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REVIEW ARTICLE

Some Mathematical and Numerical Issues in Geophysical Fluid Dynamics and Climate Dynamics

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Abstract. In this article, we address both recent advances and open questions in some mathematical and computational issues in geophysical fluid dynamics (GFD) and climate dynamics. The main focus is on 1) the primitive equations (PEs) models and their related mathematical and computational issues, 2) climate variability, predictability and successive bifurcation, and 3) a new dynamical systems theory and its applications to GFD and climate dynamics.

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1 Introduction

The atmosphere and ocean around the earth are rotating geophysical fluids, which are also two important components of the climate system. The phenomena of the atmosphere and ocean are extremely rich in their organization and complexity, and a lot of them cannot be produced by laboratory experiments. The atmosphere or the ocean or the couple atmosphere and ocean can be viewed as an initial and boundary value problem (Bjerknes [5], Rossby [125], Phillips [120]), or an infinite dimensional dynamical system. These phenomena involve a broad range of temporal and spatial scales (Charney [11]). For example, according to J. von Neumann [147], the motion of the atmosphere can be divided into three categories depending on the time scale of the prediction. They are motions corresponding respectively to the short time, medium range and long term behavior of the atmosphere. The understanding of these complicated and scientific issues necessitate a joint effort of scientists in many fields. Also, as John von Neumann [147] pointed out, this difficult problem involves a combination of modeling, mathematical theory and scientific computing.

In this article, we shall address mathematical and numerical issues in geophysical fluid dynamics and climate dynamics. The main topics include:

1. issues on the modeling, mathematical analysis and numerical analysis of the primitive equation (PEs),
2. climate variability, predictability and successive bifurcation,
3. a new dynamical systems theory and its applications to geophysical fluid dynamics.

As we know, the atmosphere is a compressible fluid and the seawater is a slightly compressible fluid. The governing equations for either the atmosphere, or the ocean, or the coupled atmosphere-ocean models are the general equations of hydrodynamic equations together with other conservation laws for such quantities as the energy, humidity and salinity, and with proper boundary and interface conditions. Most general circulation models (GCMs) are based on the PEs, which are derived using the hydrostatic assumption in the vertical direction. This assumption is due to the smallness of the aspect ratio (between the vertical and horizontal length scales). We shall present a brief survey on recent theoretical and computational developments and future studies of the PEs.

One of the primary goals in climate dynamics is to document, through careful theoretical and numerical studies, the presence of climate low frequency variability, to verify the robustness of this variability's characteristics to changes in model parameters, and to help explain its physical mechanisms. The thorough understanding of this variability is a challenging problem with important practical implications for geophysical efforts to quantify predictability, analyze error growth in dynamical models, and develop efficient forecast methods. As examples, we discuss a few sources of variability, including wind-driven (horizontal) and thermohaline (vertical) circulations, El Niño-Southern Oscillation (ENSO), and Intraseasonal oscillations (ISO).

The study of the above geophysical problems involves on the one hand applications of the existing mathematical and computational theories to the understanding of the underlying physical problems, and on the other hand the development of new mathematical theories.

We shall present briefly a dynamic bifurcation and stability theory and its applications to GFD. This theory, developed recently by Ma and Wang [100], is for both finite and infinite dimensional dynamical systems, and is centered at a new notion of bifurcation, called attractor bifurcation. The theory is briefly described by a simple system of two ordinary differential equations, and by the classical Rayleigh-Bénard convection. Applications to the stratified Boussinesq equations model and the doubly-diffusive models are also addressed.

We would like to mention that there are many important issues not covered in this article, including, for example, the ocean and atmosphere data assimilation and prediction problems, and the stochastic-dynamics studies; see, among many others, [20, 30, 31, 33, 34, 38, 39, 56, 60–64, 108] and the references therein.

The article is organized as follows. In Section 2, some basic GFD models are introduced, with some mathematical and computational issues given in Section 3. Section 4 is on predictability, and Section 5 deals with issues on climate variability. Section 6 presents the new dynamical systems theory based on attractor bifurcation and its application to Rayleigh-Bénard convection and to GFD models.

2 Modeling

2.1 The primitive equations (PEs) of the atmosphere

Physical laws governing the motion and states of the atmosphere and ocean can be described by the general equations of hydrodynamics and thermodynamics. Using a non-inertial coordinate system rotating with the earth, these equations can be written as follows:

$$\begin{aligned}
&\frac{\partial \vec{V}}{\partial t}+\vec{V}\cdot\nabla_3\vec{V}+2\vec{\Omega}\times\vec{V}-\vec{g}+\frac{1}{\rho}\mathrm{grad}_3 p=\vec{D}_M,\\
&\frac{\partial \rho}{\partial t}+\mathrm{div}_3(\rho\vec{V})=0,\\
&c_v\frac{\partial T}{\partial t}+c_v\vec{V}\cdot\nabla_3 T+\frac{p}{\rho}\nabla_3\vec{V}=Q+D_H,\\
&\frac{\partial q}{\partial t}+\vec{V}\cdot\nabla_3 q=\frac{S}{\rho}+D_q,\\
&p=R\rho T.
\end{aligned} \tag{2.1}$$

Here the first equation is the momentum equation, the second is the continuity equation, the third is the first law of thermodynamics, the fourth is the diffusion equation for the humidity, and the last is the equation of state for an ideal gas. The unknown functions

are the three-dimensional velocity field $\vec{V}$, the density function ρ, the pressure function p, the temperature function T, and the specific humidity function q. Moreover, in the above equations, $\vec{\Omega}$ stands for the angular velocity of the earth, $\vec{g}$ the gravity, R the gas constant, c_v the specific heat at constant volume, $\vec{D}_M$ the viscosity terms, D_H the temperature diffusion, Q the heat flux per unit density at the unit time interval, which includes molecule or turbulent, radiative and evaporative heating, and S the differences of the rates of the evaporation and condensation.

These equations are normally far too complicated; simplifications from both the physical and mathematical points of view are necessary. There are essentially two characteristics of both the atmosphere and ocean, which are used in simplifying the equations. The first one is that for large scale geophysical flows, the ratio between the vertical and horizontal scales is very small; this leads to the primitive equations (PEs) of both the atmosphere and the ocean, which are the basic equations for these two fluids. More precisely, the PEs are obtained from the general equations of hydrodynamics and thermodynamics of the compressible atmosphere, by approximating the momentum equation in the vertical direction with the hydrostatic equation:

$$\frac{\partial p}{\partial z}=-\rho g. \tag{2.2}$$

This hydrostatic equation is based on the ratio between the vertical and horizontal scale being small. Here ρ is the density, g the gravitational constant, and $z=r-a$ height above the sea level, r the radial distance, and a the mean radius of the earth. Equation (2.2) expresses the fact that p is a decreasing function along the vertical so that one can use p instead of z as the vertical variable. Motivated by this hydrostatic approximation, we can introduce a generalized vertical coordinate system s-system given by

$$s=s(\theta,\varphi,z,t), \tag{2.3}$$

where s is a strict monotonic function of z. Then the basic equations of the large-scale atmospheric motion in the s-system are

$$\begin{aligned}
&\frac{\partial v}{\partial t}+v\cdot\nabla_s v+\dot{s}\frac{\partial v}{\partial s}+fk\times v+\frac{1}{\rho}\nabla_s z=D_M,\\
&\frac{\partial p}{\partial s}\frac{\partial s}{\partial z}+\rho g=0,\\
&\frac{\partial}{\partial s}\left(\frac{\partial p}{\partial t}\right)_s+\nabla_s\cdot\left(v\frac{\partial p}{\partial s}\right)+\frac{\partial}{\partial s}\left(\dot{s}\frac{\partial p}{\partial s}\right)=0,\\
&c_v\frac{\partial T}{\partial t}+c_v v\cdot\nabla_s T+c_v\dot{s}\frac{\partial T}{\partial s}+\frac{1}{\rho}\left(\frac{\partial p}{\partial t}+v\cdot\nabla_s p+\dot{s}\frac{\partial p}{\partial s}\right)=Q+D_H,\\
&\frac{\partial q}{\partial t}+v\cdot\nabla_s q+\dot{s}\frac{\partial q}{\partial s}=\frac{S}{\rho}+D_q.
\end{aligned} \tag{2.4}$$

Some common s-systems in meteorology are respectively the p-system (the pressure coordinate), the σ-system (the transformed pressure coordinate), the θ-system (the isentropic coordinate), and the ζ-system (the topographic coordinate or transformed height coordinate). The above PEs appear in the literature in e.g. the books of A. E. Gill [42], G. Haltiner and R. Williams [43], J. R. Holten [44], J. Pedlosky [118], J. P. Peixoto and A. H. Oort [119], W. M. Washington and C. L. Parkinson [152], Q. C. Zeng [159]. We remark here that sometimes the pressure coordinate is denoted by η, and the terrain-following by σ.

