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Diagonalizations of Vector and Tensor Addition Theorems

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Abstract. Based on the generalizations of the Funk-Hecke formula and the Rayleigh plan-wave expansion formula, an alternative and succinct derivation of the addition theorem for general tensor field is obtained. This new derivation facilitates the diagonalization of the tensor addition theorem. In order to complete this derivation, we have carried out the evaluation of the generalization of the Gaunt coefficient for tensor fields. Since vector fields (special case of tensor fields) are very useful in practice, we discuss vector multipole fields and vector addition theorem in details. The work is important in multiple scattering and fast algorithms in wave physics.

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1 Introduction

The fast multipole method (FMM) was proposed to accelerate the method of moments (MOM) for scattering problems [1–4]. The crucial step in the fast multipole method for solving the Helmholtz equation is the diagonalization of the translation operators [1, 2, 5]. The diagonalization of a translation operator in 2D was introduced in [1]. It was extended to 3D in [2, 6]. One can also refer to [2, 6–9] for some detailed discussions on the diagonalization of the translation operator. All these discussions were based on scalar addition theorems.

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Recently, the diagonalizations of the translation operators have been extended to vector fields [10]. In this paper, we will extend the diagonalizations of the translation operators to general tensor fields[†]. To arrive at this, we shall present a succinct derivation of the tensor addition theorem. Our derivation is different from Danos and Maximon's [13]. Our derivation is based on tensorial plane-wave expansion while Danos and Maximon's is based on the scalar addition theorem. This new derivation facilitates the diagonalization of the tensor addition theorem. To the best of our knowledge, the diagonalization of the tensor addition theorem has not been discussed before. The diagonalization of the tensor addition theorem facilitates applying FMM to elastodynamics, fluid dynamics, and Dirac equations, etc., where higher-rank tensors are used.

In Section 2, we extend the Funk-Hecke formula and Rayleigh plane-wave expansion for tensor fields by the use of the irreducible tensor technique. In Section 3, we present an alternative derivation of the tensor addition theorem by introducing a generalized Gaunt coefficient. In Section 4 we will show how to diagonalize the tensor addition theorem. In Section 5, we discuss the vector addition theorem in details, since vector fields are very useful in electromagnetics and elastodynamics.

2 Tensorial spherical wave formulas and tensorial plane-wave expansion

Plane waves can be expanded in terms of spherical waves by the use of the Rayleigh plane-wave expansion [14], and conversely, spherical waves can be expanded in terms of plane waves by the use of the Funk-Hecke formula. In this section, we shall extend the Funk-Hecke formula and Rayleigh plane-wave expansion formula for tensor fields.

2.1 Tensorial spherical wave formulas

We begin with the Rayleigh plane-wave expansion formula [15][‡]

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{lm} 4\pi i^l Y_{lm}(\hat{r}) j_l(kr) Y^*_{lm}(\hat{k}), \tag{2.3}$$

[†]Here, the tensor is defined as the irreducible tensor, including spinors [11,12].

[‡]The Rayleigh plane-wave expansion reads [15]

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l} (2l+1) i^l j_l(kr) P_l(\hat{r}\cdot\hat{k}), \tag{2.1}$$

where l is summed from 0 to ∞. The identity (2.1) is also called Bauer's identity, since according to Watson [16] it was discovered by Bauer as early as 1859 (Journal für Math. LVI. (1859) pp. 104, 106). Using the Legendre's addition theorem [17] (also called spherical-harmonic addition theorem [18])

$$P_l(\hat{r}\cdot\hat{k}) = \sum_{m} \frac{4\pi}{2l+1} Y_{lm}(\hat{r}) Y^*_{lm}(\hat{k}), \tag{2.2}$$

where m is summed from $-l$ to l, one can also write the Rayleigh plane-wave expansion as (2.3). To the best of our knowledge, this double summation formulation (2.3) was first derived by Stratton [14].

where l is summed from 0 to ∞ and m is summed from $-l$ to l. After multiplying both sides by $Y_{lm}(\hat{k})$, integrating over $\hat{k}$, and using the orthogonality relation

$$\int_{\bigcirc} d\hat{k} Y^*_{l'm'}(\hat{k}) Y_{lm}(\hat{k}) = \delta_{ll'}\delta_{mm'}, \tag{2.4}$$

where the integral is over a unit circle, and we have the Funk-Hecke formula [17]

$$j_l(kr)Y_{lm}(\hat{r}) = \frac{(-i)^l}{4\pi}\int_{\bigcirc} d\hat{k} e^{i\mathbf{k}\cdot\mathbf{r}} Y_{lm}(\hat{k}). \tag{2.5}$$

Now we generalize the Funk-Hecke formula (2.5) for tensor fields. To do this, one needs to find the spin operator $\mathbf{S}$, by studying the rotational operator for tensor fields. For a rotationally symmetric system, angular momentum is conserved, and hence, the rotational operator can be constructed from the angular momentum operator [12].

Let $\mathbf{e}_\mu^{[S]}$ be the eigenvectors of the spin operator $\mathbf{S}$. They are also called irreducible unit tensors [11,12], which satisfy the orthogonality relationship

$$\mathbf{e}_\mu^{*[S]} \cdot \mathbf{e}_\nu^{[S]} = \delta_{\mu\nu}, \tag{2.6}$$

where the values of μ or ν are from $-S$ to S. For example, for $S=1$, the unit tensors are $\mathbf{e}_{-1}^{[1]}$, $\mathbf{e}_0^{[1]}$, and $\mathbf{e}_{+1}^{[1]}$, which can be written in terms of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ as [12]

$$\mathbf{e}_{-1}^{[1]} = (\hat{x} - i\hat{y})/\sqrt{2}, \tag{2.7}$$

$$\mathbf{e}_0^{[1]} = \hat{z}, \tag{2.8}$$

$$\mathbf{e}_{+1}^{[1]} = -(\hat{x} + i\hat{y})/\sqrt{2}. \tag{2.9}$$

Let $\mathbf{e}^{[S]}$ be the set of the eigenvectors of spin operator $\mathbf{S}$

$$\mathbf{e}^{[S]} \equiv \left\{ e_\mu^{[S]}; \mu = -S, -S+1, \cdots, S \right\}. \tag{2.10}$$

Coupling $\mathbf{e}^{[S]}$ to both sides of (2.5) gives

$$j_l(kr)\mathbf{Y}_{lSM}^{[J]}(\hat{r}) = \frac{(-i)^l}{4\pi}\int_{\bigcirc} d\hat{k} e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{Y}_{lSM}^{[J]}(\hat{k}), \tag{2.11}$$

where $\mathbf{Y}_{lSM}^{[J]}(\hat{k})$ is the tensor spherical harmonics defined by [12,18,19]

$$\mathbf{Y}_{lSM}^{[J]}(\hat{k}) = \sum_{m\mu} \langle lmS\mu | lSJM \rangle Y_{lm}(\hat{k}) \mathbf{e}_\mu^{[S]}, \tag{2.12}$$

and $\langle lmS\mu|lSJM\rangle$ is the Clebsch-Gordan coefficient. The tensor spherical harmonic functions $\mathbf{Y}^{[J]}_{lSM'}$, which are the eigenfunctions of the addition of angular momenta $\mathbf{J}=\mathbf{L}+\mathbf{S}$, satisfy the orthogonality and completeness relationships

$$\int_{\bigcirc} d\hat{k}\mathbf{Y}^{*[J']}_{l'SM'}(\hat{k})\cdot\mathbf{Y}^{[J]}_{lSM}(\hat{k})=\delta_{JJ'}\delta_{ll'}\delta_{MM'}, \tag{2.13}$$

$$\sum_{JlM}\mathbf{Y}^{[J]}_{lSM}(\hat{k})\mathbf{Y}^{*[J]}_{lSM}(\hat{k}')=\sum_{\mu}\mathbf{e}^{[S]}_{\mu}\mathbf{e}^{*[S]}_{\mu}\delta(\hat{k}-\hat{k}'), \tag{2.14}$$

where in (2.13) the dot $\cdot$ stands for dot products of two irreducible tensors and (2.14) implies the outer product between two tensors. For the proofs of (2.13) and (2.14), one can refer to Appendix A. Here we are only interested in the case where S is fixed, for example, $S=1$. In (2.14), the expression $\sum_{\mu}\mathbf{e}^{[S]}_{\mu}\mathbf{e}^{*[S]}_{\mu}$ represents an identity operator in the space spanned by the vectors $\mathbf{e}^{[S]}_{\mu}$. Hence, we can alternatively define

$$\sum_{\mu}\mathbf{e}^{[S]}_{\mu}\mathbf{e}^{*[S]}_{\mu}=\mathbf{I}^{[S]}. \tag{2.15}$$

Note that the Funk-Hecke formula for tensor fields (2.11) corresponds to angular part of the Fourier transform as discussed in [20] for vector fields.

