its boundary values. The trace space of $\boldsymbol{H}^1(\Omega)$ is denoted by $\boldsymbol{H}^{1/2}(\Gamma)$ and its dual by $\boldsymbol{H}^{-1/2}(\Gamma)$. Given any subset $\Gamma_s \subset \Gamma$, we denote by $\boldsymbol{H}^1_{\Gamma_s}(\Omega)$ the subspace of $\boldsymbol{H}^1(\Omega)$ containing functions with vanishing trace on Γ_s.

For all functions $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ and $q \in L^2(\Omega)$, we introduce the bilinear forms

$$
\begin{aligned}
&a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, d\boldsymbol{x}, \qquad a_m(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{v}) \, d\boldsymbol{x},\\
&b(\boldsymbol{v},q) = -\int_\Omega q \nabla \cdot \boldsymbol{v} \, d\boldsymbol{x},
\end{aligned}
$$

and the trilinear forms

$$\widetilde{c}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})=\int_\Omega(\boldsymbol{w}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x}, \qquad c(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})=\frac{1}{2}\big(\widetilde{c}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})-\widetilde{c}(\boldsymbol{w},\boldsymbol{v},\boldsymbol{u})\big).$$

We have that

$$c(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})=\widetilde{c}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})$$

for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)$ such that $\nabla\cdot\boldsymbol{w}=0$ in Ω and one of $\boldsymbol{u}=\boldsymbol{0}$, or $\boldsymbol{v}=\boldsymbol{0}$, or $\boldsymbol{w}\cdot\boldsymbol{n}=0$ hold on Γ. The form $c(\cdot,\cdot,\cdot)$ is commonly used in weak formulations of the Navier-Stokes equations instead of the more directly derived form $\widetilde{c}(\cdot,\cdot,\cdot)$ because the former leads to substantial simplifications in the analysis [23,24].

The forms $a(\cdot,\cdot)$, $a_m(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $c(\cdot,\cdot,\cdot)$ satisfy the following properties; see, e.g., [7,8,24].

a) Continuity: there exist positive constants C_a, C_m, C_d, and C_c whose values are independent of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, and q such that

$$\begin{aligned}
&|a(\boldsymbol{u},\boldsymbol{v})|\le C_a\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1, && \forall(\boldsymbol{u},\boldsymbol{v})\in\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega),\\
&|a_m(\boldsymbol{u},\boldsymbol{v})|\le C_m\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1, && \forall(\boldsymbol{u},\boldsymbol{v})\in\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega),\\
&|b(\boldsymbol{u},q)|\le C_d\|\boldsymbol{u}\|_1\|q\|_0, && \forall(\boldsymbol{u},q)\in\boldsymbol{H}^1(\Omega)\times L^2_0(\Omega),\\
&|c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|\le C_c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1, && \forall\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}^1(\Omega).
\end{aligned}$$

b) Coercivity: there exist positive constants α_a and α_m whose values are independent of $\boldsymbol{u}$ such that

$$\begin{aligned}
&a(\boldsymbol{u},\boldsymbol{u})\ge\alpha_a\|\boldsymbol{u}\|_1^2, && \forall\boldsymbol{u}\in\boldsymbol{H}^1_{\Gamma_1}(\Omega),\\
&a_m(\boldsymbol{u},\boldsymbol{u})\ge\alpha_m\|\boldsymbol{u}\|_1^2, && \forall\boldsymbol{u}\in\boldsymbol{V}(\Omega).
\end{aligned}$$

c) Weak coercivity or the inf-sup condition: there exists a positive constant β_d such that

$$\inf_{0\neq q\in L^2_0(\Omega)}\sup_{0\neq\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)}\frac{b(\boldsymbol{u},q)}{\|\boldsymbol{u}\|_1\|q\|_0}\ge\beta_d. \tag{3.1}$$

d) Antisymmetric property with respect to the last two arguments:

$$c(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})=-c(\boldsymbol{w},\boldsymbol{v},\boldsymbol{u}), \qquad \forall\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}^1(\Omega). \tag{3.2}$$

An obvious consequence of (3.2) is that

$$c(\boldsymbol{w},\boldsymbol{u},\boldsymbol{u})=0, \qquad \forall\boldsymbol{u},\boldsymbol{w}\in\boldsymbol{H}^1(\Omega). \tag{3.3}$$

The advantage of the form $c(\cdot,\cdot,\cdot)$ over the form $\widetilde{c}(\cdot,\cdot,\cdot)$ is that the latter satisfies (3.2) and (3.3) only if $\nabla\cdot\boldsymbol{w}=0$ on Ω and one of $\boldsymbol{w}\cdot\boldsymbol{n}=0$ or $\boldsymbol{u}=\boldsymbol{0}$, or, for (3.2), $\boldsymbol{v}=\boldsymbol{0}$ hold on Γ.

Remark 3.1. If w satisfies only $\nabla\cdot\boldsymbol{w}=0$, one must write

$$\widetilde{c}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})=c(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})+\frac{1}{2}\int_\Gamma \boldsymbol{w}\cdot\boldsymbol{n}\boldsymbol{u}\cdot\boldsymbol{v}ds, \qquad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in \boldsymbol{H}^1(\Omega). \tag{3.4}$$

This implies that if Dirichlet boundary conditions are imposed, then the Navier-Stokes equation can be formulated equivalently with either of the trilinear forms $c(\cdot,\cdot,\cdot)$ or $\widetilde{c}(\cdot,\cdot,\cdot)$. If Neumann or pressure boundary conditions are needed, then the surface integral in (3.4) must be added and the boundary conditions written as a function of the stress $\boldsymbol{\tau}$ defined as in (2.6).

3.2 Weak formulation

Given the notations introduced in Section 3.1, a weak formulation of the MHD system (2.15)-(2.18) is defined as follows: *given the functions* $\boldsymbol{\Phi}$ *and* $\boldsymbol{\mu}$ *satisfying* (2.12) *and* (2.14), *seek* $(\boldsymbol{w},r,\boldsymbol{Q},s)\in \boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)\times \boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)$ *such that the boundary conditions* (2.16) *and* (2.18) *are satisfied and*

$$\begin{cases} \dfrac{1}{Re}a(\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_1)+c(\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_1)-S_1c(\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_1) \\ \qquad\qquad +b(\boldsymbol{v}_1,r)=<\boldsymbol{f},\boldsymbol{v}_1>-<\boldsymbol{t},\boldsymbol{v}_1>_{\Gamma\backslash\Gamma_1}, \\ b(\boldsymbol{w},q_1)=0, \\ \dfrac{1}{Re_m}a_m(\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_2)+c(\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_2)-c(\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_2) \\ \qquad\qquad +b(\boldsymbol{v}_2,s)=0, \\ b(\boldsymbol{Q},q_2)=0 \end{cases} \tag{3.5}$$

is satisfied for all $(\boldsymbol{v}_1,q_1,\boldsymbol{v}_2,q_2)\in \boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)\times \boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)$. Note that the boundary condition (2.17) does not have to be explicitly enforced; it is, in fact, a *natural* boundary condition imposed weakly through the boundary integral appearing in the first equation in (3.5). On the other hand, (2.16) and (2.18) are *essential* boundary conditions for the weak formulation (3.5) so that they must be explicitly imposed on candidate solutions. Also note that the third equation in (3.5) does not contain any boundary integrals because an essential boundary condition for the magnetic field variable $\boldsymbol{Q}$ is imposed on all of Γ, i.e., see (2.18).

It is a straightforward matter to verify that if $(\boldsymbol{w},r,\boldsymbol{Q},s)$ is a solution of (2.15)-(2.18), then it satisfies (3.5), (2.16), and (2.18). On the other hand, the converse is true only for solutions of (3.5), (2.16), and (2.18) that are sufficiently smooth, i.e., the weak formulation (3.5), (2.16), and (2.18) admits solutions that are not sufficiently regular to satisfy (2.15)-(2.18). For this reason, solutions $(\boldsymbol{w},r,\boldsymbol{Q},s)$ of (3.5), (2.16), and (2.18) are not only referred to as *weak* solutions of the MHD system (2.15)-(2.18), but are also referred to as *generalized* solutions.

Remark 3.2. Note that our indirect approach towards finding optimal boundary controls allows for those controls to belong to larger spaces than would be possible in a straightforward direct approach. With the latter approach, one must replace the regularization volume integral in the functional (2.19) by the square of a norm of the boundary controls, such as in (2.10). For this purpose, one would like to use a norm that leads to a well-posed problem in as large a functional space as possible; these norms are the fractional Sobolev space norms, i.e., $H^{1/2}(\Gamma)$ for components of the magnetic field $\boldsymbol{B}$ and $H^{-1/2}(\Gamma)$ for components of the electric field $\boldsymbol{E}$; see [11]. The use of a fractional Sobolev space norm causes difficulties which, in practice, are avoided by using stronger boundary norms as in (2.10). In our approach, we instead deal with a distributed control $\boldsymbol{\Phi}\in\boldsymbol{H}^1(\Omega)$ and very standard norms (see (2.19)). We also remark that with $\boldsymbol{\Phi},\boldsymbol{Q},\boldsymbol{\mu}\in\boldsymbol{H}^1(\Omega)$, we have, from (2.20) and (2.21), that the components of the corresponding optimal boundary magnetic and electric field controls do belong to the appropriate spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively, so that our approach results in boundary controls that are in the appropriate natural function spaces.

3.3 Existence of a solution to the state equations

Before stating the existence theorem for the state equations, some technical lemmas are required in substitution to standard results. It is well-known that the existence (without any smallness condition on the data) of the Navier-Stokes solutions in the case of nonhomogeneous Dirichlet boundary conditions can be proved by the Hopf's lemma also known as Leray's inequality (see [8,24]). For the MHD equations, existence of solutions was proved in [15] with a smallness assumption on the boundary velocity. This smallness requirement can be removed if an extended solution from the boundary data for the velocity field is constructed with arbitrarily small $\boldsymbol{L}^3$-norm. For this purpose we recall the following result [26].

Lemma 3.1. *For any given* $\boldsymbol{g}\in\boldsymbol{H}^{1/2}(\Gamma)$ *with* $\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n}=0$ *and for any* $\epsilon>0$ *there exists a function* $\boldsymbol{u}_\epsilon\in\boldsymbol{H}^1(\Omega)$ *such that*

$$\boldsymbol{u}_\epsilon=\boldsymbol{g}, \quad \text{on } \Gamma, \qquad \nabla\cdot\boldsymbol{u}_\epsilon=0, \qquad \|\boldsymbol{u}_\epsilon\|_{\boldsymbol{L}^3(\Omega)}\leq\epsilon\|\boldsymbol{g}\|_{1/2,\Gamma}. \tag{3.6}$$

Proof. See [26]. □

We will use this result to prove the existence of the solution of our problem. This allows us to avoid having a priori bounds on the boundary conditions and controls. A generalization of this for the case of nonhomogeneous mixed Dirichlet-Neumann boundary conditions is straightforward and similar results can be found in [2,3].

Remark 3.3. For the proof of existence, we will need to perform splittings of the type $\boldsymbol{u}=\hat{\boldsymbol{u}}+\boldsymbol{u}_0$ and $\boldsymbol{B}=\hat{\boldsymbol{b}}+\boldsymbol{B}_0$, where $\boldsymbol{u}_0$ and $\boldsymbol{B}_0$ are lifting functions of the Dirichlet boundary conditions whose existence for sufficiently small values of ϵ is guaranteed by Lemma 3.1. The set of boundary conditions given by (2.3)-(2.5) does not assure that $\hat{\boldsymbol{u}}$ is homogeneous over the whole boundary Γ. Therefore, the property $\hat{\boldsymbol{u}}\cdot\boldsymbol{n}=0$ is satisfied only on Γ_1; this is the reason for the use of the trilinear form $c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ which is antisymmetric for any $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)$, independently of the values at the boundary Γ.