For simplicity, here we discuss only the case with the coordinate transformation from (θ,φ,z) to (θ,φ,p). The basic equations of the atmosphere are then the *Primitive Equations (PEs)* of the atmosphere in the p-coordinate system. As they appear in classical meteorology books (see e.g. [159] and Salby [128]), the PEs are given by

$$\begin{aligned}
&\frac{\partial v}{\partial t}+v\cdot\nabla v+\omega\frac{\partial v}{\partial p}+2\Omega\cos\theta k\times v+\nabla\Phi=D_M,\\
&\frac{\partial\Phi}{\partial p}+\frac{RT}{p}=0,\\
&\text{div } v+\frac{\partial\omega}{\partial p}=0,\\
&\frac{\partial T}{\partial t}+v\cdot\nabla T+\omega\left(\frac{\kappa T}{p}-\frac{\partial T}{\partial p}\right)=\frac{\tilde{Q}_{rad}}{c_p}+\frac{\tilde{Q}_{con}}{c_p}+D_H,\\
&\frac{\partial q}{\partial t}+v\cdot\nabla q+\omega\frac{\partial q}{\partial p}=E-C+D_q,
\end{aligned}\tag{2.5}$$

where D_M is the dissipation term for momentum and D_H and D_q are diffusion terms for heat and moisture, respectively, E and C are the rates of evaporation and condensation due to cloud processes, c_p the heat capacity, and $\tilde{Q}_{rad}$ and $\tilde{Q}_{con}$ the net radiative heating and the heating due to condensation processes, respectively. We use the pressure coordinate system (θ,φ,p), where θ $(0<\theta<\pi)$ and φ $(0<\varphi<2\pi)$ are the colatitude and longitude variables, and p the pressure of the air. The nondynamical processes $\tilde{Q}_{rad}$, $\tilde{Q}_{con}$, E and C are called model physics. Furthermore, the unknown functions are the horizontal velocity v, the vertical velocity $\omega=dp/dt$, the geopotential Φ, the temperature T, and the specific humidity q. The operators div and ∇ are the two dimensional operators on the sphere.

Of course, this set of equations is supplemented with a set of physically sound boundary conditions such as (3.3), depending on the specific form of the forcing and dissipation.

2.2 Ocean models

The sea water is almost an incompressible fluid, leading to the Boussinesq approximation, i.e., a variable density is only recognized in the buoyancy term and the equation of

state. The resulting equations are called the Boussinesq equations given as follows:

$$
\begin{aligned}
&\frac{\partial v}{\partial t}+\nabla_v v+w\frac{\partial v}{\partial z}+\frac{1}{\rho_0}\mathrm{grad}\rho+fk\times v-\mu\triangle v-\nu\frac{\partial^2 v}{\partial z^2}=0,\\
&\frac{\partial w}{\partial t}+\nabla_v w+w\frac{\partial w}{\partial z}+\frac{1}{\rho_0}\frac{\partial \rho}{\partial z}+\frac{\rho}{\rho_0}g-\mu\triangle w-\nu\frac{\partial^2 w}{\partial z^2}=0,\\
&\mathrm{div}v+\frac{\partial w}{\partial z}=0,\\
&\frac{\partial T}{\partial t}+\nabla_v T+w\frac{\partial T}{\partial z}-\mu_T\triangle T-\nu_T\frac{\partial^2 T}{\partial z^2}=0,\\
&\frac{\partial S}{\partial t}+\nabla_v S+w\frac{\partial S}{\partial z}-\mu_S\triangle S-\nu_S\frac{\partial^2 S}{\partial z^2}=0,\\
&\rho=\rho_0(1-\beta_T(T-\bar{T}_0)+\beta_S(S-\bar{S}_0)),
\end{aligned}
\tag{2.6}
$$

where v is the horizontal velocity field, w the vertical velocity, and S the salinity. The sixth equation in (2.6) is an empirical equation for the density function based on the linear approximation. In general, density ρ is a nonlinear function of T, S, and p. With higher approximations, one will encounter additional mathematical difficulties although the nonlinear equation of state is essential for some elements of ocean circulation (e.g., cabbeling).

As in the atmospheric case, the hydrostatic assumption is usually used, leading to the PEs for the large-scale ocean:

$$
\begin{aligned}
&\frac{\partial v}{\partial t}+\nabla_v v+w\frac{\partial v}{\partial z}+\frac{1}{\rho_0}\nabla p+fk\times v-\mu_v\Delta_a v-\nu_v\frac{\partial^2 v}{\partial z^2}=0,\\
&\frac{\partial p}{\partial z}=-\rho g,\\
&\mathrm{div}\ v+\frac{\partial w}{\partial z}=0,\\
&\frac{\partial T}{\partial t}+\nabla_v T+w\frac{\partial T}{\partial z}-\mu_T\Delta_a T-\nu_T\frac{\partial^2 T}{\partial z^2}=0,\\
&\frac{\partial S}{\partial t}+\nabla_v S+w\frac{\partial S}{\partial z}-\mu_s\Delta_a S-\nu\frac{\partial^2 S}{\partial z^2}=0,\\
&\rho=\rho_0(1-\beta_T(T-\bar{T}_0)+\beta_S(S-\bar{S}_0)).
\end{aligned}
\tag{2.7}
$$

Also, we note that if the hydrostatic assumption is made first, the Boussinesq approximation is not really necessary (see e.g. (2.5) – divergence-free! – or, e.g., de Szoeke and Samelson [19].

2.3 Coupled atmosphere-ocean models

Coupled atmosphere and ocean models consist of 1) models for the atmosphere component, 2) models for the ocean component, and 3) interface conditions. The interface con-

ditions are used to couple the atmosphere and ocean systems, and are usually derived based on first principles; see Gill [42], Washington and Parkinson [152]. A mathematically well-posed coupled model with physically sound interface conditions and the PEs of the atmosphere and the ocean are given in Lions, Temam and Wang [85]. We refer interested readers to these references for further studies.

As we know, the atmosphere and ocean components have quite different time scales, leading to complicated dynamics. For example, from the computational point of view, one needs to incorporate the two time scales; see e.g. [88].

3 Some theoretical and computational issues for the PEs

3.1 Dynamical systems perspective of the models

From the mathematical point of view, we can put the models addressed in the previous section in the perspective of infinite-dimensional dynamical systems as follows:

$$\begin{aligned} &\varphi_t + A\varphi + R(\varphi) = F, \\ &\varphi|_{t=0} = \varphi_0, \end{aligned} \tag{3.1}$$

defined on an infinite-dimensional phase space H. Here $A:H\to H$ is an unbounded linear operator, $R:H\to H$ is a nonlinear operator, F is the forcing term, and φ_0 is the initial data.

We remark here that the linear operator can usually be written as $A=A_1+A_2$, where A_1 stands for the irreversible diabatic linear processes of energy dissipation, and A_2 for the reversible adiabatic linear processes of energy conversation. The nonlinear term $R(\varphi)$ represents the reversible adiabatic nonlinear processes of energy conversation. The properties of these operators reflect directly the essential characteristics of two kinds of basic processes with entirely different physical meanings.

The above formulation is often achieved by (a) establishing a proper functional setting of the model, and (b) proving the existence and uniqueness of the solutions.

Hereafter we demonstrate the procedure with the PEs. Due to some technical reasons, some minor and physically reasonable modifications of the PEs are made. In particular, we assume that the model physics $\tilde{Q}_{rad}$, $\tilde{Q}_{con}$, E and C are given functions of location and time. We specify also the viscosity, diffusion terms as (see among others Lions et al. [83] and Chou [16]):

$$\begin{aligned} &D_M = -L_1 v, \\ &\frac{\tilde{Q}_{rad}}{c_p} + \frac{\tilde{Q}_{con}}{c_p} + D_H = -L_2 T + Q_T, \\ &E - C + D_q = -L_3 q + Q_q, \\ &L_i = -\mu_i \Delta - \nu_i \frac{\partial}{\partial p}\left(\left(\frac{gp}{R\bar{T}(p)}\right)^2 \frac{\partial}{\partial p}\right), \end{aligned} \tag{3.2}$$

where μ_i,ν_i are horizontal and vertical viscosity and diffusion coefficients, Δ is the Laplace operator on the sphere, Q_T and Q_q are treated as given functions, and $\bar{T}(p)$ a given temperature profile, which can be considered as the climate average of T. The boundary conditions for the PEs are given by

$$\begin{aligned}
&\frac{\partial}{\partial p}(v,T,q)=(\tilde{\gamma}_s(\tilde{v}_s-v),\quad \tilde{\alpha}_s(\tilde{T}_s-T),\quad \tilde{\alpha}_q(\tilde{q}_s-q)),\quad \omega=0 \quad \text{at } p=P,\\
&\frac{\partial}{\partial p}(v,T,q)=0,\quad \omega=0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{at } p=p_0.
\end{aligned} \tag{3.3}$$

The second and third equations (2.5) are diagnostic ones; integrating them in p-direction, we obtain

$$\begin{aligned}
&\int_{p_0}^{P} \operatorname{div} v(p')dp'=0,\\
&\omega=W(v)=-\int_{p_0}^{p} \operatorname{div} v(p')dp',\\
&\Phi=\Phi_s+\int_{p}^{P}\frac{RT(p')}{p'}dp'.
\end{aligned} \tag{3.4}$$

Then the PEs are equivalent to the following functional formulation:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\Lambda(v)u+P(u)+Lu+(\nabla\Phi_s,0,0)=(0,Q_T,Q_q),\\
&\operatorname{div}\int_{p_0}^{P} vdp=0,
\end{aligned} \tag{3.5}$$

where

$$u=(v,T,q),\quad \Lambda(v)u=\nabla_v u+W(v)\frac{\partial u}{\partial p},$$

Lu corresponds to the viscosity and diffusion terms, and Pu the lower order terms. The above new formulation was first introduced by Lions, Temam and Wang [83].