2.2 Tensorial plane-wave expansion

Right-multiplying both sides of (2.11) by $\mathbf{Y}^{*[J]}_{lSM}(\hat{k})$, summing over JlM, and using the completeness relationship (2.14), we have tensorial plane-wave expansion

$$\mathbf{I}^{[S]}e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{\mu}\mathbf{e}^{[S]}_{\mu}\mathbf{e}^{*[S]}_{\mu}e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{JlM}4\pi i^{l}\mathbf{Y}^{[J]}_{lSM}(\hat{r})j_l(kr)\mathbf{Y}^{*[J]}_{lSM}(\hat{k}). \tag{2.16}$$

One good feature of (2.16) is that the left-hand side is a single summation. Namely, the irreducible tensor is diagonalized.

Right-multiplying both sides of (2.16) by $\mathbf{e}^{[S]}_{\nu}$, using the orthogonality relation (2.6), we have

$$\mathbf{e}^{[S]}_{\nu}e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{JlM}4\pi i^{l}\mathbf{Y}^{[J]}_{lSM}(\hat{r})j_l(kr)(\mathbf{Y}^{*[J]}_{lSM}(\hat{k})\cdot\mathbf{e}^{[S]}_{\nu}), \tag{2.17}$$

which is another version of tensorial plane-wave expansion.

3 A succinct derivation of the tensor addition theorem

The addition theorem transforms the wave functions from one coordinate system into another. The addition theorem of scalar fields along an arbitrary direction was first derived by Friedman and Russek [21]. It was generalized to vector fields by Stein [22] and

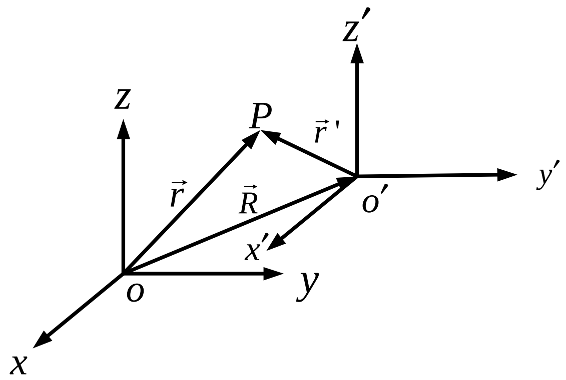

Figure 1: Addition theorem.

Cruzan [23]. It was further generalized to tensor fields by Danos and Maximon [13]. One can refer to [24] for the references and details on addition theorem. Here we suggest a new way to derive the tensor addition theorem, paralleling the derivation of the scalar addition theorem [13,25].

Letting (Fig. 1)

$$\mathbf{r}=\mathbf{R}+\mathbf{r}', \tag{3.1}$$

we have

$$\mathbf{I}^{[S]}e^{i\mathbf{k}\cdot\mathbf{r}}=e^{i\mathbf{k}\cdot\mathbf{R}}\mathbf{I}^{[S]}e^{i\mathbf{k}\cdot\mathbf{r}'}. \tag{3.2}$$

Substituting (2.16) and (2.3) into (3.2), we have

$$\begin{aligned}&\sum_{JlM}4\pi i^l\mathbf{Y}^{[J]}_{lSM}(\hat{r})j_l(kr)\mathbf{Y}^{*[J]}_{lSM}(\hat{k})\\ =&\sum_{l''m''}4\pi i^{l''}Y_{l''m''}(\hat{R})j_{l''}(kR)Y^*_{l''m''}(\hat{k})\sum_{J'l'M'}4\pi i^{l'}\mathbf{Y}^{[J']}_{l'SM'}(\hat{r}')j_{l'}(kr')\mathbf{Y}^{*[J']}_{l'SM'}(\hat{k}).\end{aligned} \tag{3.3}$$

Right-multiplying (dot product) both sides of (3.3) by $\mathbf{Y}^{[J]}_{lSM}(\hat{k})$, integrating over $\hat{k}$, and using the orthogonality (2.13), we have

$$\begin{aligned}\mathbf{Y}^{[J]}_{lSM}(\hat{r})j_l(kr)=&\sum_{J'l'M'}\sum_{l''m''}4\pi i^{l''+l'-l}Y_{l''m''}(\hat{R})j_{l''}(kR)\mathbf{Y}^{[J']}_{l'SM'}(\hat{r}')j_{l'}(kr')\\ &\times\int_{\bigcirc}d\hat{k}\mathbf{Y}^{*[J']}_{l'SM'}(\hat{k})\cdot\mathbf{Y}^{[J]}_{lSM}(\hat{k})Y^*_{l''m''}(\hat{k}).\end{aligned} \tag{3.4}$$

Let

$$G^{[S]}(l'J'M'|lJM|l''m'')=\int_{\bigcirc}d\hat{k}\mathbf{Y}^{*[J']}_{l'SM'}(\hat{k})\cdot\mathbf{Y}^{[J]}_{lSM}(\hat{k})Y^*_{l''m''}(\hat{k}), \tag{3.5}$$

which is a generalization of the Gaunt coefficient.§ Note that the Gaunt coefficient is a *scalar* constant. Thus, we obtain the matrix elements $T^{[S]}_{l'J'M',lJM}(\mathbf{R})$ of translator

$$T^{[S]}_{l'J'M',lJM}(\mathbf{R})=\sum_{l''m''}G^{[S]}(l'J'M'|lJM|l''m'')4\pi i^{l''+l'-l}Y_{l''m''}(\hat{R})j_{l''}(kR). \tag{3.6}$$

Using (2.3) and (3.5), the matrix elements $T^{[S]}_{l'J'M',lJM}(\mathbf{R})$ can also be written as

$$T^{[S]}_{l'J'M',lJM}(\mathbf{R})=i^{l'-l}\oint d\hat{k}\mathbf{Y}^{*[J']}_{l'SM'}(\hat{k})\cdot\mathbf{Y}^{[J]}_{lSM}(\hat{k})e^{i\mathbf{k}\cdot\mathbf{R}}. \tag{3.7}$$

It can be shown that [25]

$$\int_0^\infty dkk^2\frac{j_l(kr_0)j_l(kr)}{k^2-k_0^2}=\frac{\pi i}{2}k_0 j_l(k_0r_0)h_l^{(1)}(k_0r),\quad r>r_0, \tag{3.8}$$

$$\int_0^\infty dkk^2\frac{j_l(kr_0)j_{l'}(kr')j_{l''}(kR)}{k^2-k_0^2}=\frac{\pi i}{2}k_0 j_l(k_0r_0)\begin{cases} h_{l''}^{(1)}(k_0R)j_{l'}(k_0r'), & R>r'+r_0,\\ j_{l''}(k_0R)h_{l'}^{(1)}(k_0r'), & r'>R+r_0,\end{cases} \tag{3.9}$$

where k_0 has a small positive imaginary part (r_0 can be chosen arbitrarily small so that it become insignificant in (3.9)). Since (3.4) is valid for all k, multiplying it by $k^2 j_l(kr_0)/(k^2-k_0^2)$ and integrating over k from 0 to ∞ give

$$\begin{aligned}\int_0^\infty dkk^2\frac{j_l(kr_0)j_l(kr)}{k^2-k_0^2}\mathbf{Y}^{[J]}_{lSM}(\hat{r})=&\int_0^\infty dkk^2\frac{j_l(kr_0)}{k^2-k_0^2}\\ \times\sum_{J'l'M'}\sum_{l''m''}4\pi i^{l''+l'-l}Y_{l''m''}(\hat{R})j_{l''}(kR)&\mathbf{Y}^{[J']}_{l'SM'}(\hat{r}')j_{l'}(kr')G^{[S]}(l'J'M'|lJM|l''m'').\end{aligned} \tag{3.10}$$

Using (3.8) and (3.9), we can simplify (3.10) to be

$$\begin{aligned}\mathbf{Y}^{[J]}_{lSM}(\hat{r})h_l^{(1)}(k_0r)=&\sum_{J'l'M'}\sum_{l''m''}4\pi i^{l''+l'-l}G^{[S]}(l'J'M'|lJM|l''m'')\\ &\times Y_{l''m''}(\hat{R})\mathbf{Y}^{[J']}_{l'SM'}(\hat{r}')\begin{cases} h_{l''}^{(1)}(k_0R)j_{l'}(k_0r'), & R>r',\\ j_{l''}(k_0R)h_{l'}^{(1)}(k_0r'), & r'>R.\end{cases}\end{aligned} \tag{3.11}$$

Since

$$j_l(x)=\frac{1}{2}\left(h_l^{(1)}(x)+h_l^{(2)}(x)\right), \tag{3.12}$$

§Another version of the generalized Gaunt coefficient is proposed in [26], using a set of quite different tensor harmonics.