The proof of existence requires the transformation of the non-homogeneous boundary problem into a problem with homogeneous boundary conditions. Therefore, we split the velocity and magnetic fields as

$$\boldsymbol{u}=\hat{\boldsymbol{u}}+\boldsymbol{u}_0, \qquad \boldsymbol{B}=\hat{\boldsymbol{b}}+\boldsymbol{B}_0, \tag{3.7}$$

with

$$\begin{aligned} &\nabla\cdot\boldsymbol{u}_0=0, \qquad \boldsymbol{u}_0=\boldsymbol{g}, \qquad \text{on } \Gamma_1,\\ &\nabla\cdot\boldsymbol{B}_0=0, \qquad \boldsymbol{B}_0=\boldsymbol{\Phi}_0, \qquad \text{on } \Gamma. \end{aligned}$$

The MHD problem can be recast in the framework of the abstract setting for the Navier-Stokes equations as described in [7]. In order to see this, consider the bilinear forms

$$a_0((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{v}_1,\boldsymbol{v}_2))=\frac{1}{Re}a(\boldsymbol{u},\boldsymbol{v}_1)+\frac{S_1}{Re_m}a_m(\boldsymbol{B},\boldsymbol{v}_2)$$

and

$$\hat{d}((\boldsymbol{v}_1,\boldsymbol{v}_2),(r,s))=d(\boldsymbol{v}_1,r)+S_1d(\boldsymbol{v}_2,s).$$

We also define the trilinear forms

$$\begin{aligned} &a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w},\boldsymbol{C}),(\boldsymbol{v}_1,\boldsymbol{v}_2))\\ =&c(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}_1)-S_1c(\boldsymbol{B},\boldsymbol{C},\boldsymbol{v}_1)+S_1c(\boldsymbol{u},\boldsymbol{C},\boldsymbol{v}_2)-S_1c(\boldsymbol{B},\boldsymbol{w},\boldsymbol{v}_2), \end{aligned}$$

such that

$$\begin{aligned} &a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{v}_1,\boldsymbol{v}_2))\\ =&c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_1)-S_1c(\boldsymbol{B},\boldsymbol{B},\boldsymbol{v}_1)+S_1c(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v}_2)-S_1c(\boldsymbol{B},\boldsymbol{u},\boldsymbol{v}_2),\\ &\hat{a}((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w},\boldsymbol{C}),(\boldsymbol{v}_1,\boldsymbol{v}_2))\\ :=&a_0((\boldsymbol{w},\boldsymbol{C}),(\boldsymbol{v}_1,\boldsymbol{v}_2))+a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w},\boldsymbol{C}),(\boldsymbol{v}_1,\boldsymbol{v}_2)). \end{aligned}$$

For a given $(\boldsymbol{u}_0,\boldsymbol{B}_0)$, we set

$$<\hat{\boldsymbol{F}},(\boldsymbol{v}_1,\boldsymbol{v}_2)>=<\boldsymbol{f},\boldsymbol{v}_1>-\hat{a}((\boldsymbol{u}_0,\boldsymbol{B}_0),(\boldsymbol{u}_0,\boldsymbol{B}_0),(\boldsymbol{v}_1,\boldsymbol{v}_2))$$

and

$$\begin{aligned} &\tilde{a}((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w},\boldsymbol{C}),(\boldsymbol{v}_1,\boldsymbol{v}_2))\\ :=&\hat{a}((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w},\boldsymbol{C}),(\boldsymbol{v}_1,\boldsymbol{v}_2))+a_1((\boldsymbol{u}_0,\boldsymbol{B}_0),(\boldsymbol{w},\boldsymbol{C}),(\boldsymbol{v}_1,\boldsymbol{v}_2))+a_1((\boldsymbol{w},\boldsymbol{C}),(\boldsymbol{u}_0,\boldsymbol{B}_0),(\boldsymbol{v}_1,\boldsymbol{v}_2)). \end{aligned}$$

After multiplying the MHD equation by S_1, summing the equations and bringing the lifting functions to the right-hand side, the *homogenized MHD problem* in the so-called *P form* becomes [8]: seek $(\hat{\boldsymbol{u}},\hat{\boldsymbol{b}})\in \boldsymbol{H}^1_{\Gamma_1}(\Omega)\times\boldsymbol{H}^1_0(\Omega)$ and $(r,s)\in L^2_1(\Omega)\times L^2_0(\Omega)$ that satisfy

$$\begin{aligned}
&\tilde{a}((\hat{\boldsymbol{u}},\hat{\boldsymbol{b}}),(\hat{\boldsymbol{u}},\hat{\boldsymbol{b}}),(\boldsymbol{v}_1,\boldsymbol{v}_2))+\hat{d}((\boldsymbol{v}_1,\boldsymbol{v}_2),(r,s))\\
&\qquad =<\hat{\boldsymbol{F}},(\boldsymbol{v}_1,\boldsymbol{v}_2)>-<\boldsymbol{\tau}(\boldsymbol{u},r),\boldsymbol{v}_1>_{\Gamma\setminus\Gamma_1}, \quad \forall(\boldsymbol{v}_1,\boldsymbol{v}_2)\in\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times\boldsymbol{H}^1_0(\Omega),\\
&\hat{d}((\hat{\boldsymbol{u}},\hat{\boldsymbol{b}}),(z_1,z_2))=0, \qquad \forall(z_1,z_2)\in L^2_1(\Omega)\times L^2_0(\Omega),\\
&\hat{\boldsymbol{u}}=\boldsymbol{0}, \qquad \text{on } \Gamma_1,\\
&\boldsymbol{\tau}(\hat{\boldsymbol{u}},\hat{r})=\boldsymbol{t}, \qquad \text{on } \Gamma\setminus\Gamma_1,\\
&\hat{\boldsymbol{b}}=\boldsymbol{0}, \qquad \text{on } \Gamma.
\end{aligned}$$

Let $\boldsymbol{Z}$ be the kernel of $\hat{d}(\cdot,\cdot)$:

$$\boldsymbol{Z}=\{(\boldsymbol{u},\boldsymbol{B})\in\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times\boldsymbol{H}^1_0(\Omega)\,|\,\hat{d}((\boldsymbol{u},\boldsymbol{B}),(z_1,z_2))=0, \quad \forall(z_1,z_2)\in L^2_1(\Omega)\times L^2_0(\Omega)\}, \tag{3.8}$$

with the following natural norm inherited by its parent space (see [15])

$$\|(\boldsymbol{u},\boldsymbol{B})\|_{\boldsymbol{Z}}=(\|\boldsymbol{u}\|_1^2+\|\boldsymbol{B}\|_1^2)^{1/2}.$$

We can rewrite the above problem in *Q form* as

$$\tilde{a}((\hat{\boldsymbol{u}},\hat{\boldsymbol{b}}),(\hat{\boldsymbol{u}},\hat{\boldsymbol{b}}),(\boldsymbol{v}_1,\boldsymbol{v}_2))=<\hat{\boldsymbol{F}},(\boldsymbol{v}_1,\boldsymbol{v}_2)>-<\boldsymbol{\tau}(\boldsymbol{u},r),\boldsymbol{v}_1>_{\Gamma\setminus\Gamma_1}, \quad \forall(\boldsymbol{v}_1,\boldsymbol{v}_2)\in\boldsymbol{Z}. \tag{3.9}$$

It is well-known that the P and Q are equivalent problems and the latter may be used in the existence proof in the framework of the Lax-Milgram theorem.

3.4 Coercivity property

We formulate the coercivity property of the form $\tilde{a}((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w},\boldsymbol{C}),(\boldsymbol{v}_1,\boldsymbol{v}_2))$ on the space $\boldsymbol{Z}$. Note that, unlike in [15], the coercivity will be proved without any conditions on the boundary data [26].

Lemma 3.2. *For all $(\boldsymbol{u},\boldsymbol{B})\in\boldsymbol{Z}$ there exists a constant $K>0$ such that*

$$\tilde{a}((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}))\geq K\|(\boldsymbol{u},\boldsymbol{B})\|^2_{\boldsymbol{Z}}. \tag{3.10}$$

Proof. From the definition of the trilinear form we have

$$\begin{aligned}
&\tilde{a}((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}))\\
:=&a_0((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}))+a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}))\\
&\quad +a_1((\boldsymbol{u}_0,\boldsymbol{B}_0),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}))+a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u}_0,\boldsymbol{B}_0),(\boldsymbol{u},\boldsymbol{B}))\\
=&\frac{1}{Re}a(\boldsymbol{u},\boldsymbol{u})+c(\boldsymbol{u},\boldsymbol{u}_0,\boldsymbol{u})-S_1c(\boldsymbol{B},\boldsymbol{B}_0,\boldsymbol{u})\\
&\quad +\frac{S_1}{Re_m}a_m(\boldsymbol{B},\boldsymbol{B})+S_1c(\boldsymbol{u},\boldsymbol{B}_0,\boldsymbol{B})-S_1c(\boldsymbol{B},\boldsymbol{u}_0,\boldsymbol{B}),
\end{aligned}$$

where some terms vanish due to the antisymmetry property of the form c with respect to the last two arguments. The coercivity of the forms yields

$$a(\boldsymbol{u},\boldsymbol{u}) \geq \alpha_a \|\boldsymbol{u}\|_1^2, \qquad a_m(\boldsymbol{B},\boldsymbol{B}) \geq \alpha_m \|\boldsymbol{B}\|_1^2.$$

On the other hand, we can bound the remaining terms with arbitrarily small coefficients. By using the generalized Hölder inequality, the embedding of $\boldsymbol{H}^1$ into $\boldsymbol{L}^6$, the generalized Poincaré inequality for $\boldsymbol{u}$, and Lemma 3.1, we have that

$$\begin{aligned} |c(\boldsymbol{u},\boldsymbol{u}_0,\boldsymbol{u})| &= |c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}_0)| \\ &\leq \|\boldsymbol{u}\|_{\boldsymbol{L}^6} \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^3} \\ &\leq \epsilon_1 \|\boldsymbol{u}\|_1^2 \end{aligned}$$

with ϵ_1 an arbitrarily positive constant. For the second term we have

$$\begin{aligned} |c(\boldsymbol{B},\boldsymbol{B}_0,\boldsymbol{u})| &= |c(\boldsymbol{B},\boldsymbol{u},\boldsymbol{B}_0)| \\ &\leq \|\boldsymbol{B}\|_{\boldsymbol{L}^6} \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2} \|\boldsymbol{B}_0\|_{\boldsymbol{L}^3} \\ &\leq \epsilon_2 \|\boldsymbol{B}\|_1 \|\boldsymbol{u}\|_1 \end{aligned}$$

with ϵ_2 a positive constant. The third term can be treated in a similar way to obtain

$$|c(\boldsymbol{u},\boldsymbol{B}_0,\boldsymbol{B})| = |c(\boldsymbol{u},\boldsymbol{B},\boldsymbol{B}_0)| \leq \epsilon_3 \|\boldsymbol{B}\|_1 \|\boldsymbol{u}\|_1$$

for an arbitrarily positive constant ϵ_3. The fourth term yields

$$|c(\boldsymbol{B},\boldsymbol{u}_0,\boldsymbol{B})| = |c(\boldsymbol{B},\boldsymbol{B},\boldsymbol{u}_0)| \leq \epsilon_4 \|\boldsymbol{B}\|_1^2.$$