We solve then the PEs in some infinite-dimensional phase spaces H and V. In particular, we use

$$H_1=\left\{v\in L^2 \mid \operatorname{div}\int_{p_0}^{P} vdz=0\right\}$$

as the phase space for the horizontal velocity v. Then we project in the phase space. Using this projection, the unknown function Φ_s plays a role as a Lagrangian multiplier, which can be recovered by the following decomposition:

$$\begin{aligned}
&L^2=H_1\oplus H_1^{\perp},\\
&H_1^{\perp}=\left\{v\in L^2 \mid v=\nabla\Phi_s,\Phi_s\in H^1(S_a^2)\right\}.
\end{aligned}$$

Then the PEs are equivalent to an infinite-dimensional dynamical system as (3.1).

With the above formulation, for example, we encounter the following new nonlocal Stokes problem:

$$\begin{aligned} &-\triangle v+\nabla\Phi_s=f,\\ &\text{div}\int_{p_0}^{P} v dp=0, \end{aligned} \tag{3.6}$$

supplemented with suitable boundary conditions. From the mathematical point of view, all techniques for the regularity of solutions are local. But our problem here is nonlocal; the regularity of the solutions for this problem can be obtained using Nirenberg's finite difference quotient method; see [83].

Other models such as the PEs of the ocean and the coupled atmosphere-ocean models can be viewed as infinite-dimensional dynamical systems in the form of (3.1). We refer the interested readers to Lions, Temam and Wang [84] for the PEs of the ocean, and [85,87] for the coupled atmosphere-ocean models.

3.2 Well-posedness

One of the first mathematical questions is the existence, uniqueness and regularity of solutions of the models. The main results in this direction can be briefly summarized as follows, and we refer the interested readers to the related references given below for more details:

For the PEs of the large-scale atmosphere, the existence of global (in time) weak solutions of the primitive equations for the atmosphere is obtained by Lions, Temam and Wang [83], where the new formulation described above is introduced.

In fact, the key ingredient for most, if not all, existence results for the PEs of the atmosphere-only, the ocean-only, or the coupled atmosphere-ocean depend heavily on the formulation (3.4) and (3.5), first introduced by Lions, Temam and Wang [83]. We note that without using this new formulation, one can also obtain the existence of global weak solutions by introducing proper function spaces with more regularity in the p-direction; see Wang [150]. However, the new formulation is important for viewing the PEs as an infinite-dimensional system.

The existence of global strong solutions is first obtained for small data and large time or for short time by [141, 150]. Other studies include the case for thin domain case [48] (see also Ifitimie and Raugel [50] for discussions on the Navier-Stokes equations on thin domains), and for fast rotation [1].

Recently, for the primitive equations of the ocean with free top and bottom boundary conditions, an existence of long time strong solutions with general data is obtained by Cao and Titi [8], using the new formulation and the fact that the surface pressure depends on the two spatial directions, and then similar result is obtained for the Dirichlet boundary conditions by Kukavica and Ziane [65]. Furthermore, Ju [54] studied the global attractor for the primitive equations.

The corresponding results can also be obtained for the PEs of the ocean; see [8, 84,

141] and the references therein for more details. Existence of global weak solutions were obtained for the coupled atmosphere-ocean model introduced in [87].

As mentioned earlier, the hydrostatic assumption and its related formulation given by (3.5) and (3.6) are crucial in many of the existence results for both strong and weak solutions. We would like to point out that Laprise [68] suggests that hydrostatic-pressure coordinates could be used advantageously in nonhydrostatic atmospheric models based on the fully compressible equations. We believe that with the Laprise' formulation, one can extend some of the results discussed here to non-hydrostatic cases.

Another situation where the new formulation (3.5) and (3.6) does not appear to be available is related to more complex low boundary conditions. Some partial results for weak solutions are obtained in [154], and apparently many related issues are still open.

3.3 Long-time dynamics and nonlinear adjustment process

Regarding the PEs as an infinite dimensional dynamical system, the existence and finite dimensionality of the global attractors of the PEs with vertical diffusion were explored in [82–84]. The finite dimensionality of the global attractor of the PEs provides a mathematical foundation that the infinite-dynamical system can be described by a finite-dimensional dynamical system.

As we know all general circulation models of either the atmosphere-only, or the ocean-only, or the coupled ocean-atmosphere systems are based on the PEs with more detailed model physics. In both these GCM models and theoretical climate studies, the PEs are often replaced by a set of truncated ordinary differential equations (ODEs), whose asymptotic solution sets, called attractors, can be investigated in the more tractable setting of a finite-dimensional phase space, without seriously altering the essential dynamics. It is, however, not known mathematically whether this truncation is really reasonable, and moreover, how we can determine *a priori* which finite-dimensional truncations are sufficient to capture the essential features of the atmosphere or the oceans. With this objective, nonlinear adjustment process associated with the long-time dynamics of the models were carefully conducted in a series of papers by Li and Chou [71–78]; see also [82, 151].

An important consequence of the above mentioned results for the long-time dynamics is that the nonlinear adjustment process of the climate is a forced, dissipative, nonlinear system to external forcing. This nonlinear adjustment process is different from the adjustments in the traditional dynamical meteorology, including the geostrophic adjustment, rotational adjustment, potential vorticity adjustment, static adjustment, etc.

These traditional adjustments do not appear to be associated with any attracting properties that attractors usually process. It is indicated from the nonlinear adjustment process that as time increases, the information carried by the initial state will gradually be lost. In addition, there are three categories of time boundary layers (TBL) and the self-similar structure of the TBL in the adjustment and evolution processes of the forced, dissipative, nonlinear system.

An important question is how to determine the structure of the attractors and the distribution of their attractive domains under the given conditions of known external forcing. Although the information of initial state will be decayed as the time increases, it does not mean that the information of initial state is not important to long-time dynamics. The quantity of those initial values which locate very close to the boundary between two different attractive domains are very important since their local asymptotic behavior will be quite different due to slight initial error. Another open question is how to construct independent orthogonal basis of the attractor in practice based on the finite dimensionality of the global attractor. Two empirical approaches used at present are the Principal Analysis (PC) or Empirical Orthogonal Function (EOF) and Singular Vector Decomposition (SVD) based on the time series of numerical solutions or observational filed data. They are available although they are empirical. Qiu and Chou [122] and Wang et al. [149] made a valuable attempt to apply the long-time dynamics of the atmosphere mentioned above and this two empirical decomposition methods for orthogonal bases of the attractor to study 4-dimensional data assimilation.

3.4 Multiscale asymptotics and simplified models

As practiced by the earlier workers in this field such as J. Charney and John von Neumann, and from the lessons learned by the failure of Richardson's pioneering work, one tries to be satisfied with simplified models approximating the actual motions to a greater or lesser degree instead of attempting to deal with the atmosphere in all its complexity. By starting with models incorporating only what are thought to be the most important of atmospheric influences, and by gradually bringing in others, one is able to proceed inductively and thereby to avoid the pitfalls inevitably encountered when a great many poorly understood factors are introduced all at once.