based on (3.4) and (3.11), we can obtain the translation for the spherical Hankel function of the second kind $h_l^{(2)}(x)$

$$\mathbf{Y}_{lSM}^{[J]}(\hat{r})h_l^{(2)}(k_0r)=\sum_{J'l'M'}\sum_{l''m''}4\pi i^{l''+l'-l}G^{[S]}(l'J'M'|lJM|l''m'')\\ \times Y_{l''m''}(\hat{R})\mathbf{Y}_{l'SM'}^{[J']}(\hat{r}')\begin{cases} h_{l''}^{(2)}(k_0R)j_{l'}(k_0r'), & R>r', \\ j_{l''}(k_0R)h_{l'}^{(2)}(k_0r'), & r'>R. \end{cases} \tag{3.13}$$

Define $\tilde{T}_{l'J'M',lJM}^{[S]}(\mathbf{R})$ by

$$\tilde{T}_{l'J'M',lJM}^{[S]}(\mathbf{R})=\sum_{l''m''}G^{[S]}(l'J'M'|lJM|l''m'')4\pi i^{l''+l'-l}Y_{l''m''}(\hat{R})z_{l''}(kR), \tag{3.14}$$

where $z_l(x)$ is either the spherical Bessel function of the first kind $j_l(x)$, the spherical Bessel function of the second kind $y_l(x)$, the spherical Hankel function of the first kind $h_l^{(1)}(x)$, and the spherical Hankel function of the second kind $h_l^{(2)}(x)$. Note that the translator $\tilde{T}_{l'J'M',lJM}^{[S]}(\mathbf{R})$ can be obtained replacing the $j_{l''}(kR)$ with $z_{l''}(kR)$ in the translator $T_{l'J'M',lJM}^{[S]}(\mathbf{R})$.¶

Without losing any physical meaning, replacing k_0 with k in (3.11) and (3.13), we can summarize (3.4), (3.11) and (3.13) as follows

$$\mathbf{Y}_{lSM}^{[J]}(\hat{r})z_l(kr)=\begin{cases} \sum_{J'l'M'}T_{l'J'M',lJM}^{[S]}(\mathbf{R})\mathbf{Y}_{l'SM'}^{[J']}(\hat{r}')z_{l'}(kr'), & r'>R, \\ \sum_{J'l'M'}\tilde{T}_{l'J'M',lJM}^{[S]}(\mathbf{R})\mathbf{Y}_{l'SM'}^{[J']}(\hat{r}')j_{l'}(kr'), & r'<R. \end{cases} \tag{3.15}$$

Note that the translators for tensor wave fields $T_{l'J'M',lJM}^{[S]}(\mathbf{R})$ and $\tilde{T}_{l'J'M',lJM}^{[S]}(\mathbf{R})$ are *scalar* functions since the translators themselves do not carry nonzero spins. Namely, both sides of (3.15) have same spin S.

The evaluation of the generalized Gaunt coefficient is presented in Appendix B . Plugging (B.11) into (3.6) and (3.14), we have the explicit forms of translators $T_{l'J'M',lJM}^{[S]}(\mathbf{R})$ and $\tilde{T}_{l'J'M',lJM}^{[S]}(\mathbf{R})$, which are same as those derived by Danos and Maximon [13]. Here we have corrected the coefficients given by [13]. There is also a sign error before M in [28].

4 Diagonalization of the tensor addition theorem

The diagonalization of the translation operators is the crucial step in the fast multipole method for solving the wave equations [1,2,5]. We shall derive the diagonal form of the tensor addition theorem, paralleling the derivation of the diagonal form of the scalar and vector addition theorem [9,10,29].

¶Similar observation for the vector addition theorem (special case of tensor addition theorem) was made by Bruning and Lo. See the appendix in [27].

Specializing $z_{l''}(kR)$ as the spherical Hankel function of the first kind $h^{(1)}_{l''}(kR)$, we have the translator $\tilde{T}^{[S]}_{lJM,l'J'M'}(\mathbf{R})$ to be

$$\tilde{T}^{[S]}_{lJM,l'J'M'}(\mathbf{R})=\sum_{l''m''}G^{[S]}(lJM|l'J'M'|l''m'')4\pi i^{l''+l-l'}Y_{l''m''}(\hat{R})h^{(1)}_{l''}(kR). \tag{4.1}$$

In a fast multipole algorithm, the translator for tensor wave functions can be factorized by

$$\tilde{T}^{[S]}_{lJM,l'J'M'}(\mathbf{R}_{ij})=\sum_{l_1J_1M_1,l_1'J_1'M_1'}T^{[S]}_{lJM,l_1J_1M_1}(\mathbf{R}_{i\lambda})\tilde{T}^{[S]}_{l_1J_1M_1,l_1'J_1'M_1'}(\mathbf{R}_{\lambda\lambda'})T^{[S]}_{l_1'J_1'M_1',l'J'M'}(\mathbf{R}_{\lambda'j}). \tag{4.2}$$

Using (3.7) and (4.1), we compute (4.2) as

$$\begin{aligned}\tilde{T}^{[S]}_{lJM,l'J'M'}(\mathbf{R}_{ij})=&\sum_{l_1J_1M_1,l_1'J_1'M_1'}i^{l-l_1}\int_{\bigcirc}d\hat{k}\mathbf{Y}^{*[J]}_{lSM}(\hat{k})\cdot\mathbf{Y}^{[J_1]}_{l_1SM_1}(\hat{k})e^{ik\hat{k}\cdot\mathbf{R}_{i\lambda}}\\&\times\sum_{l''m''}4\pi i^{l_1+l''-l_1'}\tilde{\Psi}_{l''m''}(\mathbf{R}_{\lambda\lambda'})\int_{\bigcirc}d\hat{k}'\mathbf{Y}^{*[J_1]}_{l_1SM_1}(\hat{k}')\cdot\mathbf{Y}^{[J_1']}_{l_1'SM_1'}(\hat{k}')Y^*_{l''m''}(\hat{k}')\\&\times i^{l_1'-l'}\int_{\bigcirc}d\hat{k}''\mathbf{Y}^{*[J_1']}_{l_1'SM_1'}(\hat{k}'')\cdot\mathbf{Y}^{[J']}_{l'SM'}(\hat{k}'')e^{ik\hat{k}''\cdot\mathbf{R}_{\lambda'j}},\end{aligned} \tag{4.3}$$

where

$$\tilde{\Psi}_{l''m''}(\mathbf{R})=Y_{l''m''}(\hat{R})h^{(1)}_{l''}(kR). \tag{4.4}$$

In order to simplify (4.3), we consider

$$\begin{aligned}&\sum_{l_1J_1M_1,l_1'J_1'M_1'}\left[\mathbf{Y}^{*[J]}_{lSM}(\hat{k})\cdot\mathbf{Y}^{[J_1]}_{l_1SM_1}(\hat{k})\right]\left[\mathbf{Y}^{*[J_1]}_{l_1SM_1}(\hat{k}')\cdot\mathbf{Y}^{[J_1']}_{l_1'SM_1'}(\hat{k}')\right]\left[\mathbf{Y}^{*[J_1']}_{l_1'SM_1'}(\hat{k}'')\cdot\mathbf{Y}^{[J']}_{l'SM'}(\hat{k}'')\right]\\=&\sum_{l_1J_1M_1,l_1'J_1'M_1'}\left\{\mathbf{Y}^{*[J]}_{lSM}(\hat{k})\cdot\left[\mathbf{Y}^{[J_1]}_{l_1SM_1}(\hat{k})\mathbf{Y}^{*[J_1]}_{l_1SM_1}(\hat{k}')\right]\right\}\cdot\left\{\left[\mathbf{Y}^{[J_1']}_{l_1'SM_1'}(\hat{k}')\mathbf{Y}^{*[J_1']}_{l_1'SM_1'}(\hat{k}'')\right]\cdot\mathbf{Y}^{[J']}_{l'SM'}(\hat{k}'')\right\}\\=&\left[\mathbf{Y}^{*[J]}_{lSM}(\hat{k})\cdot\mathbf{I}^{[S]}\delta(\hat{k}-\hat{k}')\right]\cdot\left[\mathbf{I}^{[S]}\delta(\hat{k}''-\hat{k}')\cdot\mathbf{Y}^{[J']}_{l'SM'}(\hat{k}'')\right]\\=&\mathbf{Y}^{*[J]}_{lSM}(\hat{k})\cdot\mathbf{Y}^{[J']}_{l'SM'}(\hat{k}'')\delta(\hat{k}-\hat{k}')\delta(\hat{k}''-\hat{k}').\end{aligned} \tag{4.5}$$

In the above, we have used completeness relation (2.14), orthogonality relation (2.6), and the rule of associativity.