Finally, gathering all the terms and using Young's inequality $ab \leq a^2/\gamma + \gamma b^2/4$, we have

$$\begin{aligned} &\tilde{a}((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B})) \\ =&\frac{1}{Re} a(\boldsymbol{u},\boldsymbol{u}) + c(\boldsymbol{u},\boldsymbol{u}_0,\boldsymbol{u}) - S_1 c(\boldsymbol{B},\boldsymbol{B}_0,\boldsymbol{u}) \\ &\quad + \frac{S_1}{Re_m} a_m(\boldsymbol{B},\boldsymbol{B}) + S_1 c(\boldsymbol{u},\boldsymbol{B}_0,\boldsymbol{B}) - S_1 c(\boldsymbol{B},\boldsymbol{u}_0,\boldsymbol{B}) \\ \geq& Re\alpha_a \|\boldsymbol{u}\|_1^2 + \frac{Re_m}{S_1} \alpha_m \|\boldsymbol{B}\|_1^2 - |c(\boldsymbol{u},\boldsymbol{u}_0,\boldsymbol{u})| \\ &\quad - |S_1 c(\boldsymbol{B},\boldsymbol{B}_0,\boldsymbol{u})| - |S_1 c(\boldsymbol{u},\boldsymbol{B}_0,\boldsymbol{B})| - |S_1 c(\boldsymbol{B},\boldsymbol{u}_0,\boldsymbol{B})| \\ \geq& Re\alpha_a \|\boldsymbol{u}\|_1^2 + \frac{Re_m}{S_1} \alpha_m \|\boldsymbol{B}\|_1^2 \\ &\quad - \epsilon_1 \|\boldsymbol{u}\|_1^2 - \epsilon_2 \|\boldsymbol{B}\|_1 \|\boldsymbol{u}\|_1 - \epsilon_3 \|\boldsymbol{B}\|_1 \|\boldsymbol{u}\|_1 - \epsilon_4 \|\boldsymbol{B}\|_1^2 \end{aligned}$$

$$
\begin{aligned}
&\geq Re\alpha_a\|\boldsymbol{u}\|_1^2+\frac{Re_m}{S_1}\alpha_m\|\boldsymbol{B}\|_1^2\\
&\quad -\epsilon_1\|\boldsymbol{u}\|_1^2-(\epsilon_2+\epsilon_3)\|\boldsymbol{B}\|_1\|\boldsymbol{u}\|_1-\epsilon_4\|\boldsymbol{B}\|_1^2\\
&\geq Re\alpha_a\|\boldsymbol{u}\|_1^2+\frac{Re_m}{S_1}\alpha_m\|\boldsymbol{B}\|_1^2\\
&\quad -\epsilon_1\|\boldsymbol{u}\|_1^2-\frac{(\epsilon_2+\epsilon_3)}{\epsilon_5}\|\boldsymbol{B}\|_1^2-(\epsilon_2+\epsilon_3)\frac{\epsilon_5}{4}\|\boldsymbol{u}\|_1^2-\epsilon_4\|\boldsymbol{B}\|_1^2\\
&=\Big(Re\alpha_a-\epsilon_1-(\epsilon_2+\epsilon_3)\frac{\epsilon_5}{4}\Big)\|\boldsymbol{u}\|_1^2+\Big(\frac{Re_m}{S_1}\alpha_m-\frac{(\epsilon_2+\epsilon_3)}{\epsilon_5}\Big)\|\boldsymbol{B}\|_1^2\\
&\geq\min\Big\{\Big(Re\alpha_a-\epsilon_1-(\epsilon_2+\epsilon_3)\frac{\epsilon_5}{4}\Big),\Big(\frac{Re_m}{S_1}\alpha_m-\frac{(\epsilon_2+\epsilon_3)}{\epsilon_5}\Big)\Big\}\|(\boldsymbol{u},\boldsymbol{B})\|_{\boldsymbol{Z}}^2,
\end{aligned}
$$

where $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5$ are constants that can be chosen without any loss of generality so that

$$\Big(Re\alpha_a-\epsilon_1-(\epsilon_2+\epsilon_3)\frac{\epsilon_5}{4}\Big)>0,\qquad \Big(\frac{Re_m}{S_1}\alpha_m-\frac{(\epsilon_2+\epsilon_3)}{\epsilon_5}\Big)>0.$$

The proof is now complete. □

3.5 Existence

The existence theorem for Problem Q (3.9) and hence Problem P can now be summarized in the following theorem.

Theorem 3.1. *Given* $(\hat{\boldsymbol{f}}_1,\hat{\boldsymbol{f}}_2)\in\boldsymbol{H}^1(\Omega)^*\times\boldsymbol{H}^1(\Omega)^*$ *and* $\boldsymbol{t}\in\boldsymbol{H}^{-1/2}(\Gamma\backslash\Gamma_1)$, *there exists at least one solution* $(\hat{\boldsymbol{u}},\hat{\boldsymbol{b}}),(r,s)\in(\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times\boldsymbol{H}^1_0(\Omega))\times(L^2_1(\Omega)\times L^2_0(\Omega))$ *of Problem P, where* $((\hat{\boldsymbol{u}},\hat{\boldsymbol{b}})$ *solves Problem Q. Thus, the original nonhomogeneous MHD problem has a weak solution* $(\boldsymbol{u},r,\boldsymbol{B},s)\in\boldsymbol{H}^1(\Omega)\times L^2_1(\Omega)\times\boldsymbol{H}^1(\Omega)\times L^2_0(\Omega)$.

Proof. Problem Q has a solution. In fact we can apply the standard abstract setting for nonlinear mixed problems and prove the following properties (see [8]).

i) *Separability*. $\boldsymbol{Z}$ is a separable Hilbert space as a subspace of $\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)$.

ii) *Weak sequential continuity*. The mapping $\tilde{a}((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}))$ is weakly sequentially continuous on $\boldsymbol{Z}$. A similar proof can be found in [15] (see [8,15]).

iii) *Continuity*. It is an immediate consequence of the continuity of the bilinear forms a, a_m, c, d.

iv) *Coercivity*. See Lemma 3.2.

These properties imply by standard arguments that there exists at least one solution of Problem Q.

v) *LBB condition*. Following the Ladyzhenskaya-Babuska-Brezzi theory for mixed problems (see [8,15,16]) one can prove that there exists a positive constant β_2 such that

$$\inf_{0\neq(r,s)\in L^2_1(\Omega)\times L^2_0(\Omega)}\ \sup_{0\neq(\hat{\boldsymbol{u}},\hat{\boldsymbol{b}})\in\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times\boldsymbol{H}^1_0(\Omega)}\frac{\hat{d}((\hat{\boldsymbol{u}},\hat{\boldsymbol{b}}),(r,s))}{\|(\hat{\boldsymbol{u}},\hat{\boldsymbol{b}})\|_{\boldsymbol{H}^1\times\boldsymbol{H}^1}\|(r,s)\|_{L^2_1\times L^2_0}}\geq\beta_2>0. \tag{3.11}$$

The proof of i)-v) assures the existence of a solution to Problems Q and P. Therefore, the original nonhomogeneous problem has a weak solution. □

Remark 3.4. The existence can be proved by similar arguments if the Navier-Stokes equations are supplemented with mixed boundary conditions on $\Gamma\backslash\Gamma_1$ such as

$$\boldsymbol{u}\cdot\boldsymbol{n}=\boldsymbol{g}\cdot\boldsymbol{n},\boldsymbol{\tau}\times\boldsymbol{n}=\boldsymbol{t}\times\boldsymbol{n} \tag{3.12}$$

or

$$\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{g}\times\boldsymbol{n},\boldsymbol{\tau}\cdot\boldsymbol{n}=\boldsymbol{t}\cdot\boldsymbol{n}. \tag{3.13}$$

In our analysis it is required to have full Dirichlet boundary conditions for the velocity $\boldsymbol{u}=\boldsymbol{g}$ on Γ_1 and full Dirichlet boundary conditions for the magnetic field $\boldsymbol{B}=\boldsymbol{\Phi}_0$ on Γ. Neumann boundary conditions for the magnetic field are not addressed here (see [26]).

Remark 3.5. The two decompositions of the magnetic field $\boldsymbol{B}=\boldsymbol{Q}+\boldsymbol{\Phi}$ and $\boldsymbol{B}=\hat{\boldsymbol{b}}+\boldsymbol{B}_0$ serve different purposes and it is evident that there is no relationship between $\boldsymbol{\Phi}$ and $\boldsymbol{B}_0$. The former is used for the treatment of boundary conditions for the optimal control problem, the latter instead is used only for the theoretical purpose of proving the existence of a magnetic field $\boldsymbol{B}$ satisfying the nonhomogeneous state problem. In other words, the lifting $\boldsymbol{B}_0$ can be chosen in infinitely many ways with respect to a smallness parameter ϵ (see Lemma 3.1), with the only requirement that the coercivity property (3.2) holds. On the other hand, the quantity $\boldsymbol{\Phi}$ is not chosen for theoretical purposes and it is determined by the solution of the optimal control problem. Similar considerations hold for the velocity decompositions $\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{\mu}$ and $\boldsymbol{u}=\hat{\boldsymbol{u}}+\boldsymbol{u}_0$ when the boundary velocities are considered as controls.

4 Optimal control problem

4.1 Existence of an optimal solution

We first state the optimal control problem in a more precise way by defining the *set of admissible target velocities* $\boldsymbol{U}_{ad}$ and the *set of admissible solutions* $\boldsymbol{A}_{ad}$. The first is defined as

$$\boldsymbol{U}_{ad}=\{\boldsymbol{u}_d\in\boldsymbol{H}^1(\Omega)\,|\,-\nu\Delta\boldsymbol{u}_d+(\boldsymbol{u}_d\cdot\nabla)\boldsymbol{u}_d\in\boldsymbol{L}^2(\Omega)\} \tag{4.1}$$

and the second as

$$\begin{aligned}\boldsymbol{A}_{ad}=\{&(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\in\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)\times\boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)\times\boldsymbol{H}^1(\Omega)\,|\\&(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\text{ satisfies the system (3.5) and }\mathcal{J}(\boldsymbol{w},\boldsymbol{\Phi})<\infty\}.\end{aligned} \tag{4.2}$$

We also set the restriction $\bar{\boldsymbol{A}}_{ad}$ as

$$\bar{\boldsymbol{A}}_{ad}=\{(\boldsymbol{w},\boldsymbol{\Phi})\,|\,(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\in\boldsymbol{A}_{ad}\}.$$

Remark 4.1. The set of admissible target velocities may be defined on a restricted domain Ω_1 in order to match with more precision the desired solution and improve the efficiency of the optimal control. Such a restriction of the target solution domain does not change the rest of analysis.

The optimal boundary control problem can then be formulated as follows.

Problem 1. Given $\boldsymbol{u}_d \in \boldsymbol{U}_{ad}$ find a global minimum point $(\widetilde{\boldsymbol{w}},\widetilde{r},\widetilde{\boldsymbol{Q}},\widetilde{s},\widetilde{\boldsymbol{\Phi}}) \in \boldsymbol{A}_{ad}$ of the objective functional

$$\mathcal{J}(\boldsymbol{w},\boldsymbol{\Phi}) = \frac{\alpha}{2}\|\boldsymbol{w}+\boldsymbol{\mu}-\boldsymbol{u}_d\|_0^2 + \frac{\beta}{2}\|\boldsymbol{\Phi}\|_0^2 + \frac{\gamma}{2}a(\boldsymbol{\Phi},\boldsymbol{\Phi}). \tag{4.3}$$

We now state the existence of a global minimizer in $\boldsymbol{A}_{ad}$.