One of the dominant features of both the atmosphere and the ocean is the influence of the rotation of the earth. The emphasis on the importance of the rotation effects of the earth and their study should be traced back to the work of P. Laplace in the eighteenth century. The question of how a fluid adjusts in a uniformly rotating system was not completely discussed until the time of C. G. Rossby [126], when Rossby considered the process of adjustment to the geostrophic equilibrium. This process is now referred to as the *Rossby adjustment*. Roughly speaking the Rossby adjustment process explains why the atmosphere and ocean are always close to geostrophic equilibrium, for if any force tries to upset such an equilibrium, the gravitational restoring force quickly restores a near geostrophic equilibrium. Later in the 1940's, under the famous β-plane assumption, Charney [13] introduced the quasi-geostrophic (QG) equations for the *large-scale* (with horizontal scale comparable to 1000 *km*) mid-latitude atmosphere. Since then, there have been many studies from both the physical and numerical points of view. This model has been the main driving force of the much development of theoretical meteorology and oceanography.

Mathematically speaking, the QG theory is based on asymptotics in terms of a small

parameter, called Rossby number Ro, a dimensionless number relating the ratio of inertial force to Coriolis force for a given flow of a rotating fluid. The key idea in the geostrophic asymptotics, leading to the QG equations, is to approximate the spherical midlatitude region by the tangent plane, called the β-plane, at the center of the region, and to express the Coriolis parameter in terms of the Rossby number.

Thanks in particular to the vision and effort of Professor J. L. Lions, there have been extensive mathematical studies. As this topic is very well-received and studied by applied mathematicians, we do not go into details, and the interested readers are refereed to (Lions, Temam & Wang [86, 89], Bourgeois & Beale [6], Babin, Mahalov & Nicolaenko [1], Gallagher [29], Embid & Majda [25]) and the references therein. In addition, planetary geostrophic equations (PGEs) of ocean have also received a lot of attention recently following the early work by Samelson, Temam and Wang [130, 131]. However, many issues are still open.

As mentioned earlier, many geophysical processes have multiscale characteristics. One aspect of the studies requires a careful examination of the interactions of multiple temporal and spatial scales. A combination of rigorous mathematics and physical modeling together with scientific computing appear to be crucial for the understanding of these multiscale physical processes. ENSO and ISO are two such examples (see also Section 5).

3.5 Some computational issues

On the one hand, we need to develop more efficient numerical methods for general circulation models (GCMs), including atmospheric general circulation model (AGCM), oceanic general circulation model (OGCM) and coupled general circulation model (CGCM), climate system model and earth system model. On the other hand, numerical simulations are used to test the theoretical results obtained, as well as for preliminary exploration of the phenomena apparent in the governing PDEs, and to obtain guidance on the most interesting directions for theoretical studies.

Here we simply address some computational issues without discussing physical processes, which are certainly important in developing GCMs. Two basic discretization schemes commonly used in GCMs are grid-point and spectral approaches. There are many crucial issues which are not fully resolved, including 1) spherical geometry and singularity near the polar regions, 2) irregular domains, 3) multiscale (spatial and temporal) problems involving both fast and slow processes, 4) vertical stratification, 5) sub-grid processes, 6) model bias, and 7) nonhydrostatic models, etc. We note that although many existing models are formulated and solved in spherical coordinates, numerical and mathematical difficulties caused by the dependence of the meshes size on the latitude is still not fully resolved. Furthermore, high-precision computation and very fine resolutions are two tendencies in developing numerical models.

Recently a fast and efficient spectral method for the PEs of the atmosphere is introduced by Shen and Wang [133], and further studies for this method applied to more practical GCMs are needed. Another paper uses heavily the surface pressure formulation of

the PEs for the ocean is given in Samelson et al. [129]. Its incorporation to OGCMs, and its simulation for studying specific oceanic phenomena appear to be necessary.

A remarkable, but neglected, problem is on influences of round-off error on long time numerical integrations since round-off error can cause numerical uncertainty. Owing to the inherent relationship between the two uncertainties due to numerical method and finite precision of computer respectively, a computational uncertainty principle (CUP) could be definitely existed in numerical nonlinear systems (Li et al. [80, 81]), which implies a certain limitation to the computational capacity of numerical methods under the inherent property of finite machine precision. In practice, how to define the optimal time step size and optimal horizontal and vertical resolutions of a numerical model to obtain the best degree of accuracy of numerical solutions and the best simulation and prediction is an important and urgent computational issue.

4 Nonlinear error growth dynamics and predictability

For a complex nonlinear chaotic system such as the atmosphere or the climate the intrinsic randomness in the system sets a theoretical limit to its predictability; see Lorenz [90, 92]. Beyond the predictability limit, the system becomes unpredictable. In the studies of predictability, the Lyapunov stability theory has been used to determine the predictability limit of a nonlinear dynamical system. The Lyapunov exponents give a basic measure of the mean divergence or convergence rates of nearby trajectories on a strange attractor, and therefore may be used to study the mean predictability of chaotic system; see Eckmann and Ruelle [24], Wolf et al. [153] and Fraedrich [28]. Recently, local or finite-time Lyapunov exponents have been defined for a prescribed finite-time interval to study the local dynamics on an attractor Kazantsev [57], Ziehmann et al. [161] and Yoden and Nomura [155]. However, the existing local or finite-time Lyapunov exponents, which are same as the global Lyapunov exponent, are established on the basis of the fact that the initial perturbations are sufficiently small such that the evolution of them can be governed approximately by the tangent linear model (TLM) of the nonlinear model, which essentially belongs to linear error growth dynamics. Clearly, as long as an uncertainty remains infinitesimal in the framework of linear error growth dynamics it cannot pose a limit to predictability. To determine the limit of predictability, any proposed local Lyapunov exponent must be defined with the respect to the nonlinear behaviors of nonlinear dynamical systems Lacarra and Talagrand [67] and Mu [114].

In view of the limitations of linear error growth dynamics, it is necessary to propose a new approach based on nonlinear error growth dynamics for quantifying the predictability of chaotic systems. Ding and Li [22] and Li et al. [79] have presented a nonlinear error growth dynamics which applies fully nonlinear growth equations of nonlinear dynamical systems instead of linear approximation to error growth equations to discuss the evolution of initial perturbations and employed it to study predictability.

For an n-dimensional nonlinear system, the dynamics of small initial perturbation

$\delta_0=\delta(t_0)\in\mathbb{R}^n$ about an initial point $x_0=x(t_0)$ in the n-dimensional phase space are governed by the nonlinear propagator $\eta(x_0,\delta_0,\tau)$, which propagates the initial error forward to the error at the time $t=t_0+\tau$:

$$\delta(t_0+\tau)=\eta(x_0,\delta_0,\tau)\delta_0.$$

Then the nonlinear local Lyapunov exponent (NLLE) is defined by

$$\lambda_1(x_0,\delta_0,\tau)=\frac{1}{\tau}\ln\frac{\|\delta(t_0+\tau)\|}{\|\delta_0\|}.$$

This indicates that $\lambda_1(x_0,\delta_0,\tau)$ depends generally on the initial state x_0 in the phase space, the initial error δ_0, and evolution time τ. The NLLE is quite different from the global Lyapunov exponent (GLE) or the local Lyapunov exponent (LLE) based on linear error dynamics. If we study the average predictability of the whole system, the whole ensemble mean of the NLLE, $\bar{\lambda}_1(\delta_0,\tau)=<\lambda_1(x_0,\delta_0,\tau)>$ should be introduced, where the symbol $<\cdot>$ denotes the ensemble average. Then the average predictability limit of a chaotic system could be quantitatively determined using the evolution of the mean relative growth of the initial error (RGIE)

$$\bar{E}(\delta_0,\tau)=\exp(\bar{\lambda}_1(\delta_0,\tau)\tau).$$

According to the chaotic dynamical system theory and probability theory, the saturation theorem of RGIE [22] may be obtained as follows.

Theorem 4.1. [22] *For a chaotic dynamic system, the mean relative growth of the initial error (RGIE) will necessarily reach a saturation value in a finite time interval.*

Once the RGIE reaches the saturation, at the moment almost all predictability of chaotic dynamic systems is lost. Therefore, the predictability limit can be defined as the time at which the RGIE reaches its saturation level.