The use of (4.5) simplifies (4.3) to be

$$\tilde{T}^{[S]}_{lJM,l'J'M'}(\mathbf{R}_{ij})=\int_{\bigcirc}d\hat{k}i^{l}\mathbf{Y}^{*[J]}_{lSM}(\hat{k})\cdot e^{i\mathbf{k}\cdot\mathbf{R}_{i\lambda}}\breve{T}(\mathbf{k},\mathbf{R}_{\lambda\lambda'})i^{-l'}\mathbf{Y}^{[J']}_{l'SM'}(\hat{k})e^{i\mathbf{k}\cdot\mathbf{R}_{\lambda'j}}, \tag{4.6}$$

where

$$\breve{T}(\mathbf{k},\mathbf{R}_{\lambda\lambda'})=\sum_{l''m''}4\pi i^{l''}\tilde{\Psi}_{l''m''}(\mathbf{R}_{\lambda\lambda'})Y^*_{l''m''}(\hat{k}). \tag{4.7}$$

Note that the factor $\breve{T}(\mathbf{k},\mathbf{R}_{\lambda\lambda'})$ is exactly same as the factor for the diagonalization of the scalar addition theorem.

5 Vector spherical harmonics, vector multipole fields, and vector addition theorem

Since vector fields (special case of tensor fields with spin $S=1$) are very useful in practice, we will discuss vector multipole fields and vector addition theorem in details. Two sets of vector spherical harmonics and the corresponding vector multipole fields are discussed. One set is the eigenstates of the angular momentum, each of which involves a single orbital angular momentum. The other set naturally describes the divergence and curl properties of the vector fields, so it is often used in electromagnetics and elastodynamics. Similar to [10, 29], we shall present a succinct derivation of the vector addition theorem.

5.1 Vector spherical harmonics

The tensor spherical harmonics becomes usual vector spherical harmonics $\mathbf{Y}_{l1M}^{[J]}(\hat{r})$ by setting $S=1$ in (2.12), where the value l can be only $J-1$, J and $J+1$. It follows that these 3 vector spherical harmonics can be denoted as

$$\mathbf{Y}_{J-1,M}^{[J]}(\hat{r}),\ \mathbf{Y}_{J,M}^{[J]}(\hat{r}),\ \mathbf{Y}_{J+1,M}^{[J]}(\hat{r}). \tag{5.1}$$

Another set of vector spherical harmonics called Hansen spherical harmonics can be denoted as [20].

$$\mathbf{P}_{JM}(\hat{r})=\hat{\mathbf{e}}_r Y_{lm}(\hat{r}),\quad \mathbf{B}_{JM}(\hat{r})=\frac{\nabla_\alpha Y_{lm}(\hat{r})}{\sqrt{l(l+1)}},\quad \mathbf{C}_{JM}(\hat{r})=-\hat{\mathbf{e}}_r\times\mathbf{B}_{JM}(\hat{r}), \tag{5.2}$$

where ∇_α is defined by

$$\nabla_\alpha=\hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta}+\frac{\hat{\mathbf{e}}_\varphi}{sin\theta}\frac{\partial}{\partial\varphi}. \tag{5.3}$$

From the definition (5.2), it can be shown that the vector spherical harmonics and Hansen spherical harmonics have the relation

$$\mathbf{P}_{JM}(\hat{r})=\frac{1}{\sqrt{2J+1}}\left[-\sqrt{J+1}\mathbf{Y}_{J+1,M}^{[J]}(\hat{r})+\sqrt{J}\mathbf{Y}_{J-1,M}^{[J]}(\hat{r})\right], \tag{5.4}$$

$$\mathbf{B}_{JM}(\hat{r})=\frac{1}{\sqrt{2J+1}}\left[\sqrt{J}\mathbf{Y}_{J+1,M}^{[J]}(\hat{r})+\sqrt{J+1}\mathbf{Y}_{J-1,M}^{[J]}(\hat{r})\right], \tag{5.5}$$

$$\mathbf{C}_{JM}(\hat{r})=-i\mathbf{Y}_{J,M}^{[J]}(\hat{r}). \tag{5.6}$$

Replacing $\hat{r}$ with $\hat{k}$ in (5.1), (5.2), (5.4)-(5.6), one also can define these spherical harmonics in momentum space. Letting $S=1$ in (2.13) and using (5.4)-(5.6), we have the orthogonality of Hansen spherical harmonic functions

$$\int_{\bigcirc} d\hat{k}\mathbf{P}^*_{J'M'}(\hat{k})\cdot\mathbf{P}_{JM}(\hat{k})=\delta_{JJ'}\delta_{MM'}, \tag{5.7}$$

$$\int_{\bigcirc} d\hat{k}\mathbf{B}^*_{J'M'}(\hat{k})\cdot\mathbf{B}_{JM}(\hat{k})=\delta_{JJ'}\delta_{MM'}, \tag{5.8}$$

$$\int_{\bigcirc} d\hat{k}\mathbf{C}^*_{J'M'}(\hat{k})\cdot\mathbf{C}_{JM}(\hat{k})=\delta_{JJ'}\delta_{MM'}. \tag{5.9}$$

Letting $S=1$ in (2.14) and using (5.4)-(5.6), we have the completeness of Hansen spherical harmonic functions

$$\begin{aligned}&\sum_{JM}\mathbf{P}_{JM}(\hat{k})\mathbf{P}^*_{JM}(\hat{k}')+\mathbf{B}_{JM}(\hat{k})\mathbf{B}^*_{JM}(\hat{k}')+\mathbf{C}_{JM}(\hat{k})\mathbf{C}^*_{JM}(\hat{k}')\\ =&\sum_{\mu}\mathbf{e}^{[1]}_{\mu}\mathbf{e}^{*[1]}_{\mu}\delta(\hat{k}-\hat{k}')=\mathbf{I}^{[1]}\delta(\hat{k}-\hat{k}').\end{aligned} \tag{5.10}$$

5.2 Vector multipole fields

Vector multipole fields can be defined by

$$z_{J-1}(kr)\mathbf{Y}^{[J]}_{J-1,M}(\hat{r}),\ z_J(kr)\mathbf{Y}^{[J]}_{J,M}(\hat{r}),\ z_{J+1}(kr)\mathbf{Y}^{[J]}_{J+1,M}(\hat{r}). \tag{5.11}$$

In electromagnetics and elastodynamics, in order to describe the divergence and curl properties of the vector fields, Hansen multipole fields is often used . Hansen multipole fields can be defined by

$$\begin{aligned}&\mathbf{M}_{JM}(\mathbf{r})=\nabla\times[\mathbf{r}z_l(kr)Y_{lm}(\hat{r})],\quad \mathbf{N}_{JM}(\mathbf{r})=\frac{1}{k}\nabla\times\mathbf{M}_{JM}(\mathbf{r}),\\ &\mathbf{L}_{JM}(\mathbf{r})=\frac{1}{k}\nabla[z_l(kr)Y_{lm}(\hat{r})].\end{aligned} \tag{5.12}$$

Hansen multipole fields $\mathbf{M}_{JM}(\mathbf{r})$ and $\mathbf{N}_{JM}(\mathbf{r})$ describe solenoidal waves and $\mathbf{L}_{JM}(\mathbf{r})$ describes longitudinal waves. Working out the *curl* and *grad* in (5.12), we can express Hansen multipole fields as

$$\mathbf{M}_{JM}(\mathbf{r})=-i\sqrt{J(J+1)}z_J(kr)\mathbf{Y}^{[J]}_{J,M}(\hat{r}), \tag{5.13}$$

$$\mathbf{N}_{JM}(\mathbf{r})=-\frac{J\sqrt{J+1}}{\sqrt{2J+1}}z_{J+1}(kr)\mathbf{Y}^{[J]}_{J+1,M}(\hat{r})+\frac{(J+1)\sqrt{J}}{\sqrt{2J+1}}z_{J-1}(kr)\mathbf{Y}^{[J]}_{J-1,M}(\hat{r}), \tag{5.14}$$

$$\mathbf{L}_{JM}(\mathbf{r})=\frac{\sqrt{J+1}}{\sqrt{2J+1}}z_{J+1}(kr)\mathbf{Y}^{[J]}_{J+1,M}(\hat{r})+\frac{\sqrt{J}}{\sqrt{2J+1}}z_{J-1}(kr)\mathbf{Y}^{[J]}_{J-1,M}(\hat{r}). \tag{5.15}$$