Theorem 4.1. *Given $\boldsymbol{u}_d \in \boldsymbol{U}_{ad}$, there exists a solution $(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi}) \in \boldsymbol{A}_{ad}$ of the optimal control problem.*

Proof. The proof follows standard techniques. Theorem 3.1 states the existence of a solution of the state MHD system, therefore the set of admissible solutions $\boldsymbol{A}_{ad}$ is nonempty. We define

$$M := \inf_{(\boldsymbol{w},\boldsymbol{\Phi}) \in \bar{A}_{ad}} \mathcal{J}(\boldsymbol{w},\boldsymbol{\Phi}).$$

Clearly M exists and $M \geq 0$ since the set $\boldsymbol{A}_{ad}$ is nonempty and the functional is nonnegative. Thus let $\{(\boldsymbol{w}_n,r_n,\boldsymbol{Q}_n,s_n,\boldsymbol{\Phi}_n)\}$ be a minimizing sequence in $\boldsymbol{A}_{ad}$ for the objective functional, i.e.,

$$\lim_{n\to\infty} \mathcal{J}(\boldsymbol{w}_n,\boldsymbol{\Phi}_n) = M.$$

Now we show that the minimizing sequence $\{(\boldsymbol{w}_n,r_n,\boldsymbol{Q}_n,s_n,\boldsymbol{\Phi}_n)\}$ is uniformly bounded. As every convergent sequence of real numbers, the sequence $\mathcal{J}(\boldsymbol{w}_n,\boldsymbol{\Phi}_n)$ is bounded, hence the sequence $\{\boldsymbol{\Phi}_n\}$ is uniformly bounded in $\boldsymbol{V}(\Omega)$. By the continuous dependence of the state solutions on the data and by using the triangle inequality, we also have that the sequences $\{\boldsymbol{w}_n\}$, $\{\boldsymbol{Q}_n\}$, $\{r_n\}$, and $\{s_n\}$ are uniformly bounded. Hence, we can extract a subsequence $\{(\boldsymbol{w}_m,r_m,\boldsymbol{Q}_m,s_m,\boldsymbol{\Phi}_m)\}$ that converges weakly to some $(\widetilde{\boldsymbol{w}},\widetilde{r},\widetilde{\boldsymbol{Q}},\widetilde{s},\widetilde{\boldsymbol{\Phi}})$. We write

$$\begin{aligned}
\boldsymbol{w}_m &\to \widetilde{\boldsymbol{w}} && \text{weakly in} \quad \boldsymbol{V}(\Omega),\\
r_m &\to \widetilde{r} && \text{weakly in} \quad L_1^2(\Omega),\\
\boldsymbol{Q}_m &\to \widetilde{\boldsymbol{Q}} && \text{weakly in} \quad \boldsymbol{V}(\Omega),\\
s_m &\to \widetilde{s} && \text{weakly in} \quad L_0^2(\Omega),\\
\boldsymbol{\Phi}_m &\to \widetilde{\boldsymbol{\Phi}} && \text{weakly in} \quad \boldsymbol{V}(\Omega),\\
\boldsymbol{w}_m &\to \widetilde{\boldsymbol{w}} && \text{strongly in} \quad \boldsymbol{L}^2(\Omega),\\
\boldsymbol{Q}_m &\to \widetilde{\boldsymbol{Q}} && \text{strongly in} \quad \boldsymbol{L}^2(\Omega),\\
\boldsymbol{\Phi}_m &\to \widetilde{\boldsymbol{\Phi}} && \text{strongly in} \quad \boldsymbol{L}^2(\Omega).
\end{aligned}$$

The strong convergence in $\boldsymbol{L}^2(\Omega)$ stems from the compact imbedding $\boldsymbol{H}^1(\Omega)\hookrightarrow\hookrightarrow\boldsymbol{L}^2(\Omega)$.

Now we pass to the limit in order to show that $(\widetilde{\boldsymbol{w}},\widetilde{r},\widetilde{\boldsymbol{Q}},\widetilde{s},\widetilde{\boldsymbol{\Phi}})\in\boldsymbol{A}_{ad}$, i.e., the weak limit of the subsequences satisfies the constraints. We can pass to the limit inside the linear and the nonlinear terms of the MHD equation. The result for the nonlinear term can be proved by compactness arguments (see for example [8, 15, 24]). Therefore, the set $\boldsymbol{A}_{ad}$ is closed in the weak topology. In order to see that $(\widetilde{\boldsymbol{w}},\widetilde{r},\widetilde{\boldsymbol{Q}},\widetilde{s},\widetilde{\boldsymbol{\Phi}})\in\boldsymbol{A}_{ad}$ is a solution for the optimal control problem we finally have to show that $(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\Phi}})\in\bar{\boldsymbol{A}}_{ad}$ minimizes the functional. In fact, by the weak lower semicontinuity of the functional, we have

$$M\leq\mathcal{J}(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\Phi}})\leq\liminf_{m\to\infty}\mathcal{J}(\boldsymbol{w}_m,\boldsymbol{\Phi}_m)\leq\lim_{k\to\infty}\big(\inf_{p\geq k}\mathcal{J}(\boldsymbol{w}_p,\boldsymbol{\Phi}_p)\big)=M.$$

Therefore the infimum M of the functional is attained at the point $(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\Phi}})$, which is indeed a minimizer. □

Remark 4.2. The existence of a minimizer to the optimal control problem has been proved. In the following, the technique of Lagrange multipliers will be used in order to turn the constrained problem into an unconstrained one. Unfortunately, this method only allows us to search for local minima [14], i.e., to find solutions of the local problem:

Problem 2. Given $\boldsymbol{u}_d\in\boldsymbol{U}_{ad}$, find a $(\widetilde{\boldsymbol{w}},\widetilde{r},\widetilde{\boldsymbol{Q}},\widetilde{s},\widetilde{\boldsymbol{\Phi}})\in\boldsymbol{A}_{ad}$ such that the objective functional

$$\mathcal{J}(\boldsymbol{w},\boldsymbol{\Phi})=\frac{\alpha}{2}\|\boldsymbol{w}+\boldsymbol{\mu}-\boldsymbol{u}_d\|_0^2+\frac{\beta}{2}\|\boldsymbol{\Phi}\|_0^2+\frac{\gamma}{2}a(\boldsymbol{\Phi},\boldsymbol{\Phi}) \tag{4.4}$$

is locally minimized, i.e., such that there exists $\epsilon>0$ so that

$$\mathcal{J}(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\Phi}})\leq\mathcal{J}(\boldsymbol{w},\boldsymbol{\Phi}),\quad\forall(\boldsymbol{w},\boldsymbol{\Phi})\in\bar{\boldsymbol{A}}_{ad}\ \text{ with }\ \|\boldsymbol{w}-\widetilde{\boldsymbol{w}}\|_1+\|\boldsymbol{\Phi}-\widetilde{\boldsymbol{\Phi}}\|_1<\epsilon. \tag{4.5}$$

4.2 First-order necessary condition

In order to obtain the first-order necessary conditions and the optimality system for the optimal control problem, we introduce the nonlinear continuous constraint operator $M:\boldsymbol{B}_1\to\boldsymbol{B}_2$ defined between the spaces $\boldsymbol{B}_1=\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)\times\boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)\times\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{B}_2=(\boldsymbol{H}^1_{\Gamma_1}(\Omega))^*\times L^2_1(\Omega)\times\boldsymbol{H}^{-1}(\Omega)\times L^2_0(\Omega)\times L^2_0(\Omega)$ so that $M(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})=(\boldsymbol{f}_1,q_1,\boldsymbol{f}_2,q_2,q_3)$, where

$$\begin{aligned}
<\boldsymbol{f}_1,\boldsymbol{v}_1>:=&\frac{1}{Re}a(\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_1)+c(\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_1)-S_1c(\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_1)\\
&+d(\boldsymbol{v}_1,r)+<\boldsymbol{t},\boldsymbol{v}_1>_{\Gamma\backslash\Gamma_1},\\
(q_1,z_1):=&d(\boldsymbol{w},z_1),\\
<\boldsymbol{f}_2,\boldsymbol{v}_2>:=&\frac{1}{Re_m}a_m(\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_2)+c(\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_2)-c(\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_2)+d(\boldsymbol{v}_2,s),\\
(q_2,z_2):=&d(\boldsymbol{Q},z_2),\\
(q_3,z_3):=&d(\boldsymbol{\Phi},z_3)
\end{aligned}$$

for all functions $(\boldsymbol{v}_1,z_1,\boldsymbol{v}_2,z_2,z_3)\in \boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)\times \boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)\times L^2_0(\Omega)$ and boundary conditions

$$\begin{aligned} \boldsymbol{w}&=\boldsymbol{0}, && \text{on } \Gamma_1,\\ \boldsymbol{Q}&=\boldsymbol{0}, && \text{on } \Gamma,\\ \boldsymbol{\Phi}&=\boldsymbol{\Phi}_{0,d}, && \text{on } \Gamma_d,\\ \boldsymbol{\Phi}&=\boldsymbol{\Phi}_{0,c}, && \text{on } \Gamma_c. \end{aligned}$$

With the definition of the mapping $M(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})$, the constraints can be expressed as $M(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})=(\boldsymbol{f},0,\boldsymbol{0},0,0)$. Through the usual method of Lagrange multipliers, we turn the constrained minimization problem into an unconstrained one. The new problem is then:

Problem 3. Find a stationary point $(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}},\bar{a},\bar{\boldsymbol{\lambda}},\bar{\pi}_1,\bar{\boldsymbol{\xi}},\bar{\pi}_2,\bar{\pi}_3)$ of the Lagrangian functional

$$\begin{aligned} &\mathcal{J}_{aug}(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi},a,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)\\ =&a\mathcal{J}(\boldsymbol{w},\boldsymbol{\Phi})+<(\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3),M(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})>. \end{aligned} \tag{4.6}$$

Clearly, one does not know whether stationary points of $\mathcal{J}_{aug}$, i.e., points with vanishing Fréchet differential, yield a local minimum of the original cost functional. In the following, we will only derive a first-order necessary condition.

At every point $(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\in \boldsymbol{B}_1$ we introduce the bounded linear mapping $M'\in \mathcal{L}(\boldsymbol{B}_1,\boldsymbol{B}_2)$ as $M'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}}) = (\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)$ for $(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})\in \boldsymbol{B}_1$ and $(\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)\in \boldsymbol{B}_2$ if and only if

$$\begin{aligned} <\hat{\boldsymbol{f}}_1,\boldsymbol{v}_1>:=&\frac{1}{Re}a(\hat{\boldsymbol{w}},\boldsymbol{v}_1)+c(\hat{\boldsymbol{w}},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_1)+c(\boldsymbol{w}+\boldsymbol{\mu},\hat{\boldsymbol{w}},\boldsymbol{v}_1)-S_1c(\hat{\boldsymbol{Q}},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_1)\\ &-S_1c(\boldsymbol{Q}+\boldsymbol{\Phi},\hat{\boldsymbol{Q}},\boldsymbol{v}_1)-S_1c(\hat{\boldsymbol{\Phi}},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_1)-S_1c(\boldsymbol{Q}+\boldsymbol{\Phi},\hat{\boldsymbol{\Phi}},\boldsymbol{v}_1)+d(\boldsymbol{v}_1,\hat{r}),\\ (\hat{q}_1,z_1):=&d(\hat{\boldsymbol{w}},z_1),\\ <\hat{\boldsymbol{f}}_2,\boldsymbol{v}_2>:=&\frac{1}{Re_m}a_m(\hat{\boldsymbol{Q}},\boldsymbol{v}_2)+\frac{1}{Re_m}a_m(\hat{\boldsymbol{\Phi}},\boldsymbol{v}_2)+c(\hat{\boldsymbol{w}},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_2)+c(\boldsymbol{w}+\boldsymbol{\mu},\hat{\boldsymbol{Q}},\boldsymbol{v}_2)\\ &+c(\boldsymbol{w}+\boldsymbol{\mu},\hat{\boldsymbol{\Phi}},\boldsymbol{v}_2)-c(\hat{\boldsymbol{Q}},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_2)-c(\hat{\boldsymbol{\Phi}},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_2)-c(\boldsymbol{Q}+\boldsymbol{\Phi},\hat{\boldsymbol{w}},\boldsymbol{v}_2)+d(\boldsymbol{v}_2,\hat{s}),\\ (\hat{q}_2,z_2):=&d(\hat{\boldsymbol{Q}},z_2),\\ (\hat{q}_3,z_3):=&d(\hat{\boldsymbol{\Phi}},z_3) \end{aligned}$$

for all functions $(\boldsymbol{v}_1,z_1,\boldsymbol{v}_2,z_2,z_3)\in \boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)\times \boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)\times L^2_0(\Omega)$ and homogeneous boundary conditions

$$\begin{aligned} \hat{\boldsymbol{w}}&=\boldsymbol{0}, && \text{on } \Gamma_1,\\ \hat{\boldsymbol{Q}}&=\boldsymbol{0}, && \text{on } \Gamma,\\ \hat{\boldsymbol{\Phi}}&=\boldsymbol{0}, && \text{on } \Gamma_d. \end{aligned}$$

We have the following result.