If the first NLLE $\lambda_1(x_0,\delta_0,\tau)$ along the most rapidly growing direction has been obtained, for an n-dimensional nonlinear dynamic system the first m NLLE spectra along other orthogonal directions can be successively determined by the growth rate of the volume V_m of an m-dimensional subspace spanned by the m initial error vectors $\delta_m(t_0)=(\delta_1(t_0),\cdots,\delta_m(t_0)$:

$$\sum_{i=1}^{m}\lambda_i=\frac{1}{\tau}\ln\frac{V_m(\delta_m(t_0+\tau))}{V_m(\delta_m(t_0))},$$

for $m=2,3,\cdots,n$, where $\lambda_i=\lambda_i(x_0,\delta_0,\tau)$ is the i-th NLLE of the dynamical system. Correspondingly the whole ensemble mean of the i-th NLLE is defined as

$$\bar{\lambda}_i(\delta_0,\tau)=<\lambda_i(x_0,\delta_0,\tau)>.$$

For a chaotic system, each error vector tends to fall along the local direction of the most rapid growth. Due to the finite precision of the computer, the collapse toward a

common direction causes the orientation of all error vectors to become indistinguishable. This problem can be overcome by the repeated use of the Gram-Schmidt reorthogonalization (GSR) procedure on the vector frame. Giving a set of error vectors $\{\delta_1,\cdots,\delta_n\}$, the GSR provides the following orthogonal set $\{\delta'_1,\cdots,\delta'_n\}$:

$$\begin{aligned}
&\delta'_1=\delta_1,\\
&\delta'_2=\delta_2-\frac{(\delta_2,\delta'_2)}{(\delta'_1,\delta'_1)}\delta'_1,\\
&\cdots\\
&\delta'_n=\delta_n-\frac{(\delta_n,\delta'_{n-1})}{(\delta'_{n-1},\delta'_{n-1})}\delta'_{n-1}-\cdots-\frac{(\delta_n,\delta'_1)}{(\delta'_1,\delta'_1)}\delta'_1.
\end{aligned}$$

The growth rate of the m-dimensional volume can be calculated by the use of the first m orthogonal error vectors, and then the first m NLLE spectra can be obtained correspondingly.

On the other hand, we introduce the local ensemble mean of the NLLE in order to measure predictability of specified state with certain initial uncertainties in the phase space and to investigate distribution of predictability limit in the phase space [22]. Assuming that all initial perturbations with the amplitude and random directions are on a spherical surface centered at an initial point x_0, the local ensemble mean of the NLLE relative to x_0 is defined as

$$\bar{\lambda}_L(x_0,\tau)=<\lambda(x_0,\varepsilon,\tau)>_N \quad \text{for } N\to\infty,$$

and then the local average predictability limit of a chaotic system at the point x_0 could be quantitatively determined by examining the evolution of the local mean relative growth of initial error (LRGIE) $\bar{E}(x_0,\tau)=\exp(\bar{\lambda}_L(x_0,\tau)\tau)$. The local ensemble mean of the NLLE different from the whole ensemble mean of the NLLE could show local error growth dynamics of subspace on an attractor in the phase space. Moreover, in practice the local average predictability limit itself might be regarded as a predict and to provide an estimation of accuracy of prediction results.

The nonlinear error growth theory mentioned above provides a new idea for predictability study. However, a great deal of work, including the theory itself, is needed. For a real system such as the atmosphere and ocean, the further studies related to the following questions are needed:

1. Quantitative estimates of the temporal-spatial characteristics of the predictability limit of different variables of the atmosphere and ocean by use of observation data (Chen et al. [14] made a preliminary attempt to this aspect).
2. Relationships among the predictability limits of motion on various time and space scales.
3. Disclosure of the mechanisms influencing predictability from the view of nonlinear error growth dynamics.

4. Predictability limit varying with changes of initial perturbations.
5. Decadal change of the predictability limit.
6. Predictability of extreme events.
7. Prediction of predictability.

5 Climate variability and successive bifurcation

Understanding climate variability and related physical mechanisms and their applications to climate prediction and projection are the primary goals in the study of climate dynamics. One of problems of climate variability research is to understand and predict the periodic, quasi-periodic, aperiodic, and fully turbulent characteristics of large-scale atmospheric and oceanic flows. Bifurcation theory enables one to determine how qualitatively different flow regimes appear and disappear as control parameters vary; it provides us, therefore, with an important method to explore the theoretical limits of predicting these flow regimes.

For this purpose, the ideas of dynamical systems theory and nonlinear functional analysis have been applied so far to climate dynamics mainly by careful numerical studies. These were pioneered by Lorenz [90,91], Stommel [139], and Veronis [144,146] among others, who explored the bifurcation structure of low-order models of atmospheric and oceanic flows.

Recently, pseudo-arclength continuation methods have been applied to atmospheric (Legras and Ghil [70]) and oceanic (Speich et al. [137] and Dijkstra [21]) models with increasing horizontal resolution. These numerical bifurcation studies have produced so far fairly reliable results for two classes of geophysical flows: (i) atmospheric flows in a periodic mid-latitude channel, in the presence of bottom topography and a forcing jet; and (ii) oceanic flows in a rectangular mid-latitude basin, subject to wind stress on its upper surface; see among others Charney and DeVore [12], Pedlosky [117], Legras and Ghil [70] and Jin and Ghil [53] for saddle-node and Hopf bifurcations in the atmospheric channel, and [9, 49, 55, 110–112, 115, 127, 134, 135, 137] for saddle-node, pitchfork or Hopf or global bifurcation in the oceanic basin.

Apparently, further numerical bifurcation studies are inevitably necessary. Typical problems include 1) continuation algorithms (pseudo-arclength methods and stability analysis) applied to large-dimensional dynamical systems (discretized PDEs), 2) Galerkin approach using finite-element discretization together with 'homotopic' meshes that can deform continuously from a domain to another.

Another important further direction is to rigorously conduct bifurcation and stability analysis for the original partial differential equations models associated with typical phenomena. Some progresses have been made in this direction; see among others [15,45–47]. It is clear that much more effort is needed; see also Section 6 below.

Furthermore, very little is known for theoretical and numerical investigations on the bifurcations of coupled systems, which are of practical significance for the coupled dy-

namics. It is also practically important to study the variability and dynamics of systems under varying external forcing, e.g., the features, processes and dynamics of weather and climate varies with the global warming.

Hereafter in next few subsections, we present some issues on a few specific physical phenomena.

5.1 Wind-driven and thermohaline circulations

For the ocean, basin-scale motion is dominated by wind-driven (horizontal) and thermohaline (vertical) circulations. Their variability, independently and interactively, may play a significant role in climate changes, past and future. The wind-driven circulation plays a role mostly in the oceans' subannual-to-interannual variability, while the thermohaline circulation is most important in decadal-to-millenial variability.

The thermohaline circulation (THC) is highly nonlinear due to the combined effects of the temperature and the salinity on density (Meinckel et al. [113]; Rahmstorf [124]), which cause the existence of multiple equilibria and thresholds in the THC. The abrupt climate is related to the shift between multiple equilibria flow regimes in the THC. The sensitivity of the THC to anthropogenic climate forcing is still an open question (Rahmstorf [124]; Meincke et al. [113]; Thorpe et al. [142]). This is closely related to the question on whether an abrupt breakdown of the THC can result from global warming. In particular, there are two different but connected the stability and transitions associated with the problem. The first is stability and transitions of the solutions of the partial differential equation models in the phase space, and the second is the structure of the solutions and its transitions in the physical spaces. Issues related to these transitions appear to be very important. One such example is the western boundary current separation. The physics of the separation of western boundary currents is a long standing problem in physical oceanography. The Gulf Stream in the North Atlantic and Kuroshio in the North Pacific have a fairly similar behavior with separation from the coast occurring at or close to a fixed latitude. The Agulhas Current in the Indian Ocean, however, behaves differently by showing a retroflection accompanied by ring formation. The current rushes southward along the east coast of the African, overshoots the southern latitude of this continent and then suddenly it turns eastward and flows backward into the Indian Ocean. The North Brazil Current in the equatorial Atlantic shows a similar, but weaker, retroflection.

Mathematically speaking, the boundary-layer separation problem is crucial for understanding the transition to turbulence and stability properties of fluid flows. This problem is also closely linked to structural and dynamical bifurcation of the flow through a topological change of its spatial and phase-space structure. This program of research has been initiated by Tian Ma and one of the authors in this article, in collaboration in part with Michael Ghil; see [35–37, 93–96, 99, 101]. A great deal of further studies in this direction are needed.

5.2 Intraseasonal Oscillations (ISOs)

Another important source of variability is related to ISOs such as the Madden-Julian Oscillation (MJO). MJO is a large-scale oscillation (wave) in the equatorial region (Madden and Julian [106, 107]) and is the dominant component of the intraseasonal (30-90 days) variability in the tropical atmosphere (Zhang [160]). Although some theories and hypotheses have been proposed to understand the MJO, a completely satisfactory dynamical theory for the MJO has not yet been established (Holton [44]).

The basic governing equations used to theoretically analyze and simulate the MJO could be found in Wang [148]. From the mathematical point of view, the well-posedness, asymptotic behavior of the equations, and bifurcation and stability analysis are the first theoretical questions to be answered.

The observation diagnosis indicates that there is a rich multiscale structure of the MJO, and scale interactions might play an important role in the MJO (Slingo et al. [136]). However, whether the scale interactions are essential for the scale selection of the MJO is an important open question (Wang [148, 160]). It is therefore necessary to develop a multiscale analysis theory of the multiscale model to study the upscale energy transfer and to recognize the formation of large-scale structure of the system through multiscale interactions. Two basic aspects might be involved into this area. One is the significance of short-term cycle in the life cycle of the inherent large-scale structure of a dynamical system, e.g., the diurnal cycle to the MJO. The other is how the mesoscale and synoptic-scale systems go through certain organized action or stochastic dynamics to form a massive behavior and influence movement of this large-scale structure.