5.3 Vector Funk-Hecke formulas

Letting $S=1$ in (2.11) gives the formulas for vector fields as follows

$$j_{J-1}(kr)\mathbf{Y}^{[J]}_{J-1,M}(\hat{r}) = \frac{(-i)^{J-1}}{4\pi}\int_{\bigcirc} d\hat{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{Y}^{[J]}_{J-1,M}(\hat{k}), \tag{5.16}$$

$$j_{J}(kr)\mathbf{Y}^{[J]}_{J,M}(\hat{r}) = \frac{(-i)^{J}}{4\pi}\int_{\bigcirc} d\hat{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{Y}^{[J]}_{J,M}(\hat{k}), \tag{5.17}$$

$$j_{J+1}(kr)\mathbf{Y}^{[J]}_{J+1,M}(\hat{r}) = \frac{(-i)^{J+1}}{4\pi}\int_{\bigcirc} d\hat{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{Y}^{[J]}_{J+1,M}(\hat{k}). \tag{5.18}$$

Using (5.4)-(5.6) and (5.13)-(5.15) in (5.16)-(5.18), we can prove the Funk-Hecke formulas for vector fields

$$\Re_g\mathbf{M}_{JM}(\mathbf{r}) = \sqrt{J(J+1)}\frac{(-i)^{J}}{4\pi}\int_{\bigcirc} d\hat{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{C}_{JM}(\hat{k}), \tag{5.19}$$

$$\Re_g\mathbf{N}_{JM}(\mathbf{r}) = i\sqrt{J(J+1)}\frac{(-i)^{J}}{4\pi}\int_{\bigcirc} d\hat{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{B}_{JM}(\hat{k}), \tag{5.20}$$

$$\Re_g\mathbf{L}_{JM}(\mathbf{r}) = i\frac{(-i)^{J}}{4\pi}\int_{\bigcirc} d\hat{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{P}_{JM}(\hat{k}), \tag{5.21}$$

where the $\Re_g$ operator implies taking the regular part of the function, that is, a spherical Hankel function is replaced by a spherical Bessel function. The above proof is different from those in [10, 20, 29, 30].

It will be discussed in Subsection 5.5 that the wave number of the solenoidal waves can be different from that of longitudinal waves because of the decoupling the translations of the solenoidal waves and longitudinal waves. Thus, in general, one may express the vector Funk-Hecke formulas as follows [20]

$$\Re_g\mathbf{M}_{JM}(\mathbf{r}) = \sqrt{J(J+1)}\frac{(-i)^{J}}{4\pi}\int_{\bigcirc} d\hat{k}e^{ik_s\mathbf{e}_k\cdot\mathbf{r}}\mathbf{C}_{JM}(\hat{k}), \tag{5.22}$$

$$\Re_g\mathbf{N}_{JM}(\mathbf{r}) = i\sqrt{J(J+1)}\frac{(-i)^{J}}{4\pi}\int_{\bigcirc} d\hat{k}e^{ik_s\mathbf{e}_k\cdot\mathbf{r}}\mathbf{B}_{JM}(\hat{k}), \tag{5.23}$$

$$\Re_g\mathbf{L}_{JM}(\mathbf{r}) = i\frac{(-i)^{J}}{4\pi}\int_{\bigcirc} d\hat{k}e^{ik_p\mathbf{e}_k\cdot\mathbf{r}}\mathbf{P}_{JM}(\hat{k}), \tag{5.24}$$

where k_s and k_p are the wave numbers of the solenoidal waves and the longitudinal waves, respectively.

5.4 Vector plane-wave expansion

Letting $S=1$ in (2.16), we have the vectorial plane-wave expansion

$$\begin{aligned}\mathbf{I}^{[1]}e^{i\mathbf{k}\cdot\mathbf{r}}=&\sum_{JM}4\pi i^{J-1}j_{J-1}(kr)\mathbf{Y}^{[J]}_{J-1,M}(\hat{r})\mathbf{Y}^{*[J]}_{J-1,M}(\hat{k})\\&+\sum_{JM}4\pi i^{J}j_{J}(kr)\mathbf{Y}^{[J]}_{J,M}(\hat{r})\mathbf{Y}^{*[J]}_{J,M}(\hat{k})\\&+\sum_{JM}4\pi i^{J+1}j_{J+1}(kr)\mathbf{Y}^{[J]}_{J+1,M}(\hat{r})\mathbf{Y}^{*[J]}_{J+1,M}(\hat{k}).\end{aligned}\tag{5.25}$$

Using (5.4)-(5.6) and (5.13)-(5.15) in (5.25), we have the vectorial plane-wave expansion in terms of Hansen multipole fields

$$\begin{aligned}\mathbf{I}^{[1]}e^{i\mathbf{k}\cdot\mathbf{r}}=&\sum_{JM}4\pi i^{J}\left[\frac{1}{\sqrt{J(J+1)}}\Re_g\mathbf{M}_{JM}(\mathbf{r})\mathbf{C}^*_{JM}(\hat{k})\right.\\&\left.-\frac{i}{\sqrt{J(J+1)}}\Re_g\mathbf{N}_{JM}(\mathbf{r})\mathbf{B}^*_{JM}(\hat{k})-i\Re_g\mathbf{L}_{JM}(\mathbf{r})\mathbf{P}^*_{JM}(\hat{k})\right].\end{aligned}\tag{5.26}$$

It can be shown that

$$\mathbf{I}^{[1]}=\sum_{\mu}\mathbf{e}^{[1]}_{\mu}\mathbf{e}^{*[1]}_{\mu}=\mathbf{e}_k\mathbf{e}_k+\mathbf{e}_\theta\mathbf{e}_\theta+\mathbf{e}_\varphi\mathbf{e}_\varphi.\tag{5.27}$$

Using (5.27) in (5.26), we have

$$\begin{aligned}(\mathbf{e}_k\mathbf{e}_k+\mathbf{e}_\theta\mathbf{e}_\theta+\mathbf{e}_\varphi\mathbf{e}_\varphi)e^{i\mathbf{k}\cdot\mathbf{r}}=&\sum_{JM}4\pi i^{J}\left[\frac{1}{\sqrt{J(J+1)}}\Re_g\mathbf{M}_{JM}(\mathbf{r})\mathbf{C}^*_{JM}(\hat{k})\right.\\&\left.-\frac{i}{\sqrt{J(J+1)}}\Re_g\mathbf{N}_{JM}(\mathbf{r})\mathbf{B}^*_{JM}(\hat{k})-i\Re_g\mathbf{L}_{JM}(\mathbf{r})\mathbf{P}^*_{JM}(\hat{k})\right].\end{aligned}\tag{5.28}$$

Taking dot product of both sides of (5.28) with $\mathbf{e}_k$, we have the plane-wave expansion for the longitudinal waves

$$\mathbf{e}_ke^{i\mathbf{k}\cdot\mathbf{r}}=-i\sum_{JM}4\pi i^{J}\Re_g\mathbf{L}_{JM}(\mathbf{r})Y^*_{JM}(\hat{k}).\tag{5.29}$$

Taking cross product of both sides of (5.28) with $\mathbf{e}_k$, we have the plane-wave expansion for the solenoidal waves

$$\begin{aligned}(-\mathbf{e}_\theta\mathbf{e}_\varphi+\mathbf{e}_\varphi\mathbf{e}_\theta)e^{i\mathbf{k}\cdot\mathbf{r}}=&\sum_{JM}4\pi i^{J}\left[\frac{1}{\sqrt{J(J+1)}}\Re_g\mathbf{M}_{JM}(\mathbf{r})\mathbf{C}^*_{JM}(\hat{k})\right.\\&\left.-\frac{i}{\sqrt{J(J+1)}}\Re_g\mathbf{N}_{JM}(\mathbf{r})\mathbf{B}^*_{JM}(\hat{k})\right]\times\mathbf{e}_k.\end{aligned}\tag{5.30}$$

For detailed discussions on the vectorial plane-wave expansion in terms of Hansen multipole fields, one can refer to [10,14,29,31,32].

5.5 Vector addition theorem

Letting $S=1$ in (3.15) gives vector addition theorem immediately. However, in practice, Hansen multipole fields are often used. Here we present a derivation of vector addition theorem based on Hansen multipole fields. This derivation is similar to that presented in [10,29], but here we include longitudinal waves. This derivation clearly shows that the translations of longitudinal waves and solenoidal waves are decoupled.