Theorem 4.2. *Let* $(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\in\boldsymbol{B}_1$. *The operator* $M'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})$ *has closed range and a finite-dimensional kernel.*

Proof. i) We can show that the operator $M'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})$ is a compact perturbation of an isomorphism, i.e., it can be decomposed as $M'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})=K'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})+S'$, where S' is an isomorphism and K' is a compact operator. The operator $S'\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})=(\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)$ for $(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})\in\boldsymbol{B}_1$ and $(\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)\in\boldsymbol{B}_2$ is defined as

$$
\begin{aligned}
<\hat{\boldsymbol{f}}_1,\boldsymbol{v}_1>&:=\frac{1}{Re}a(\hat{\boldsymbol{w}},\boldsymbol{v}_1)+d(\boldsymbol{v}_1,\hat{r}),\\
(\hat{q}_1,z_1)&:=d(\hat{\boldsymbol{w}},z_1),\\
<\hat{\boldsymbol{f}}_2,\boldsymbol{v}_2>&:=\frac{1}{Re_m}a_m(\hat{\boldsymbol{Q}},\boldsymbol{v}_2)+\frac{1}{Re_m}a_m(\hat{\boldsymbol{\Phi}},\boldsymbol{v}_2)+d(\boldsymbol{v}_2,\hat{s}),\\
(\hat{q}_2,z_2)&:=d(\hat{\boldsymbol{Q}},z_2),\\
(\hat{q}_3,z_3)&:=d(\hat{\boldsymbol{\Phi}},z_3)
\end{aligned}
$$

for all functions $(\boldsymbol{v}_1,z_1,\boldsymbol{v}_2,z_2,z_3)\in\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)\times\boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)\times L^2_0(\Omega)$ and homogeneous boundary conditions. Given the unique solvability of the Stokes problem, we have that S' is an isomorphism. The operator $K'\in\mathcal{L}(\boldsymbol{B}_1,\boldsymbol{B}_2)$ is defined as $K'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})=(\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)$ for $(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})\in\boldsymbol{B}_1$ and $(\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)\in\boldsymbol{B}_2$ if and only if

$$
\begin{aligned}
<\hat{\boldsymbol{f}}_1,\boldsymbol{v}_1>:=&+c(\hat{\boldsymbol{w}},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_1)+c(\boldsymbol{w}+\boldsymbol{\mu},\hat{\boldsymbol{w}},\boldsymbol{v}_1)-S_1c(\hat{\boldsymbol{Q}},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_1)\\
&-S_1c(\boldsymbol{Q}+\boldsymbol{\Phi},\hat{\boldsymbol{Q}},\boldsymbol{v}_1)-S_1c(\hat{\boldsymbol{\Phi}},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_1)-S_1c(\boldsymbol{Q}+\boldsymbol{\Phi},\hat{\boldsymbol{\Phi}},\boldsymbol{v}_1),\\
(\hat{q}_1,z_1):=&0,\\
<\hat{\boldsymbol{f}}_2,\boldsymbol{v}_2>:=&+c(\hat{\boldsymbol{w}},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{v}_2)+c(\boldsymbol{w}+\boldsymbol{\mu},\hat{\boldsymbol{Q}},\boldsymbol{v}_2)+c(\boldsymbol{w}+\boldsymbol{\mu},\hat{\boldsymbol{\Phi}},\boldsymbol{v}_2)\\
&-c(\hat{\boldsymbol{Q}},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_2)-c(\hat{\boldsymbol{\Phi}},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{v}_2)-c(\boldsymbol{Q}+\boldsymbol{\Phi},\hat{\boldsymbol{w}},\boldsymbol{v}_2),\\
(\hat{q}_2,z_2):=&0,\\
(\hat{q}_3,z_3):=&0
\end{aligned}
$$

for all functions $(\boldsymbol{v}_1,z_1,\boldsymbol{v}_2,z_2,z_3)\in\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)\times\boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)\times L^2_0(\Omega)$ and homogeneous boundary conditions. The compactness of Sobolev embeddings and the properties of the trilinear form $c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ imply that the operator K' is compact [9,16].

Finally, by the Fredholm alternative, the operator M' is a semi-Fredholm operator, i.e., it has a closed range and a finite-dimensional kernel. □

Let $(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})$ denote an optimal solution in the local sense and $N:\boldsymbol{B}_1\to\mathbb{R}\times\boldsymbol{B}_2$ be the nonlinear mapping defined by $N(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})=(a,\boldsymbol{f}_1,q_1,\boldsymbol{f}_2,q_2,q_3)$ for $(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\in\boldsymbol{B}_1$ and $(a,\boldsymbol{f}_1,q_1,\boldsymbol{f}_2,q_2,q_3)\in\mathbb{R}\times\boldsymbol{B}_2$ if and only if

$$
\begin{aligned}
a&:=\mathcal{J}(\boldsymbol{w},\boldsymbol{\Phi})-\mathcal{J}(\bar{\boldsymbol{w}},\bar{\boldsymbol{\Phi}}),\\
(\boldsymbol{f}_1,q_1,\boldsymbol{f}_2,q_2,q_3)&:=M(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi}).
\end{aligned}
$$

As we did before, for any $(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\in\boldsymbol{B}_1$ we introduce the bounded linear mapping $N'\in\mathcal{L}(\boldsymbol{B}_1,\mathbb{R}\times\boldsymbol{B}_2)$ defined as $N'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})=(a_0,\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)$ for $(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})\in\boldsymbol{B}_1$ and $(a_0,\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)\in\mathbb{R}\times\boldsymbol{B}_2$ such that

$$\begin{aligned}&a_0:=\mathcal{J}'(\boldsymbol{w},\boldsymbol{\Phi})\cdot(\hat{\boldsymbol{w}},\hat{\boldsymbol{\Phi}}),\\&(\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3):=M'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}}).\end{aligned}$$

Theorem 4.3. *Let $(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})$ denote an optimal solution in the local sense.*

i) The operator $N'(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})$ has closed range.

ii) The operator $N'(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})$ is not onto.

Proof. i) Since the kernel of a continuous linear operator is closed in its domain, we have that $\mathrm{Ker}(M')$ is closed in $\boldsymbol{B}_1$. Clearly $\mathrm{Ker}(M')$ is a Banach space as it is a closed subspace of a Banach space. It is also known that if f is a linear functional on a Banach space X, then either $f\equiv 0$ or $\mathrm{Ran}(f)=\mathbb{R}$. Applying these results, one has that $J'\cdot\mathrm{Ker}(M')$ is either 0 or $\mathbb{R}$, therefore $J'\cdot\mathrm{Ker}(M')$ is closed in $\mathbb{R}$. Now we recall a well-known result [25]. Let X,Y,Z be Banach spaces, $A:X\to Y$ and $B:X\to Z$ be continuous linear operators. Let $C:X\to Y\times Z$ be defined as $C(x)=(A(x),B(x))$. If $\mathrm{Ran}(A)$ is closed in Y and $B\cdot\mathrm{Ker}(A)$ is closed in Z, then $\mathrm{Ran}(C)$ is closed in $Y\times Z$. Setting $A=M'$, $B=J'$, $C=N'$, $X=\boldsymbol{B}_1$, $Y=\boldsymbol{B}_2$ and $Z=\mathbb{R}$ we have the result.

ii) By contradiction, if the operator N' were onto, there would exist by the implicit function theorem another optimal solution $(\widetilde{\boldsymbol{w}},\widetilde{r},\widetilde{\boldsymbol{Q}},\widetilde{s},\widetilde{\boldsymbol{\Phi}})\in\boldsymbol{A}_{ad}$, different from the assumed optimal solution $(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})$, such that

$$\|\bar{\boldsymbol{w}}-\widetilde{\boldsymbol{w}}\|_1+\|\bar{\boldsymbol{\Phi}}-\widetilde{\boldsymbol{\Phi}}\|_1<\epsilon$$

and $\mathcal{J}(\widetilde{\boldsymbol{w}},\widetilde{\boldsymbol{\Phi}})<\mathcal{J}(\bar{\boldsymbol{w}},\bar{\boldsymbol{\Phi}})$. This is in contradiction of the hypothesis that $(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})$ is an optimal solution. ☐

Remark 4.3. We remark that the operator $N'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})$ is not onto in the more general case $(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})\in\boldsymbol{B}_1$, but the previous result is sufficient for our purposes.

Now we derive the first-order necessary condition from which an optimality system may be derived.

Theorem 4.4 (First-order necessary condition)**.** *Let $(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})$ denote an optimal solution in the local sense. There exists a nonzero Lagrange multiplier $(a,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)\in\mathbb{R}\times\boldsymbol{B}_2^*$ that satisfies the Euler equations*

$$\begin{aligned}&\mathcal{J}'_{aug}(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}},a,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}},0,\boldsymbol{0},0,\boldsymbol{0},0,0)\\=&a\mathcal{J}'(\bar{\boldsymbol{w}},\bar{\boldsymbol{\Phi}})\cdot(\hat{\boldsymbol{w}},\hat{\boldsymbol{\Phi}})+<(\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3),M'(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})>\\=&0,\qquad\forall(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})\in\boldsymbol{B}_1.\end{aligned}\tag{4.7}$$

Furthermore, if the operator M' is onto, we have $a\neq 0$; thus we may choose $a=1$ so that there exists a nonzero Lagrange multiplier $(\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)\in\boldsymbol{B}_2^*$ that satisfies the Euler equations

$$\begin{aligned}&\mathcal{J}'_{aug}(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}},1,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}},0,\boldsymbol{0},0,\boldsymbol{0},0,0)\\ =&\mathcal{J}'(\bar{\boldsymbol{w}},\bar{\boldsymbol{\Phi}})\cdot(\hat{\boldsymbol{w}},\hat{\boldsymbol{\Phi}})+<(\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3),M'(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})>\\ =&0,\qquad\forall(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})\in\boldsymbol{B}_1.\end{aligned}\tag{4.8}$$

Proof. Since $\mathrm{Ran}(N')$ is a closed and a proper subspace of $\mathbb{R}\times\boldsymbol{B}_2$, the Hahn-Banach theorem (see [27], p. 109) implies that there exists a nonzero element of $(\mathbb{R}\times\boldsymbol{B}_2)^*$ that annihilates the range of $N'(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})$, i.e., there exists $(a,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)\in(\mathbb{R}\times\boldsymbol{B}_2)^*$ such that

$$\begin{aligned}&<(a,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3),(a_0,\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)>=0,\\ &\qquad\forall(a_0,\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)\in\mathrm{Ran}(N'(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})).\end{aligned}\tag{4.9}$$

Hence the result (4.7) follows from the definition of N'. Moreover, if the operator M' is onto, one has that $a\neq 0$; in fact, by contradiction, one would have $<(\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3),(\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)>=0$ for all $(\hat{\boldsymbol{f}}_1,\hat{q}_1,\hat{\boldsymbol{f}}_2,\hat{q}_2,\hat{q}_3)\in\mathrm{Ran}(M')\equiv\boldsymbol{B}_2$, which would imply $(\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)=(\boldsymbol{0},0,\boldsymbol{0},0,0)$ since M' is onto. This contradicts the previous result stating that $(a,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)\neq(0,\boldsymbol{0},0,\boldsymbol{0},0,0)$. Without any loss of generality, we may normalize so as to have $a=1$, which yields (4.8). ☐

Now we briefly discuss the case $a=0$. Setting $a=0$ in the first-order necessary condition 4.4, one would have $(\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)\neq(\boldsymbol{0},0,\boldsymbol{0},0,0)$ such that

$$<(\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3),M'(\bar{\boldsymbol{w}},\bar{r},\bar{\boldsymbol{Q}},\bar{s},\bar{\boldsymbol{\Phi}})\cdot(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})>=0,\qquad\forall(\hat{\boldsymbol{w}},\hat{r},\hat{\boldsymbol{Q}},\hat{s},\hat{\boldsymbol{\Phi}})\in\boldsymbol{B}_1.\tag{4.10}$$

This would bring to an optimality system without any contribution from the Fréchet differential of the cost functional $\mathcal{J}'(\bar{\boldsymbol{w}},\bar{\boldsymbol{\Phi}})\cdot(\hat{\boldsymbol{w}},\hat{\boldsymbol{\Phi}})$. Therefore, such a system would retain no information about the objective of the control problem. Its solution would have no physical meaning. Thus, the case $a=0$ must be avoided in practice. As we know, the surjectivity of $M'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})$ is a sufficient condition for having $a\neq 0$. One is interested in finding whether surjectivity of $M'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})$ occurs without any ad hoc assumption. Unfortunately, this is not always possible and some appropriate assumptions may be required for specific optimal control problems [9,16].