Recently, extratropical ISO or mid-high latitude low-frequency variability (LFV) has been revealed, e.g., the 70-day oscillation found over the North Atlantic (Felik, Gihl and Simonnet, [26]; Keppenne, Marcus, Kimoto and Ghil [58]; Lau, Sheu and Kang [69]) that is related to LFV of North Atlantic Oscillation (NAO). The dynamics of extratropical ISO or LFV is clearly an attractive field in the future.

5.3 ENSO

The El Niño-Southern Oscillation (ENSO) is the known strongest interannual climate variability associated with strong atmosphere-ocean coupling, which has significant impacts on global climate. ENSO is in fact a phenomenon that warm events (El Niño phase) and clod events (La Niña phase) in the equatorial eastern Pacific SST anomaly occur by turns, which associated with persistent weakening or strengthening in the trade winds by turns.

It is convenient, effective, and easily understandable to employ the simplified coupled dynamical models to investigate some essential behaviors of ENSO dynamics. The simplest and leading theoretical models for ENSO are the delayed oscillator model (Schopf and Suarez [132], Battisti and Hirst [4]) and the recharge oscillator model (Jin [52]). However, the basic shortcoming of those highly simplified models cannot account for the ob-

served irregularity of ENSO, although they could qualitatively explain the average features of an ENSO cycle. Hence further study is inevitably necessary. Another simplified coupled ocean-atmosphere model which can be used to predict ENSO event is the Zebiak and Cane (ZC) model (Zebiak and Cane [158]). The atmosphere model is the Gill-type (Gill [41]; Neelin et al. [116]), and the ocean model consists of a shallow-water layer with an embedded mixed layer. Also, we would like to mention the intriguing behavior of Boolean Delay Equations (BDE) in the ENSO context; see Ghil, Zaliapin and Coluzzi [40] and the references therein. It is worth studying the dynamical bifurcation and stability of solutions of this kind of simplified models to understand the phase transitions of ENSO and its low-frequency variability; see also the dynamical bifurcation theory in the next section.

The ENSO coupling processes and dynamics under the global warming by using successive bifurcation theory and the predictability of ENSO by using of nonlinear error growth theory are also of considerable practical importance. However, an interesting current debate is whether ENSO is best modeled as a stochastic or chaotic system - linear and noise-forced, or nonlinear oscillatory and unstable? It is obvious that a careful fundamental level examination of the problem is crucial.

6 New dynamical systems theories and geophysical applications

6.1 Introduction and motivation

As mentioned earlier, most studies on bifurcation issues in geophysical fluid dynamics so far have only considered systems of ordinary differential equations (ODEs) that are obtained by projecting the PDEs onto a finite-dimensional solution space, either by finite differencing or by truncating a Fourier expansion (see Ghil and Childress [32] and further references there).

A challenging mathematical problem is to conduct rigorous bifurcation and stability analysis for the original partial differential equations (PDEs) that govern geophysical flows. Progresses in this area should allow us to overcome some of the inherent limitations of the numerical bifurcation results that dominate the climate dynamics literature up to this point, and to capture the essential dynamics of the governing PDE systems.

Recently, Ma and Wang initiated a study on a new dynamic bifurcation and stability theory for dynamical systems. This bifurcation theory is centered at a new notion of bifurcation, called attractor bifurcation for dynamical systems, both finite dimensional and infinite dimensional. The main ingredients of the theory include a) the attractor bifurcation theory, b) steady state bifurcation for a class of nonlinear problems with even order non-degenerate nonlinearities, regardless of the multiplicity of the eigenvalues, and c) new strategies for the Lyapunov-Schmidt reduction and the center manifold reduction procedures. The general philosophy is that we first derive general existence of bifurcation to attractors, and then we classify the bifurcated attractors to derive detailed dynamics including for instance stability of the bifurcated solutions.

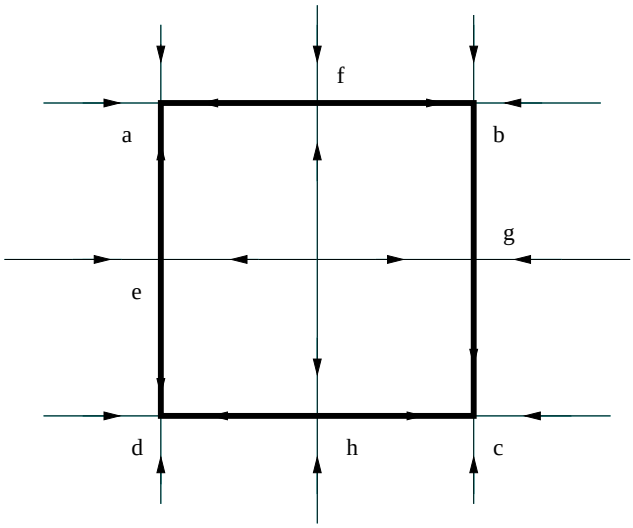

Figure 1: A bifurcated attractor containing 4 nodes (the points a, b, c, and d), 4 saddles (the points e, f, g, h), and orbits connecting these 8 points.

The bifurcation theory has been applied to various problems from science and engineering, including, in particular, the Kuramoto-Sivashinshy equation, the Cahn-Hillard equation, the Ginzburg-Landau equation, Reaction-Diffusion equations in Biology and Chemistry, and the Bénard convection problem and the Taylor problem in classical fluid mechanics; see a recent book by Ma and Wang [100]. For applications to geophysical fluid dynamics problems, we have carried out the detailed bifurcation and stability analysis for 1) the stratified Boussinesq equations [47], 2) the doubly-diffusive modes (both 2D and 3D) [45,46].

We proceed with a simple example to illustrate the basic motivation and ideas behind the attractor bifurcation theory. For $x=(x_1,x_2)\in\mathbb{R}^2$, the system

$$\dot{x}=\lambda x-(x_1^3,x_2^3)+o(|x|^3)$$

bifurcates from $(x,\lambda)=(0,0)$ to an attractor $\Sigma_\lambda=S^1$. This bifurcated attractor is as shown in Fig. 1, and contains exactly 4 nodes (the points a, b, c, and d), 4 saddles (the points e, f, g, h), and orbits connecting these 8 points. From the physical transition point of view, as λ crosses 0, the new state after the system undergoes a transition is represented by the **whole bifurcated attractor** Σ_λ, rather than any of the steady states or any of the connecting orbits. The connecting orbits represents transient states. Note that the global attractor is the 2D region enclosed by Σ_λ. We point out here that the bifurcated attractor is different from the study on global attractors of a dissipative dynamical system-both finite and infinite dimensional. Global attractor studies the global long time dynamics (see among others [2,17,18,27]), while the bifurcated attractor provides a natural object for studying dynamical transitions [100,103,105].

One important characteristic of the new attractor bifurcation theory is related to the asymptotic stability of the bifurcated solutions. This characteristic can be viewed as follows. First, as an attractor itself, the bifurcated attractor has a basin of attractor, and consequently is a useful object to describe local transitions. Second, with detailed classification of the solutions in the bifurcated attractor, we are able to access not only the asymptotic stability of the bifurcated attractor, but also the stability of different solutions in the bifurcated attractor, providing a more complete understanding of the transitions of the physical system as the system parameter varies. Third, as Kirchgässner [59] indicated, an ideal stability theorem would include all physically meaningful perturbations and today we are still far from this goal. In addition, fluid flows are normally time dependent. Therefore bifurcation analysis for steady state problems provides in general only partial answers to the problem, and is not enough for solving the stability problem. Hence from the physical point of view, attractor bifurcation provides a nature tool for studying transitions for deterministic systems.

6.2 A brief account of the attractor bifurcation theory

We now briefly present this new attractor bifurcation theory and refer the interested readers to [100,105] for details of the theory and its various applications.

We start with a basic state $\bar{\varphi}$ of the system, a steady state solution of (3.1). Then consider $\varphi = u + \bar{\varphi}$. Then (3.1) becomes

$$\frac{du}{dt} = L_\lambda u + G(u,\lambda), \tag{6.1}$$

$$u(0) = u_0, \tag{6.2}$$

where H and H_1 are two Hilbert spaces such that $H_1 \to H$ be a dense and compact inclusion.