Consider the tensor addition theorem by letting (Fig. 1)

$$\mathbf{r}=\mathbf{R}+\mathbf{r}', \tag{5.31}$$

then

$$\mathbf{I}^{[1]}e^{i\mathbf{k}\cdot\mathbf{r}}=e^{i\mathbf{k}\cdot\mathbf{R}}\mathbf{I}^{[1]}e^{i\mathbf{k}\cdot\mathbf{r}'}. \tag{5.32}$$

Plugging (5.26) and (2.3) into (5.32), we have

$$\begin{aligned}
&\sum_{JM}4\pi i^J\left[\frac{1}{\sqrt{J(J+1)}}\Re_g\mathbf{M}_{JM}(\mathbf{r})\mathbf{C}^*_{JM}(\hat{k})-\frac{i}{\sqrt{J(J+1)}}\Re_g\mathbf{N}_{JM}(\mathbf{r})\mathbf{B}^*_{JM}(\hat{k})-i\Re_g\mathbf{L}_{JM}(\mathbf{r})\mathbf{P}^*_{JM}(\hat{k})\right]\\
&=\sum_{l''m''}4\pi i^{l''}Y_{l''m''}(\hat{R})j_{l''}(kR)Y^*_{l''m''}(\hat{k})\sum_{J'M'}4\pi i^{J'}\left[\frac{1}{\sqrt{J'(J'+1)}}\Re_g\mathbf{M}_{J'M'}(\mathbf{r}')\mathbf{C}^*_{J'M'}(\hat{k})\right.\\
&\quad\left.-\frac{i}{\sqrt{J'(J'+1)}}\Re_g\mathbf{N}_{J'M'}(\mathbf{r}')\mathbf{B}^*_{J'M'}(\hat{k})-i\Re_g\mathbf{L}_{J'M'}(\mathbf{r}')\mathbf{P}^*_{J'M'}(\hat{k})\right].
\end{aligned} \tag{5.33}$$

Taking dot product of both sides of (5.33) with $\mathbf{C}_{JM}(\hat{k})$, integrating over $\hat{k}$, and using orthogonality (5.7)-(5.9), we have

$$\begin{aligned}
\Re_g\mathbf{M}_{JM}(\mathbf{r})=\sum_{J'M'}\sum_{l''m''}4\pi i^{J'+l''-J}&\left[\frac{\sqrt{J(J+1)}}{\sqrt{J'(J'+1)}}\Re_g\mathbf{M}_{J'M'}(\mathbf{r}')Y_{l''m''}(\hat{R})j_{l''}(kR)G_{CC}\right.\\
&\left.+\frac{-i\sqrt{J(J+1)}}{\sqrt{J'(J'+1)}}\Re_g\mathbf{N}_{J'M'}(\mathbf{r}')Y_{l''m''}(\hat{R})j_{l''}(kR)G_{BC}\right],
\end{aligned} \tag{5.34}$$

where

$$G_{CC}=\int_{\bigcirc}d\hat{k}\mathbf{C}^*_{J'M'}(\hat{k})\cdot\mathbf{C}_{JM}(\hat{k})Y^*_{l''m''}(\hat{k}), \tag{5.35}$$

$$G_{BC}=\int_{\bigcirc}d\hat{k}\mathbf{B}^*_{J'M'}(\hat{k})\cdot\mathbf{C}_{JM}(\hat{k})Y^*_{l''m''}(\hat{k}). \tag{5.36}$$

Taking dot product both sides of (5.33) with $\mathbf{B}_{JM}(\hat{k})$, integrating over $\hat{k}$, using orthogonality (5.7)-(5.9), we have

$$\begin{aligned}\Re_g \mathbf{N}_{JM}(\mathbf{r}) = \sum_{J'M'} \sum_{l''m''} 4\pi i^{J'+l''-J} \Bigg[\frac{i\sqrt{J(J+1)}}{\sqrt{J'(J'+1)}} \Re_g \mathbf{M}_{J'M'}(\mathbf{r}') Y_{l''m''}(\hat{R}) j_{l''}(kR) G_{CB} \\ + \frac{\sqrt{J(J+1)}}{\sqrt{J'(J'+1)}} \Re_g \mathbf{N}_{J'M'}(\mathbf{r}') Y_{l''m''}(\hat{R}) j_{l''}(kR) G_{BB} \Bigg],\end{aligned} \tag{5.37}$$

where

$$G_{CB} = \int_{\bigcirc} d\hat{k} \mathbf{C}^*_{J'M'}(\hat{k}) \cdot \mathbf{B}_{JM}(\hat{k}) Y^*_{l''m''}(\hat{k}), \tag{5.38}$$

$$G_{BB} = \int_{\bigcirc} d\hat{k} \mathbf{B}^*_{J'M'}(\hat{k}) \cdot \mathbf{B}_{JM}(\hat{k}) Y^*_{l''m''}(\hat{k}). \tag{5.39}$$

Taking dot product of both sides of (5.33) with $\mathbf{P}_{JM}(\hat{k})$, integrating over $\hat{k}$, and using orthogonality (5.7)-(5.9), we have

$$\Re_g \mathbf{L}_{JM}(\mathbf{r}) = \sum_{J'M'} \sum_{l''m''} 4\pi i^{J'+l''-J} \Re_g \mathbf{L}_{J'M'}(\mathbf{r}') Y_{l''m''}(\hat{R}) j_{l''}(kR) G_{PP}, \tag{5.40}$$

where

$$G_{PP} = \int_{\bigcirc} d\hat{k} \mathbf{P}^*_{J'M'}(\hat{k}) \cdot \mathbf{P}_{JM}(\hat{k}) Y^*_{l''m''}(\hat{k}). \tag{5.41}$$

The Hansen spherical harmonics (5.2) have the following properties [10]

$$\mathbf{C}^*_{J'M'}(\hat{k}) \cdot \mathbf{C}_{JM}(\hat{k}) = \mathbf{B}^*_{J'M'}(\hat{k}) \cdot \mathbf{B}_{JM}(\hat{k}), \tag{5.42}$$

$$\mathbf{B}^*_{J'M'}(\hat{k}) \cdot \mathbf{C}_{JM}(\hat{k}) = -\mathbf{C}^*_{J'M'}(\hat{k}) \cdot \mathbf{B}_{JM}(\hat{k}). \tag{5.43}$$

It follows that

$$G_{BB} = G_{CC}, \quad G_{BC} = -G_{CB}. \tag{5.44}$$

With (5.44), we can summarize (5.34), (5.37), and (5.40) as

$$\begin{bmatrix} \Re_g \mathbf{M}_{JM}(\mathbf{r}) \\ \Re_g \mathbf{N}_{JM}(\mathbf{r}) \\ \Re_g \mathbf{L}_{JM}(\mathbf{r}) \end{bmatrix} = \sum_{J'M'} \sum_{l''m''} 4\pi i^{J'+l''-J} Y_{l''m''}(\hat{R}) j_{l''}(kR) [T_{JM,J'M'}] \begin{bmatrix} \Re_g \mathbf{M}_{J'M'}(\mathbf{r}') \\ \Re_g \mathbf{N}_{J'M'}(\mathbf{r}') \\ \Re_g \mathbf{L}_{J'M'}(\mathbf{r}') \end{bmatrix}, \tag{5.45}$$

where $[T_{JM,J'M'}]$ is a 3×3 matrix

$$[T_{JM,J'M'}] = \begin{pmatrix} \frac{\sqrt{J(J+1)}}{\sqrt{J'(J'+1)}} G_{CC} & \frac{-i\sqrt{J(J+1)}}{\sqrt{J'(J'+1)}} G_{BC} & 0 \\ \frac{-i\sqrt{J(J+1)}}{\sqrt{J'(J'+1)}} G_{BC} & \frac{\sqrt{J(J+1)}}{\sqrt{J'(J'+1)}} G_{CC} & 0 \\ 0 & 0 & G_{PP} \end{pmatrix}. \tag{5.46}$$

The Gaunt coefficients G_{CC} and G_{BC} (in slightly different formulations) were introduced in [10]. See Appendix B for the evaluations of the Gaunt coefficient G_{CC}, G_{BC} and G_{PP}. Plugging (B.12), (B.17) and (B.20) into (5.46), we have the explicit forms of the vector addition theorem. The translations of solenoidal fields (the parts determined by G_{CC} and G_{BC}) of the above explicit forms are same as those presented in Appendix D of reference [25], except for a sign error before the last 1 in the fourth line of (D.23a) of [25].