Here we follow an approach proposed in [9] and derive the conditions for M' to be surjective.

Theorem 4.5. *Except for a countable set of values $(Re,Re_m)\subset\mathbb{R}^2$, the operator $M'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})$ is an isomorphism. Moreover, $M'(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})$ is an isomorphism whenever Re and Re_m are sufficiently small.*

Proof. The proof follows the same lines as in [9, Proposition 3.7]. The second result comes from a Neumann series argument. ☐

Remark 4.4. If the values of Re and Re_m are sufficiently small, the operator M' is onto and one may choose $a=1$ in the optimality system. This assumption is in agreement with the physical model of the Navier-Stokes equations in the absence of a turbulence model, which is known to be appropriate for small Reynolds numbers Re.

4.3 The optimality system

With the assumption of small Re and Re_m, one may choose $a=1$. By studying the stationary points of the augmented functional, one has a possible candidate for the local optimal solution.

Theorem 4.6 (Optimality system). *Let* $(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi},1,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)$ *be a stationary point of the Lagrangian functional* $\mathcal{J}_{aug}(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi},1,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)$. *The variables* $(\boldsymbol{\Phi},\pi_3)$ *in* $\boldsymbol{H}^1(\Omega)\times L^2_0(\Omega)$ *satisfy the system*

$$\begin{aligned}&\beta(\boldsymbol{\Phi},\delta\boldsymbol{\Phi})+\gamma a(\boldsymbol{\Phi},\delta\boldsymbol{\Phi})-S_1c(\delta\boldsymbol{\Phi},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{\lambda})-S_1c(\boldsymbol{Q}+\boldsymbol{\Phi},\delta\boldsymbol{\Phi},\boldsymbol{\lambda})\\&\qquad+\frac{1}{Re_m}a_m(\delta\boldsymbol{\Phi},\boldsymbol{\xi})+c(\boldsymbol{w}+\boldsymbol{\mu},\delta\boldsymbol{\Phi},\boldsymbol{\xi})-c(\delta\boldsymbol{\Phi},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{\xi})+d(\delta\boldsymbol{\Phi},\pi_3)=0, \qquad (4.11a)\\&d(\boldsymbol{\Phi},\delta\pi_3)=0 \qquad (4.11b)\end{aligned}$$

for all test functions $(\delta\boldsymbol{\Phi},\delta\pi_3)\in\boldsymbol{H}^1_{\Gamma_d}(\Omega)\times L^2_0(\Omega)$ *along with boundary conditions* $\boldsymbol{\Phi}=\boldsymbol{\Phi}_{0,d}$ *on* Γ_d. *The variables* $(\boldsymbol{\lambda},\pi_1)$ *in* $\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)$ *are solutions of*

$$\begin{aligned}&\frac{1}{Re}a(\delta\boldsymbol{w},\boldsymbol{\lambda})+c(\delta\boldsymbol{w},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{\lambda})+c(\boldsymbol{w}+\boldsymbol{\mu},\delta\boldsymbol{w},\boldsymbol{\lambda})+d(\delta\boldsymbol{w},\pi_1)+c(\delta\boldsymbol{w},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{\xi})\\&\qquad-c(\boldsymbol{Q}+\boldsymbol{\Phi},\delta\boldsymbol{w},\boldsymbol{\xi})+\alpha(\boldsymbol{w}+\boldsymbol{\mu}-\boldsymbol{u}_d,\delta\boldsymbol{w})=0, \qquad (4.12a)\\&d(\boldsymbol{\lambda},\delta r)=0, \qquad (4.12b)\end{aligned}$$

for all test functions $(\delta\boldsymbol{w},\delta r)\in\boldsymbol{H}^1_{\Gamma_1}(\Omega)\times L^2_1(\Omega)$. *The variables* $(\boldsymbol{\xi},\pi_2)$ *in* $\boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)$ *satisfy*

$$\begin{aligned}&\frac{1}{Re_m}a_m(\delta\boldsymbol{Q},\boldsymbol{\xi})+c(\boldsymbol{w}+\boldsymbol{\mu},\delta\boldsymbol{Q},\boldsymbol{\xi})-c(\delta\boldsymbol{Q},\boldsymbol{w}+\boldsymbol{\mu},\boldsymbol{\xi})-S_1c(\delta\boldsymbol{Q},\boldsymbol{Q}+\boldsymbol{\Phi},\boldsymbol{\lambda})\\&\qquad-S_1c(\boldsymbol{Q}+\boldsymbol{\Phi},\delta\boldsymbol{Q},\boldsymbol{\lambda})+d(\delta\boldsymbol{Q},\pi_2)=0, \qquad (4.13a)\\&d(\boldsymbol{\xi},\delta s)=0 \qquad (4.13b)\end{aligned}$$

for all test functions $(\delta\boldsymbol{Q},\delta s)\in\boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)$.

Proof. It suffices to set to zero the Fréchet differential of the augmented functional $\mathcal{J}_{aug}(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi},1,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)$ as in Theorem 4.4 and then split the independent variations. One has

$$\mathcal{J}'(\boldsymbol{w},\boldsymbol{\Phi})\cdot(\delta\boldsymbol{w},\delta\boldsymbol{\Phi})=\alpha(\boldsymbol{w}+\boldsymbol{\mu}-\boldsymbol{u}_d,\delta\boldsymbol{w})+\beta(\boldsymbol{\Phi},\delta\boldsymbol{\Phi})+\gamma a(\boldsymbol{\Phi},\delta\boldsymbol{\Phi}). \qquad (4.14)$$

Clearly, the variations $(\delta\boldsymbol{\lambda},\delta\pi_1,\delta\boldsymbol{\xi},\delta\pi_2,\delta\pi_3)$ with respect to the Lagrange multipliers yield the constraints defined by $M(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})=(\boldsymbol{f},0,\boldsymbol{0},0,0)$, i.e., the state equations and the divergence-free constraint (2.12) on the control variable $\boldsymbol{\Phi}$. For the other variables one can proceed in a standard way and obtain the corresponding Euler equations [9–11, 15, 17]. □

Remark 4.5. From the first-order necessary condition, we know that to every local optimal solution $(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi})$ there corresponds a stationary point $(\boldsymbol{w},r,\boldsymbol{Q},s,\boldsymbol{\Phi},1,\boldsymbol{\lambda},\pi_1,\boldsymbol{\xi},\pi_2,\pi_3)$ of the augmented functional which satisfies the optimality system. On the other hand, a solution of the optimality system may or may not correspond to a solution of the original optimal control problem. In order to ensure that, one would require a second-order sufficient condition. In this paper we choose to search for an optimal solution by numerically solving the optimality system and then by checking if the solution is a good candidate for solving the optimal control problem.

The optimality system is a very complex system and its numerical solution is a difficult and time-consuming task.

5 Numerical results

5.1 Finite element approximation of the optimality system

In this section, we approximate the optimality system by using a finite element method. We consider only conforming finite element approximations. Let $\boldsymbol{X}_h\subset\boldsymbol{H}^1(\Omega)$ and $S_h\subset L^2(\Omega)$ be two families of finite dimensional subspaces parametrized by h that tends to zero. We denote $\boldsymbol{X}_{h,0}=\boldsymbol{X}_h\cap\boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{X}_{h,\Gamma_s}=\boldsymbol{X}_h\cap\boldsymbol{H}_{\Gamma_s}^1(\Omega)$ for any non-empty subset $\Gamma_s\subset\Gamma$. We also set $S_{h0}=S_h\cap L_0^2(\Omega)$ and $S_{h1}=S_h\cap L_1^2(\Omega)$. The following assumptions on $\boldsymbol{X}_h$ and S_h are considered:

i) the *approximation hypotheses*: there exist an integer l and a constant C, independent of h, $\boldsymbol{u}$, and p, such that for $1\le k\le l$ we have

$$\inf_{\boldsymbol{u}_h\in\boldsymbol{X}_h}\|\boldsymbol{u}_h-\boldsymbol{u}\|_1\le Ch^k\|\boldsymbol{u}\|_{k+1},\quad \forall\boldsymbol{u}\in\boldsymbol{H}^{k+1}(\Omega)\cap\boldsymbol{H}^1(\Omega),$$
$$\inf_{p_h\in S_h}\|p-p_h\|\le Ch^k\|p\|_k,\qquad \forall p\in H^k(\Omega)\cap L_0^2(\Omega);$$

ii) the *inf-sup condition* or *LBB condition*: there exists a constant C', independent of h, such that [1, 8, 24]

$$\inf_{0\neq q_h\in S_{h0}}\sup_{0\neq\boldsymbol{u}_h\in\boldsymbol{X}_h}\frac{\int_\Omega q_h\nabla\cdot\boldsymbol{u}_h d\boldsymbol{x}}{\|\boldsymbol{u}_h\|_1\|q_h\|}\ge C'>0.$$

Next, let $\boldsymbol{P}_h=\gamma_0(\boldsymbol{X}_h)$, i.e., $\boldsymbol{P}_h$ consists of the restriction to the boundary Γ of functions $\boldsymbol{u}\in\boldsymbol{X}_h$. For all choices of conforming finite element spaces $\boldsymbol{X}_h$ we then have that $\boldsymbol{P}_h\subset\boldsymbol{H}^{-1/2}(\Gamma)$. For the subspaces $\boldsymbol{P}_h=\gamma_0(\boldsymbol{X}_h)$, we assume

iii) *boundary approximation property*: there exist an integer l and a constant C, independent of h and $\boldsymbol{s}$, such that for $1\leq k\leq l$ we have [4]

$$\inf_{\boldsymbol{s}_h\in\boldsymbol{P}_h}\|\boldsymbol{s}_h-\boldsymbol{s}\|_{-1/2,\Gamma}\leq Ch^k\|\boldsymbol{u}\|_{k-1/2},\quad \forall \boldsymbol{s}\in\boldsymbol{H}^{k-1/2}(\Gamma).$$