The mapping $u:[0,\infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ is a family of linear completely continuous fields depending continuously on $\lambda \in \mathbb{R}$, such that

$$\begin{aligned} &L_\lambda = -A + B_\lambda && \text{a sectorial operator,} \\ &A : H_1 \to H && \text{a linear homeomorphism,} \\ &B_\lambda : H_1 \to H && \text{a linear compact operator.} \end{aligned} \tag{6.3}$$

It is known that L_λ generates an analytic semigroup $\{e^{-tL_\lambda}\}_{t\geq 0}$ and we can define fractional power operators L_λ^α for $\alpha \in \mathbb{R}$ with domain $H_\alpha = D(L_\lambda^\alpha)$ such that $H_{\alpha_1} \subset H_{\alpha_2}$ is compact if $\alpha_1 > \alpha_2$, $H = H_0$ and $H_1 = H_{\alpha=1}$.

Furthermore, we assume that for some $\theta < 1$ the nonlinear operator $G(\cdot,\lambda) : H_\theta \to H_0$ is a C^r bounded operator $(r \geq 1)$, and

$$G(u,\lambda) = o(\|u\|_\theta), \qquad \forall \lambda \in \mathbb{R}. \tag{6.4}$$

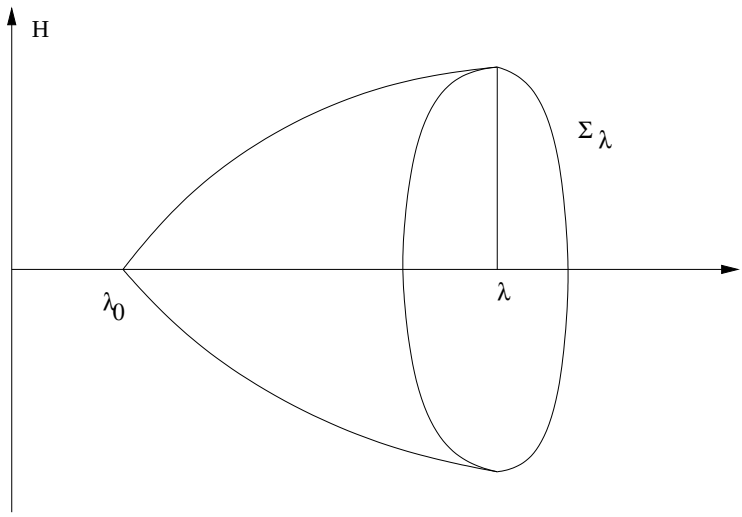

Figure 2: Definition of attractor bifurcation.

Definition 6.1. (Ma & Wang [98,100])

1. *We say that the equation* (6.1) *bifurcates from* $(u,\lambda)=(0,\lambda_0)$ *to an invariant set* Σ_λ*, if there exists a sequence of invariant sets* $\{\Sigma_{\lambda_n}\}$ *of* (6.1)*, such that* $0\notin\Sigma_{\lambda_n}$*, and*

$$\lim_{n\to\infty}\lambda_n=\lambda_0,\qquad \lim_{n\to\infty}\max_{x\in\Sigma_{\lambda_n}}\|x\|=0.$$

2. *If the invariant sets* Σ_λ *are attractors of* (6.1)*, then the bifurcation is called an attractor bifurcation (see Fig.* 2*).*

Let $\{\beta_k(\lambda)\in\mathbb{C}\,|\,k=1,2,\cdots\}$ be the eigenvalues of $L_\lambda=-A+B_\lambda$ (counting the multiplicities). Suppose that

$$Re\beta_i(\lambda)\begin{cases}<0 & \text{if } \lambda<\lambda_0\\ =0 & \text{if } \lambda=\lambda_0\\ >0 & \text{if } \lambda>\lambda_0\end{cases}\qquad \text{for } 1\le i\le m, \tag{6.5}$$

$$Re\beta_j(\lambda_0)<0\qquad \text{for } j\ge m+1. \tag{6.6}$$

Let E_0 be the eigenspace of L_λ at λ_0

$$E_0=\bigcup_{1\le i\le m}\Big\{u\in H\,|\,(L_{\lambda_0}-\beta_i(\lambda_0))^k u=0,k\in\mathbb{N}\Big\}.$$

By (6.5) we know that $\dim E_0=m$.

In physical terms, the above properties of eigenvalues are called principle of exchange of stabilities (PES). They can be verified in many physical systems. It is obvious that they are necessary for linear instability and the following attractor bifurcation theorem demonstrates that they lead to bifurcation to an attractor.

Theorem 6.1 (Ma & Wang [98, 100]). *Assume that* (6.5) *and* (6.6) *hold true, and let* $u=0$ *be a locally asymptotically stable equilibrium point of* (6.1) *at* $\lambda=\lambda_0$. *Then we have the following assertions.*

1. *Eq.* (6.1) *bifurcates from* $(u,\lambda)=(0,\lambda_0)$ *to an attractor* Σ_λ *for* $\lambda>\lambda_0$ *with* $m-1\leq \dim\Sigma_\lambda\leq m$, *which is an* $(m-1)$*-dimensional homological sphere.*
2. *For any* $u_\lambda\in\Sigma_\lambda$, u_λ *can be expressed as* $u_\lambda=v_\lambda+o(\|v_\lambda\|)$, *where* $v_\lambda\in E_0$.
3. *There exists a neighborhood* $U\subset H$ *of* $u=0$ *such that* Σ_λ *attracts* $U\backslash\Gamma$, *where* Γ *is the stable manifold of* $u=0$ *with codimension* m. *In particular, if* $u=0$ *is global asymptotically stable for* (6.1) *at* $\lambda=\lambda_0$ *and* (6.1) *has a global attractor for any* λ *near* λ_0, *then* Σ_λ *attracts* $H\backslash\Gamma$.

With this general theorem, along with other important ingredients addressed in Ma and Wang [100,105], at our disposal, the main strategy to conduct the bifurcation analysis consists of 1) the existence of attractor bifurcation, and 2) classification of solutions in the bifurcated attractor. We refer the interested readers to the books [100, 105] for details; see also the discussion below on the Rayleigh-Bénard convection.

6.3 Dynamic bifurcation in classical fluid dynamics

The new dynamic bifurcation theory has been naturally applied to problems in fluid dynamics, including the Rayleigh-Bénard convection, the Taylor problem, and the parallel shear flows, by Ma and Wang and their collaborators. The study of these basic problems on the one hand plays an important role in understanding the turbulent behavior of fluid flows, and on the other hand often leads to new insights and methods toward solutions of other problems in sciences and engineering.

To illustrate the main ideas of the applications of the theory, we consider now the Rayleigh-Bénard convection problem. Linear theory of the Rayleigh-Bénard problem were essentially derived by physicists; see, among others, Chandrasekhar [10] and Drazin and Reid [23]. Bifurcating solutions of the nonlinear problem were first constructed formally by Malkus and Veronis [109]. The first rigorous proofs of the existence of bifurcating solutions were given by Yudovich [156, 157] and Rabinowitz [123]. Yudovich proved the existence of bifurcating solutions by a topological degree argument. Earlier, however, Velte [143] had proved the existence of branching solutions of the Taylor problem by a topological degree argument as well. The have been many studies since the above mentioned early works. However, as we shall see, a rigorous and complete understanding of the problem is not available until the recent work by Ma and Wang using the attractor bifurcation theory.

To illustrate the ideas, we start by recalling the basic set-up of the problem. Let the Rayleigh number be

$$R=\frac{g\alpha(\bar{T}_0-\bar{T}_1)h^3}{(\kappa\nu)}.$$

The Bénard convection is modeled by the Boussinesq equations. In their nondimensional form, these equations are written as follows:

$$\begin{aligned}
&\frac{1}{P_r}\left[\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\nabla p\right]-\Delta u-\sqrt{R}Tk=0,\\
&\frac{\partial T}{\partial t}+(u\cdot\nabla)T-\sqrt{R}u_3-\Delta T=0,\\
&\operatorname{div} u=0.
\end{aligned} \tag{6.7}$$

Here the unknown functions are (u,T,p), which are the deviations from the basic field. The non-dimensional domain is $\Omega=D\times(0,1)\subset\mathbb{R}^3$, where $D\subset\mathbb{R}^2$ is an open set. The coordinate system is given by $x=(x_1,x_2,x_3)\in\mathbb{R}^3$. They are supplemented with the following initial value conditions

$$(u,T)=(u_0,T_0) \qquad \text{at } t=0. \tag{6.8}$$

Boundary conditions are needed at the top and bottom and at the lateral boundary $\partial D\times(0,1)$. At the top and bottom boundary ($x_3=0,1$), either the so-called rigid or free boundary conditions are given

$$\begin{aligned}
&T=0, \quad u=0 && \text{(rigid boundary)},\\
&T=0, \quad u_3=0, \quad \frac{\partial(u_1,u_2)}{\partial x_3}=0 && \text{(free boundary)}.
\end{aligned} \tag{6.9}$$

Different combinations of top and bottom boundary conditions are normally used in different physical setting such as *rigid-rigid*, *rigid-free*, *free-rigid*, and *free-free*.