Using (5.13), (5.14), and (5.15), (3.8) and (3.9), and following the method in Section 3, we can extend the vector addition theorem (5.45) to include singular Hansen multipole fields

$$\begin{bmatrix} \mathbf{M}_{JM}(\mathbf{r}) \\ \mathbf{N}_{JM}(\mathbf{r}) \\ \mathbf{L}_{JM}(\mathbf{r}) \end{bmatrix} = \sum_{J'M'}\sum_{l''m''} 4\pi i^{J'+l''-J} Y_{l''m''}(\hat{R}) j_{l''}(kR)[T_{JM,J'M'}] \begin{bmatrix} \mathbf{M}_{J'M'}(\mathbf{r}') \\ \mathbf{N}_{J'M'}(\mathbf{r}') \\ \mathbf{L}_{J'M'}(\mathbf{r}') \end{bmatrix}, \; r'>R, \tag{5.47}$$

$$\begin{bmatrix} \mathbf{M}_{JM}(\mathbf{r}) \\ \mathbf{N}_{JM}(\mathbf{r}) \\ \mathbf{L}_{JM}(\mathbf{r}) \end{bmatrix} = \sum_{J'M'}\sum_{l''m''} 4\pi i^{J'+l''-J} Y_{l''m''}(\hat{R}) z_{l''}(kR)[T_{JM,J'M'}] \begin{bmatrix} \Re_g\mathbf{M}_{J'M'}(\mathbf{r}') \\ \Re_g\mathbf{N}_{J'M'}(\mathbf{r}') \\ \Re_g\mathbf{L}_{J'M'}(\mathbf{r}') \end{bmatrix}, \; r'<R. \tag{5.48}$$

Note that zero entries in (5.46) suggest that the solenoidal waves $\mathbf{M}_{JM}(\mathbf{r})$, and $\mathbf{N}_{JM}(\mathbf{r})$, and longitudinal waves, $\mathbf{L}_{JM}(\mathbf{r})$, are translated separately. This is because

$$\mathbf{P}^*_{J'M'}(\hat{k})\cdot\mathbf{B}_{JM}(\hat{k})=0, \quad \mathbf{P}^*_{J'M'}(\hat{k})\cdot\mathbf{C}_{JM}(\hat{k})=0 \tag{5.49}$$

holds for *all* $J'M',JM$. This decoupling of the solenoidal waves and longitudinal waves also implies that the wave numbers can be different for the solenoidal waves and longitudinal waves, just as in elastodynamics. One can refer to [33] for the detailed properties of elastodynamics. Also note that the translation of the longitudinal waves is just as that of scalar wave since *gradient* operator is translationally invariant [34, 35]. Furthermore, the block for solenoidal waves of the matrix (5.46) also shows that the diagonal elements are same, and off-diagonal elements are equal to each other [10], because

$$\mathbf{N}_{JM}(\mathbf{r}) = \frac{1}{k}\nabla\times\mathbf{M}_{JM}(\mathbf{r})$$

is translationally invariant [35].

It can be shown that the dyadic Green's function for electromagnetics can be written in terms of Hansen multipole fields as [25]

$$\bar{\mathbf{G}}(\mathbf{r},\mathbf{r}') = ik\sum_{JM}\frac{1}{J(J+1)}[\Re_g\mathbf{M}_{JM}(\mathbf{r})\mathbf{M}^*_{JM}(\mathbf{r}') + \Re_g\mathbf{N}_{JM}(\mathbf{r})\mathbf{N}^*_{JM}(\mathbf{r}')], \tag{5.50}$$

where $\mathbf{M}^*_{JM}(\mathbf{r}')$, $\mathbf{N}^*_{JM}(\mathbf{r}')$, and $\mathbf{L}^*_{JM}(\mathbf{r}')$ are specialized by letting $z_l(kr) = h^{(1)}_l(kr)$. This formulation of the dyadic Green's function (5.50) can be factorized in terms of the vector field translator [10]. The diagonalizations of the vector addition theorem based on Hansen multipole fields has been discussed in great detail in [10, 29].

6 Concluding remarks

An alternative derivation of the tensor addition theorem has been obtained based on the generalization of the Funk-Hecke formula and Rayleigh plane-wave expansion for tensor fields. This new derivation facilitates the diagonalization of the tensor addition theorem. Our diagonal form unifies scalar, vector, and general tensor fields such that the implementation of scalar, vector, and tensor fields can be treated in a unified fashion. Moreover, we have discussed vector multipole fields and vector addition theorem from a different angle.

In a FMM, the summation in (4.7) should be truncated, and the integration in (4.6) is approximated by quadrature rules. The error analysis for the truncations and integrations will be studied in future work.

A The orthogonality and completeness relationships of the tensor spherical harmonics

In this Appendix, we shall prove the orthogonality (2.13) and completeness (2.14) relationships of the tensor spherical harmonics.

A.1 The orthogonality of the tensor spherical harmonics

$$\begin{aligned}
&\int_{\bigcirc} d\hat{k}\mathbf{Y}^{*[J']}_{l'SM'}(\hat{k})\cdot\mathbf{Y}^{[J]}_{lSM}(\hat{k})\\
&=\int_{\bigcirc} d\hat{k}\sum_{m'\mu'}(\langle l'm'S\mu'|l'SJ'M'\rangle)^{*}Y^{*}_{l'm'}(\hat{k})\mathbf{e}^{*[S]}_{\mu'}\cdot\sum_{m\mu}\langle lmS\mu|lSJM\rangle Y_{lm}(\hat{k})\mathbf{e}^{[S]}_{\mu}\\
&=\int_{\bigcirc} d\hat{k}\sum_{m'\mu'}\langle l'SJ'M'|l'm'S\mu'\rangle Y^{*}_{l'm'}(\hat{k})\sum_{m\mu}\langle lmS\mu|lSJM\rangle Y_{lm}(\hat{k})\delta_{\mu'\mu}\\
&=\sum_{m'\mu m}\langle l'SJ'M'|l'm'S\mu\rangle\langle lmS\mu|lSJM\rangle\delta_{l'l}\delta_{m'm}\\
&=\sum_{\mu m}\langle lSJ'M'|lmS\mu\rangle\langle lmS\mu|lSJM\rangle\delta_{l'l}=\delta_{JJ'}\delta_{ll'}\delta_{MM'}.
\end{aligned}\tag{A.1}$$

In the above, we have used (2.6), the orthogonality of the spherical harmonics

$$\int_{\bigcirc} d\hat{k}Y^{*}_{l'm'}(\hat{k})Y_{lm}(\hat{k})=\delta_{l'l}\delta_{m'm},\tag{A.2}$$

and the unitary property of the Clebsch-Gordan coefficient [12]

$$\sum_{\mu m}\langle lSJ'M'|lmS\mu\rangle\langle lmS\mu|lSJM\rangle=\delta_{JJ'}\delta_{MM'}\delta(lSJ)\tag{A.3}$$

with $\delta(lSJ)=1$.

To the best of our knowledge, the orthogonality of the vector spherical harmonics (special case of (2.13) with $S=1$) was first presented in [36].

A.2 The completeness of the tensor spherical harmonics

$$
\begin{aligned}
\sum_{JlM} \mathbf{Y}_{lSM}^{[J]}(\hat{k})\mathbf{Y}_{lSM}^{*[J]}(\hat{k}') &= \sum_{JlM}\sum_{m\mu} \langle lmS\mu|lSJM\rangle Y_{lm}(\hat{k})\mathbf{e}_{\mu}^{[S]} \sum_{m'\mu'} (\langle lm'S\mu'|lSJM\rangle)^* Y_{lm'}^*(\hat{k}')\mathbf{e}_{\mu'}^{*[S]} \\
&= \sum_{JlM}\sum_{m\mu m'\mu'} \langle lmS\mu|lSJM\rangle\langle lSJM|lm'S\mu'\rangle Y_{lm}(\hat{k})Y_{lm'}^*(\hat{k}')\mathbf{e}_{\mu}^{[S]}\mathbf{e}_{\mu'}^{*[S]} \\
&= \sum_{l}\sum_{m\mu m'\mu'} \delta_{mm'}\delta_{\mu\mu'} Y_{lm}(\hat{k})Y_{lm'}^*(\hat{k}')\mathbf{e}_{\mu}^{[S]}\mathbf{e}_{\mu'}^{*[S]} \\
&= \sum_{l}\sum_{m\mu} Y_{lm}(\hat{k})Y_{lm}^*(\hat{k}')\mathbf{e}_{\mu}^{[S]}\mathbf{e}_{\mu}^{*[S]} \\
&= \sum_{\mu} \mathbf{e}_{\mu}^{[S]}\mathbf{e}_{\mu}^{*[S]}\delta(\hat{k}-\hat{k}') = \mathbf{I}^{[S]}\delta(\hat{k}-\hat{k}'). \qquad \text{(A.4)}
\end{aligned}
$$

In the above, we have used the completeness of the spherical harmonics

$$\sum_{lm} Y_{lm}(\hat{k})Y_{lm}^*(\hat{k}') = \delta(\hat{k}-\hat{k}'), \tag{A.5}$$

and the unitary property of the Clebsch-Gordan coefficient [12]

$$\sum_{JM} \langle lmS\mu|lSJM\rangle\langle lSJM|lm'S\mu'\rangle = \delta_{mm'}\delta\mu\mu'. \tag{A.6}$$

Since LHS of (2.14) can be written as

$$\sum_{JlM}\sum_{\mu\nu} [\mathbf{Y}_{lSM}^{[J]}(\hat{k})]^{\mu} [\mathbf{Y}_{lSM}^{*[J]}(\hat{k}')]^{\nu} \mathbf{e}_{\mu}^{[S]}\mathbf{e}_{\nu}^{*[S]} \tag{A.7}$$

and RHS of (2.14) can be written as

$$\sum_{\mu\nu} \mathbf{e}_{\mu}^{[S]}\mathbf{e}_{\nu}^{*[S]}\delta_{\mu\nu}\delta(\hat{k}-\hat{k}'), \tag{A.8}$$

we have

$$\sum_{JlM}\sum_{\mu\nu} [\mathbf{Y}_{lSM}^{[J]}(\hat{k})]^{\mu} [\mathbf{Y}_{lSM}^{*[J]}(\hat{k}')]^{\nu} \mathbf{e}_{\mu}^{[S]}\mathbf{e}_{\nu}^{*[S]} = \sum_{\mu\nu} \mathbf{e}_{\mu}^{[S]}\mathbf{e}_{\nu}^{*[S]}\delta_{\mu\nu}\delta(\hat{k}-\hat{k}'), \tag{A.9}$$

which suggests another formulation of the completeness of the tensor spherical harmonics

$$\sum_{JlM} [\mathbf{Y}_{lSM}^{[J]}(\hat{k})]^{\mu} [\mathbf{Y}_{lSM}^{*[J]}(\hat{k}')]^{\nu} = \delta_{\mu\nu}\delta(\hat{k}-\hat{k}'). \tag{A.10}$$

To the best of our knowledge, the formulation (A.10) was first presented in reference [37] without proof.