In order to solve the optimal control problem we must solve the optimality system in the variables $(\boldsymbol{w}_h,r_h,\boldsymbol{Q}_h,s_h,\boldsymbol{\Phi}_h,\boldsymbol{\lambda}_h,\pi_{1h},\boldsymbol{\xi}_h,\pi_{2h},\pi_{3h})$. As in the infinite-dimensional case, we can divide the discrete optimality system into three parts: the state system, the adjoint system and the control equation. The discrete state system for the state variables $(\boldsymbol{w}_h,r_h,\boldsymbol{Q}_h,s_h)$ can be written as

$$\begin{aligned}
&\frac{1}{Re}a(\boldsymbol{w}_h+\boldsymbol{\mu}_h,\boldsymbol{v}_{1h})+c(\boldsymbol{w}_h+\boldsymbol{\mu}_h,\boldsymbol{w}_h+\boldsymbol{\mu}_h,\boldsymbol{v}_{1h})-S_1c(\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\boldsymbol{v}_{1h})+d(\boldsymbol{v}_{1h},r_h)\\
&\qquad =<\boldsymbol{f}_h,\boldsymbol{v}_{1h}>-<\boldsymbol{t}_h,\boldsymbol{v}_{1h}>_{\Gamma\backslash\Gamma_1}, &&(5.1a)\\
&d(\boldsymbol{w}_h,q_{1h})=0, &&(5.1b)\\
&\frac{1}{Re_m}a_m(\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\boldsymbol{v}_{2h})+c(\boldsymbol{w}_h+\boldsymbol{\mu}_h,\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\boldsymbol{v}_{2h})\\
&\qquad -c(\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\boldsymbol{w}_h+\boldsymbol{\mu}_h,\boldsymbol{v}_{2h})+d(\boldsymbol{v}_{2h},s_h)=0, &&(5.1c)\\
&d(\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,q_{2h})=0 &&(5.1d)
\end{aligned}$$

for all test functions $(\boldsymbol{v}_{1h},q_{1h},\boldsymbol{v}_{2h},q_{2h})\in\boldsymbol{X}_{h,\Gamma_1}\times S_{h1}\times\boldsymbol{X}_{h,0}\times S_{h0}$ and boundary conditions

$$\boldsymbol{w}_h=\boldsymbol{0},\quad \text{on } \Gamma_1, \tag{5.2}$$
$$\boldsymbol{Q}_h=\boldsymbol{0},\quad \text{on } \Gamma. \tag{5.3}$$

The adjoint system, in $(\boldsymbol{\lambda}_h,\pi_{1h},\boldsymbol{\xi},\pi_{2h})$, can be written as

$$\begin{aligned}
&\frac{1}{Re}a(\delta\boldsymbol{w}_h,\boldsymbol{\lambda}_h)+c(\delta\boldsymbol{w}_h,\boldsymbol{w}_h+\boldsymbol{\mu}_h,\boldsymbol{\lambda}_h)+c(\boldsymbol{w}_h+\boldsymbol{\mu}_h,\delta\boldsymbol{w}_h,\boldsymbol{\lambda}_h)+d(\delta\boldsymbol{w}_h,\pi_{1h})\\
&\qquad +c(\delta\boldsymbol{w}_h,\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\boldsymbol{\xi}_h)-c(\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\delta\boldsymbol{w}_h,\boldsymbol{\xi}_h)+\alpha(\boldsymbol{w}_h+\boldsymbol{\mu}_h-\boldsymbol{u}_d,\delta\boldsymbol{w}_h)=0, &&(5.4a)\\
&d(\boldsymbol{\lambda}_h,\delta r_h)=0, &&(5.4b)\\
&\frac{1}{Re_m}a_m(\delta\boldsymbol{Q}_h,\boldsymbol{\xi}_h)+c(\boldsymbol{w}_h+\boldsymbol{\mu}_h,\delta\boldsymbol{Q}_h,\boldsymbol{\xi}_h)-c(\delta\boldsymbol{Q}_h,\boldsymbol{w}_h+\boldsymbol{\mu}_h,\boldsymbol{\xi}_h)\\
&\qquad -S_1c(\delta\boldsymbol{Q}_h,\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\boldsymbol{\lambda}_h)-S_1c(\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\delta\boldsymbol{Q}_h,\boldsymbol{\lambda}_h)+d(\delta\boldsymbol{Q}_h,\pi_{2h})=0, &&(5.4c)\\
&d(\boldsymbol{\xi}_h,\delta s_h)=0 &&(5.4d)
\end{aligned}$$

for all test functions $(\delta\boldsymbol{w}_h,\delta r_h,\delta\boldsymbol{Q}_h,\delta s_h)\in\boldsymbol{X}_{h,\Gamma_1}\times S_{h1}\times\boldsymbol{X}_{h,0}\times S_{h0}$ with boundary conditions $\boldsymbol{\lambda}_h=\boldsymbol{0}$ on Γ_1 and $\boldsymbol{\xi}_h=\boldsymbol{0}$ on Γ.

The control equations for the variables $(\boldsymbol{\Phi}_h,\pi_{3h})$ take the form

$$\begin{aligned}
&\beta(\boldsymbol{\Phi}_h,\delta\boldsymbol{\Phi}_h)+\gamma a(\boldsymbol{\Phi}_h,\delta\boldsymbol{\Phi}_h)+\frac{1}{Re_m}a_m(\boldsymbol{\xi}_h,\delta\boldsymbol{\Phi}_h)+c(\boldsymbol{w}_h+\boldsymbol{\mu}_h,\delta\boldsymbol{\Phi}_h,\boldsymbol{\xi}_h)-c(\delta\boldsymbol{\Phi}_h,\boldsymbol{w}_h+\boldsymbol{\mu}_h,\boldsymbol{\xi}_h)\\
&\qquad -S_1c(\delta\boldsymbol{\Phi}_h,\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\boldsymbol{\lambda}_h)-S_1c(\boldsymbol{Q}_h+\boldsymbol{\Phi}_h,\delta\boldsymbol{\Phi}_h,\boldsymbol{\lambda}_h)+d(\delta\boldsymbol{\Phi}_h,\pi_{3h})=0, &&(5.5a)\\
&d(\boldsymbol{\Phi}_h,\delta\pi_{3h})=0 &&(5.5b)
\end{aligned}$$

for all test functions $(\delta\mathbf{\Phi}_h,\delta\pi_{3h})\in\boldsymbol{X}_{h,\Gamma_d}\times S_{h0}$ and boundary conditions $\mathbf{\Phi}_h=\mathbf{\Phi}_{h0,d}$ on Γ_d. The optimal boundary control $\mathbf{\Phi}_{0h}$ for the magnetic field is then extracted directly as

$$\mathbf{\Phi}_{0h}=\gamma_0(\mathbf{\Phi}_h), \tag{5.6}$$

where γ_0 is the trace operator.

Remark 5.1. In order to obtain the discrete version of the optimality system, one could also choose to start with the finite element approximation of the state system and introduce discrete constraint operators M_h and N_h corresponding to M and N but acting on finite-dimensional spaces. Then, one can compute the Fréchet differentials of such operators on the finite-dimensional spaces. This possibility would bring some theoretical differences, for instance some proofs would become trivial in the finite-dimensional framework. Nevertheless, this issue is mainly a matter of taste from a theoretical point of view and the numerical approximations are expected to be similar. Our choice follows the so-called *differentiate-then-discretize* approach [13].

Remark 5.2. It is clear that the distributed control approach leads to an optimality condition that is defined on the domain Ω, whereas a standard boundary control approach such as that induced by (2.10) leads to an optimality condition defined only on the boundary control region Γ_c. Nevertheless, the additional computational cost of our distributed approach is well justified by the discussion in Remark 3.2.

5.2 Computational example

In this section we report the results about the numerical solution of the optimality system. The obtained solution is a candidate solution for the discrete optimal control problem. Let us consider a two-dimensional square channel $\Omega_h=[0,1]\times[0,1]$, discretized by a grid of 32×32 standard quadrilateral Taylor-Hood elements. The boundary conditions for a channel-like configuration are as follows. For the inlet and outlet sides $y=0$ and $y=1$ we enforce

$$\boldsymbol{u}\times\boldsymbol{n}=0, \qquad p=p_i, \qquad \text{on } y=0, \tag{5.7}$$

$$\boldsymbol{u}\times\boldsymbol{n}=0, \qquad p=p_0, \qquad \text{on } y=1, \tag{5.8}$$

where $p_i-p_0=1$ is the pressure jump driving the flow. We enforce no-slip boundary conditions on the wall sides $x=0$ and $x=1$

$$\boldsymbol{u}\cdot\boldsymbol{n}=0, \qquad \boldsymbol{u}\times\boldsymbol{n}=0, \qquad \text{on } x=0,\ \ x=1. \tag{5.9}$$

The target flow $\boldsymbol{u}_d$ is a desired constant velocity $\boldsymbol{u}_d=(0,0.075)$ on the target region $\Omega_{h1}=\{(x,y)|x\in[0.25,0.75],y\in[0.90625,1]\}$. The subregion Ω_{h1} for the desired velocity is shown in Fig. 1 together with the boundary conditions associated with the state variables

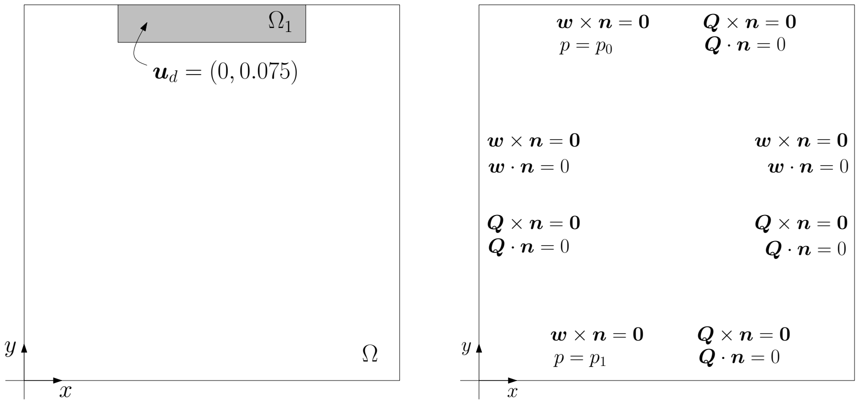

Figure 1: Computational domain Ω with target velocity region Ω_1 (left) and boundary conditions for state variables (right).

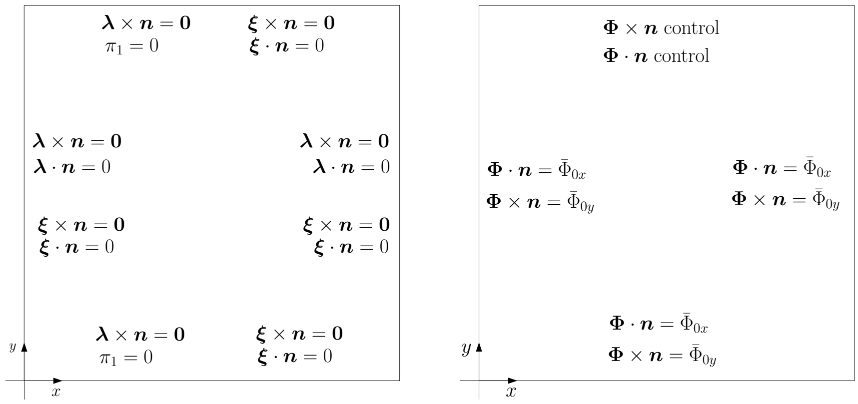

Figure 2: Boundary conditions for adjoint variables (left) and control variable (right).

$(\boldsymbol{w}_h, r_h, \boldsymbol{Q}_h, s_h)$. The boundary conditions for the adjoint variables $(\boldsymbol{\lambda}_h, \pi_{1h}, \boldsymbol{\xi}_h, \pi_{2h})$ and the control variables are shown in Fig. 2. We study the minimization of the functional

$$\mathcal{J}(\boldsymbol{u}_h, \boldsymbol{\Phi}_h) = \frac{\alpha}{2}\int_{\Omega_{h1}} (\boldsymbol{u}_h - \boldsymbol{u}_d)^2 d\boldsymbol{x} + \frac{1}{2}\int_{\Omega_h} (\nabla \boldsymbol{\Phi}_h)^2 d\boldsymbol{x}, \tag{5.10}$$

which is the discrete version of the objective functional (2.19) with $\beta = 0$ and $\gamma = 1$. In this case the integral of the velocity error is limited to the subregion Ω_{h1} of the domain Ω_h.

In the control portion of the boundary, Γ_c, the values of the boundary magnetic field change in order to minimize the objective functional. Indeed, various choices of the sides for the control portion Γ_c can be considered, which may depend on different issues like the position of the target region Ω_{h1}. Given our choice of the target region, we consider the control portion $\Gamma_c = \{(x,y) \in \Omega | y = 1\}$ for a first numerical investigation. The control is therefore the magnetic field $\boldsymbol{\Phi}$ on the outflow side. On the portion $\Gamma_d = \Gamma \backslash \Gamma_c$ a fixed value $\boldsymbol{\Phi}_h = \bar{\boldsymbol{\Phi}}_h$ is given.