On the lateral boundary $\partial D\times[0,1]$, one of the following boundary conditions are usually used:

1. Periodic condition:

$$(u,T)(x_1+k_1L_1,x_2+k_2L_2,x_3)=(u,T)(x_1,x_2,x_3), \tag{6.10}$$

for any $k_1,k_2\in\mathbb{Z}$.

2. Dirichlet boundary condition:

$$u=0, \quad T=0 \quad \left(\text{or } \frac{\partial T}{\partial n}=0\right); \tag{6.11}$$

3. Free boundary condition:

$$T=0, \quad u_n=0, \quad \frac{\partial u_\tau}{\partial n}=0, \tag{6.12}$$

where n and τ are the unit normal and tangent vectors on $\partial D\times[0,1]$ respectively, and $u_n=u\cdot n$, $u_\tau=u\cdot\tau$.

By using the attractor bifurcation theory, the following results have been obtained by Ma and Wang in [97, 100, 104, 105].

(1) When the Rayleigh number R crosses the first critical Rayleigh number R_c, the Rayleigh-Bénard problem bifurcates from the basic state to an attractor A_R, homologic to S^{m-1}, where m is the multiplicity of R_c as an eigenvalue of the linearized problem near the basic solution, for all physically sound boundary conditions, regardless of the geometry of the domain and the multiplicity of the eigenvalue R_c for the linear problem

(2) Consider the 3D Bénard convection in $\Omega=(0,L_1)\times(0,L_2)\times(0,1)$ with free top-bottom and periodic horizontal boundary conditions, and with

$$\frac{k_1^2}{L_1^2}+\frac{k_2^2}{L_2^2}=\frac{1}{8} \qquad \text{for some } k_1,k_2\in\mathbb{Z}. \tag{6.13}$$

Then

$$A_R=\begin{cases} S^5 & \text{if} \quad L_2=\sqrt{k^2-1}L_1, \quad k=2,3,\cdots, \\ S^3 & \text{otherwise.} \end{cases} \tag{6.14}$$

(3) For the 3D Bénard convection in $\Omega=(0,L)^2\times(0,1)$ with free boundary conditions and with

$$0<L^2<\frac{2-2^{1/3}}{2^{1/3}-1}. \tag{6.15}$$

The bifurcated attractor A_R consists of exactly eight singular points and eight heteroclinic orbits connecting the singular points, as shown in Figure 1, with 4 of them being minimal attractors, and the other 4 saddle points.

The proof of these results is carried out in the following steps:

1) It is classical to put the Boussinesq equations (6.7) in the abstract form (6.1). Then it is easy to see that the linearized operator is a self-adjoint operator., and both conditions (6.3) and (6.4) are satisfied.

2) As the linearized operator is symmetric, it is not hard to verify the PES (6.5) and (6.6) for the Boussinesq equations.

3) To derive a general attractor theorem for the Benard convection regardless of the domain and the boundary conditions, we need to verify the asymptotic stability of the basic solution $(u,T)=0$ at the critical Reynolds number. This can be achieved by a general stability theorem, derived by Ma and Wang in [97].

4) The detailed structure of the solutions in the bifurcated attractor can be derived using a new approximation formula for the center manifold function; see Ma and Wang [100, 104, 105].

We remark here that the high dimensional sphere S^{m-1} contains not only the steady states generated by symmetry groups inherited in the problem, but also many transient states, which were completely missed by any classical theories. These transient states are highly relevant in geophysical fluid dynamics and climate dynamics. We believe the climate low frequency climate variabilities discussed in the previous section are related to certain transient states, and further exploration in this direction are certainly important and necessary.

6.4 Dynamic bifurcation and stability in geophysical fluid dynamics

As mentioned earlier, the theory has been applied to models in geophysical fluid dynamics models including the doubly-diffusive models, and rotating Boussinesq equations. We now briefly address these applications in turn.

ROTATING BOUSSINESQ EQUATIONS: Rotating Boussinesq equations are basic model in atmosphere and ocean dynamical models. In [47], for the case where the Prandtl number is greater than one, a complete stability and bifurcation analysis near the first critical Rayleigh number is carried out. Second, for the case where the Prandtl number is smaller than one, the onset of the Hopf bifurcation near the first critical Rayleigh number is established, leading to the existence of nontrivial periodic solutions. The analysis is based on a newly developed bifurcation and stability theory for nonlinear dynamical systems as mentioned above.

DOUBLE-DIFFUSIVE OCEAN MODEL: Double-diffusion was first originally discovered in the 1857 by Jevons [51], forgotten, and then rediscovered as an "oceanographic curiosity" a century later; see among others Stommel, Arons and Blanchard [140], Veronis [145], and Baines and Gill [3]. In addition to its effects on oceanic circulation, double-diffusion convection has wide applications to such diverse fields as growing crystals, the dynamics of magma chambers and convection in the sun.

The best known doubly-diffusive instabilities are "salt-fingers" as discussed in the pioneering work by Stern [138]. These arise when hot salty water lies over cold fresh water of a higher density and consist of long fingers of rising and sinking water. A blob of hot salty water which finds itself surrounded by cold fresh water rapidly loses its heat while retaining its salt due to the very different rates of diffusion of heat and salt. The blob becomes cold and salty and hence denser than the surrounding fluid. This tends to make the blob sink further, drawing down more hot salty water from above giving rise to sinking fingers of fluid.

In [45,46], we present a bifurcation and stability analysis on the doubly-diffusive convection. The main objective is to study 1) the mechanism of the saddle-node bifurcation and hysteresis for the problem, 2) the formation, stability and transitions of the typical convection structures, and 3) the stability of solutions. It is proved in particular that there are two different types of transitions: continuous and jump, which are determined explicitly using some physical relevant nondimensional parameters. It is also proved that the jump transition always leads to the existence of a saddle-node bifurcation and hysteresis phenomena.

However, there are many issues still open, including in particular to use the dynamical systems tools developed to study some of the issues raised in the previous sections.

6.5 Stability and transitions of geophysical flows in the physical space

Another important area of studies in geophysical fluid dynamics is to study the structure and its stability and transitions of flows in the physical spaces.

A method to study these important problems in geophysical fluid dynamics is a recently developed geometric theory for incompressible flows by Ma and Wang [101]. This theory consists of research in directions: 1) the study of the structure and its transitions/evolutions of divergence-free vector fields, and 2) the study of the structure and its transitions of velocity fields for 2-D incompressible fluid flows governed by the Navier-Stokes equations or the Euler equations. The study in the first direction is more kinematic in nature, and the results and methods developed can naturally be applied to other problems of mathematical physics involving divergence-free vector fields. In fluid dynamics context, the study in the second direction involves specific connections between the solutions of the Navier-Stokes or the Euler equations and flow structure in the physical space. In other words, this area of research links the kinematics to the dynamics of fluid flows. This is unquestionably an important and difficult problem.

Progresses have been made in several directions. First, a new rigorous characterization of boundary layer separations for 2-D viscous incompressible flows is developed recently by Ma and Wang, in collaboration in part with Michael Ghil [101]. The nature of flow's boundary layer separation from the boundary plays a fundamental role in many physical problems, and often determines the nature of the flow in the interior as well. The main objective of this section is to present a rigorous characterization of the boundary layer separations of 2D incompressible fluid flows. This is a long standing problem in fluid mechanics going back to the pioneering work of Prandtl [121] in 1904. No known theorem, which can be applied to determine the separation, is available until the recent work by Ghil, Ma and Wang [36,37], and Ma and Wang [93,96], which provides a first rigorous characterization. Interior separations are studied rigorously by Ma and Wang [99]. The results for both the interior and boundary layer separations are used in Ma and Wang [102] for the transitions of the Couette-Poiseuille and the Taylor-Couette-Poiseuille flows.

Another example in this area is the justification of the roll structure (e.g. rolls) in the physical space in the Rayleigh-Bénard convection by Ma and Wang [97,104]. We note that a special structure with rolls separated by a cross channel flow derived in [104] has not been rigorously examined in the Bénard convection setting although it has been observed in other physical contexts such as the Branstator-Kushnir waves in the atmospheric dynamics [7,66].

With this theory in our disposal, the structure/patterns and their stability and transitions in the underlying physical spaces for those problems in fluid dynamics and in geophysical fluid dynamics can be classified. In particular, this theory has been used to

study the formation, persistence and transitions of flow structures including boundary layer separation including the Gulf separation, the Hadley circulation, and the Walker circulation. Further investigation appears to be utterly important and necessary. In addition, it appears that such theoretical and numerical studies will lead to better predications on weather and climate regimes.

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