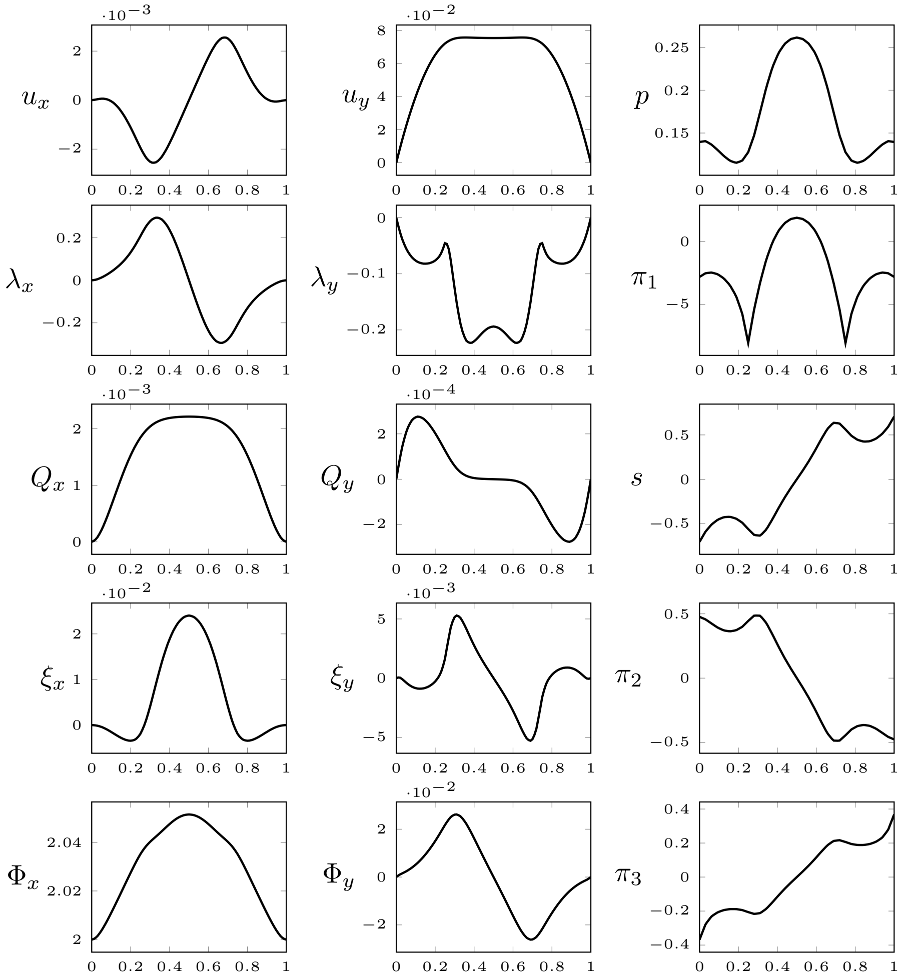

Figure 3: State, costate and control variables along the line $y=0.95$ with $\bar{\mathbf{\Phi}}_h=(2,0)$ and $\alpha=10^5$.

In Figs. 3 and 4 we show the state, adjoint and control variables along the lines $y=0.95$ and $x=0.5$ for $\bar{\mathbf{\Phi}}_h=(2,0)$ and $\alpha=10^5$.

In order to study the effect of the parameter α on the minimization of the functional, we plot in Fig. 5 the velocity profile along the line $y=0.95$ for the values $\alpha=0,10^3,10^4,10^5$. In Fig. 6 the boundary control $\mathbf{\Phi}$ on the line $y=1$ for $\alpha=0,10^3,10^4,10^5$ is shown. Notice that the case $\alpha=0$ corresponds to the well-known Poiseuille parabolic flow profile. For $\alpha=10^5$, the control and the controlled solutions converge to the profiles shown; the limiting behavior is not reported in these figures but it can be seen from the functional values; see Table 1 in which we report the values of the error $\int_{\Omega_{h1}}\|\boldsymbol{u}_h-\boldsymbol{u}_d\|^2 d\boldsymbol{x}$ for various α. It is evident from the figures and table that a higher value of the parameter α yields a more accurate control, as the velocity profile gets closer to the desired velocity $\boldsymbol{u}_d$ with increasing α.

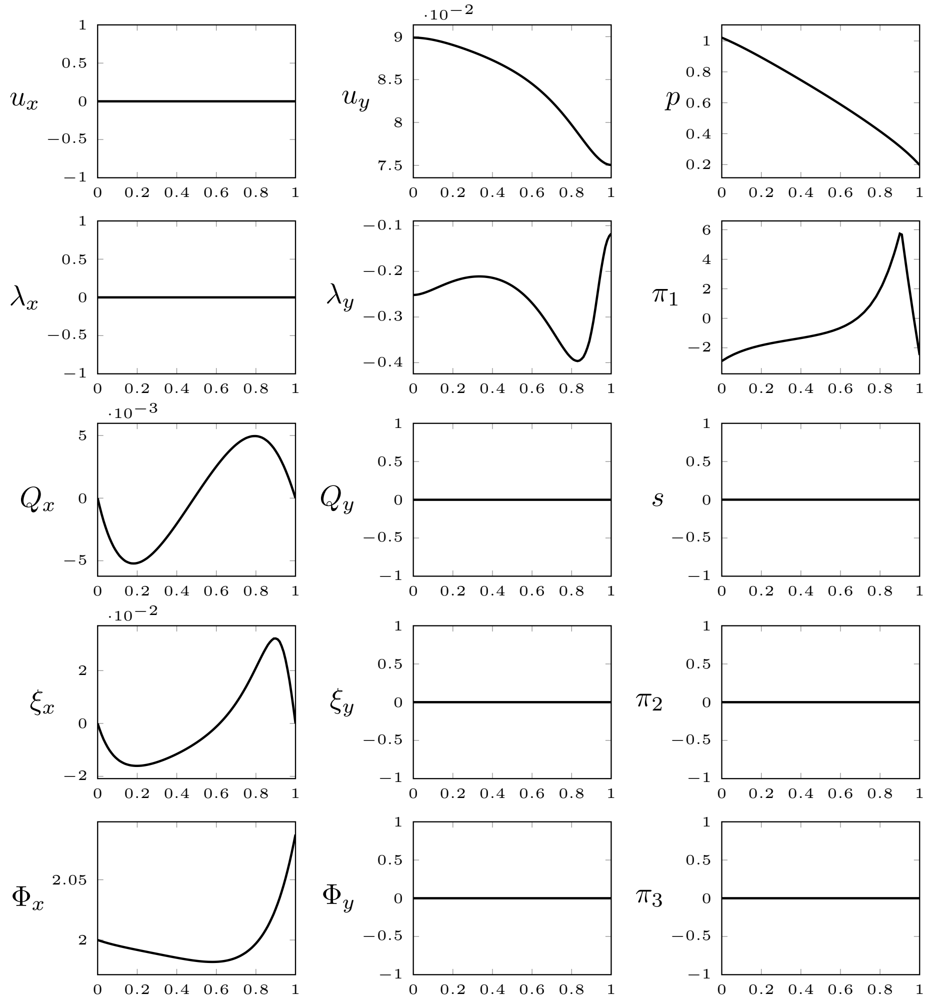

Figure 4: State, costate and control variables along the line $x=0.5$ with $\bar{\mathbf{\Phi}}_h=(2,0)$ and $\alpha=10^5$.

Table 1: Some values of the functional $F_0=\int_{\Omega_{h1}}\|\boldsymbol{u}_h-\boldsymbol{u}_d\|^2 d\boldsymbol{x}$ for different values of α.

α	F_0
0	$7.89865\cdot 10^{-5}$
10^3	$1.34407\cdot 10^{-6}$
10^4	$6.86789\cdot 10^{-7}$
$5\cdot 10^4$	$2.08805\cdot 10^{-7}$
$7.5\cdot 10^4$	$2.03182\cdot 10^{-7}$
10^5	$2.01474\cdot 10^{-7}$

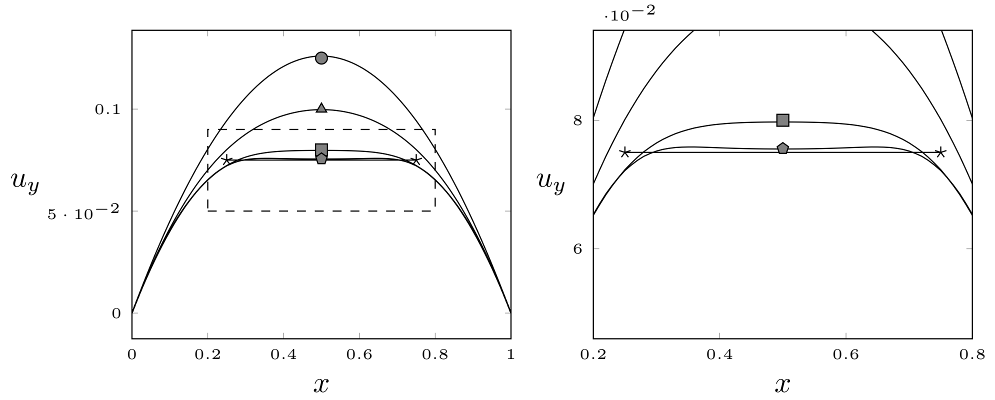

Figure 5: Left: velocity profile along the line $y=0.95$ with $\bar{\mathbf{\Phi}}_{0h}=(2,0)$ and $\alpha=0,10^3,10^4,10^5$ (circle, triangle, square, and pentagon marks, respectively) and comparison with the target (horizontal line between the two star marks). Right: a zoom on the target zone.

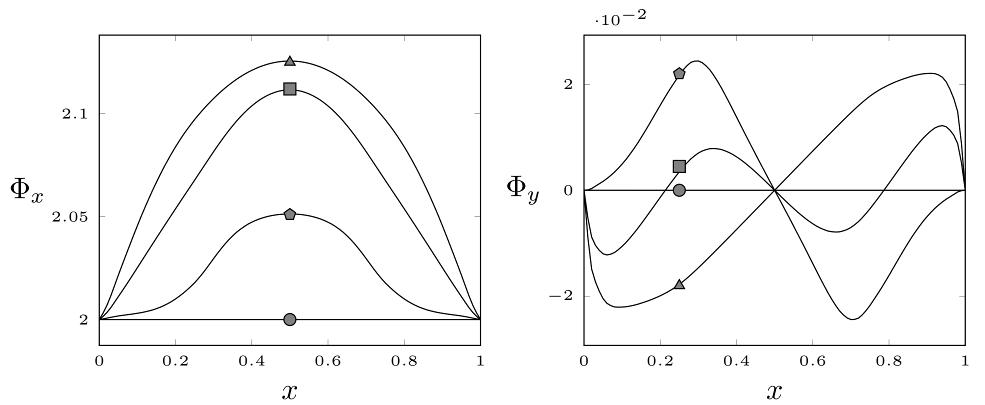

Figure 6: Boundary control functions $\mathbf{\Phi}_x$ (left) and $\mathbf{\Phi}_y$ (right) on $y=1$ for $\alpha=0$, 10^3, 10^4, and 10^5 (circle, triangle, square, and pentagon marks, respectively).

6 Conclusions

A new approach to the boundary optimal control of the incompressible steady MHD equations has been presented. With the introduction of the lifting function for the boundary conditions on the magnetic field, the boundary problem can be formulated as an extended distributed problem. We have formulated a weak form of the steady MHD equations whose existence can be proved without any condition on the data for the velocity and magnetic fields. This formulation can therefore take into account arbitrary nonhomogeneous values of the velocity on the boundary, so that channel flow problems can be studied. The Lagrange multiplier technique has been used to derive an optimality system whose solutions are candidate solutions for the optimal control problem. Numerical results of some computational tests show that a possible local minimum for the optimal control problem can be computed.

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