B Evaluations of the Gaunt coefficients

As discussed in Sections 3, 4 and 5, the generalized Gaunt coefficients are introduced to facilitate the diagonalizations of the vector and tensor addition theorems. In this Appendix, for complete derivations of the vector and tensor addition theorems, we shall carry out the evaluations of the Gaunt coefficients. Moreover, the evaluations of the Gaunt coefficients are useful for the applications of the vector and tensor addition theorems.

B.1 Evaluation of the generalized Gaunt coefficient

The tensor spherical harmonics $\mathbf{Y}_{lSM}^{[J]}(\hat{k})$ can be written in terms of Wigner 3-j symbol as

$$\mathbf{Y}_{lSM}^{[J]}(\hat{k})=\sum_{m\mu}(-1)^{l-S+M}(2J+1)^{\frac{1}{2}}\begin{pmatrix} l & S & J \\ m & \mu & -M \end{pmatrix} Y_{lm}(\hat{k})\mathbf{e}_{\mu}^{[S]}. \tag{B.1}$$

Similarly,

$$\mathbf{Y}_{l'SM'}^{*[J']}(\hat{k})=\sum_{m'\nu}(-1)^{l'-S+M'}(2J'+1)^{\frac{1}{2}}\begin{pmatrix} l' & S & J' \\ m' & \nu & -M' \end{pmatrix} Y_{l'm'}^{*}(\hat{k})\mathbf{e}_{\nu}^{*[S]}. \tag{B.2}$$

Plugging (B.1) and (B.2) into (3.5), we have

$$\begin{aligned} G^{[S]}(l'J'M'|lJM|l''m'')=&\sum_{m\mu m'\nu}(-1)^{l+M+l'+M'}(2J+1)^{\frac{1}{2}}(2J'+1)^{\frac{1}{2}}\mathbf{e}_{\nu}^{*[S]}\cdot\mathbf{e}_{\mu}^{[S]} \\ &\times\begin{pmatrix} l & S & J \\ m & \mu & -M \end{pmatrix}\begin{pmatrix} l' & S & J' \\ m' & \nu & -M' \end{pmatrix} G^{[0]}(l'm'|lm|l''m''), \end{aligned} \tag{B.3}$$

where $G^{[0]}(l'm'|lm|l''m'')$ is the usual Gaunt coefficient (the solid angle integral of the product of triple spherical harmonics)

$$G^{[0]}(l'm'|lm|l''m'')=\oint d\hat{k}Y_{l'm'}^{*}(\hat{k})Y_{lm}(\hat{k})Y_{l''m''}^{*}(\hat{k}). \tag{B.4}$$

The use of (2.6) simplifies (B.3) to be

$$\begin{aligned} G^{[S]}(l'J'M'|lJM|l''m'')=&\sum_{m\mu m'}(-1)^{l+M+l'+M'}(2J+1)^{\frac{1}{2}}(2J'+1)^{\frac{1}{2}} \\ &\times\begin{pmatrix} l & S & J \\ m & \mu & -M \end{pmatrix}\begin{pmatrix} l' & S & J' \\ m' & \mu & -M' \end{pmatrix} G^{[0]}(l'm'|lm|l''m''). \end{aligned} \tag{B.5}$$

The Gaunt coefficient $G^{[0]}(l'm'|lm|l''m'')$ can be evaluated by either interpreting one of the triple spherical harmonics as a spherical harmonics operator and using the Wigner-

Eckart theorem [19,37] or specializing the integral of a triple product of rotation D functions [12]. Its value written in terms of Wigner 3-j symbols is

$$G^{[0]}(l'm'|lm|l''m'') = (-1)^m[(2l+1)(2l'+1)(2l''+1)/4\pi]^{\frac{1}{2}} \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & m'' \end{pmatrix}. \tag{B.6}$$

Plugging (B.6) into (B.5), we have

$$\begin{aligned} G^{[S]}(l'J'M'|lJM|l''m'') = &\sum_{m\mu m'} (-1)^{l+M+l'+M'}(2J+1)^{\frac{1}{2}}(2J'+1)^{\frac{1}{2}} \\ &\times \begin{pmatrix} l & S & J \\ m & \mu & -M \end{pmatrix} \begin{pmatrix} l' & S & J' \\ m' & \mu & -M' \end{pmatrix} (-1)^m[(2l+1)(2l'+1)(2l''+1)/4\pi]^{\frac{1}{2}} \\ &\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & m'' \end{pmatrix}. \end{aligned} \tag{B.7}$$

We rewrite the summation of the triple product of the Wigner 3-j symbols in (B.7) as follows

$$\begin{aligned} &\sum_{m,\mu,m'} (-1)^m \begin{pmatrix} l & S & J \\ m & \mu & -M \end{pmatrix} \begin{pmatrix} l' & S & J' \\ m' & \mu & -M' \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & m'' \end{pmatrix} \\ = &\sum_{m,\mu,m'} (-1)^m \begin{pmatrix} J & l & S \\ -M & m & \mu \end{pmatrix} \begin{pmatrix} l' & J' & S \\ -m' & M' & -\mu \end{pmatrix} \begin{pmatrix} l' & l & l'' \\ m' & -m & m'' \end{pmatrix}. \end{aligned} \tag{B.8}$$

In the above, we have used the symmetry properties of Wigner 3-j symbols, and $l+l'+l''$ being an even integer. Let $\tilde{\mu} = -\mu$, (B.8) can be written as

$$\sum_{m,\tilde{\mu},m'} (-1)^m \begin{pmatrix} J & l & S \\ -M & m & -\tilde{\mu} \end{pmatrix} \begin{pmatrix} l' & J' & S \\ -m' & M' & \tilde{\mu} \end{pmatrix} \begin{pmatrix} l' & l & l'' \\ m' & -m & m'' \end{pmatrix}. \tag{B.9}$$

Wigner 3-j and 6-j symbols have the relation

$$\begin{aligned} &\sum_{m,\tilde{\mu},m'} (-1)^{l'+l+S+m'+m+\tilde{\mu}} \begin{pmatrix} J & l & S \\ -M & m & -\tilde{\mu} \end{pmatrix} \begin{pmatrix} l' & J' & S \\ -m' & M' & \tilde{\mu} \end{pmatrix} \begin{pmatrix} l' & l & l'' \\ m' & -m & m'' \end{pmatrix} \\ = &\begin{pmatrix} J & J' & l'' \\ -M & M' & m'' \end{pmatrix} \begin{Bmatrix} J & J' & l'' \\ l' & l & S \end{Bmatrix}. \end{aligned} \tag{B.10}$$

Notice that left-hand side of (B.10) is not the product of *arbitrary* triple Wigner 3-j symbols. Using (B.10) and (B.9), we have the generalized Gaunt coefficient in terms of Wigner 3-j and 6-j symbols

$$\begin{aligned} G^{[S]}(l'J'M'|lJM|l''m'') = &(-1)^{M+S}[(2J+1)(2J'+1)(2l+1)(2l'+1)(2l''+1)/4\pi]^{\frac{1}{2}} \\ &\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J & J' & l'' \\ -M & M' & m'' \end{pmatrix} \begin{Bmatrix} J & J' & l'' \\ l' & l & S \end{Bmatrix}. \end{aligned} \tag{B.11}$$

B.2 Evaluation of the Gaunt coefficient G_{CC}

$$
\begin{aligned}
G_{CC}=&\int_{\bigcirc} d\hat{k}\mathbf{C}^*_{J'M'}(\hat{k})\cdot\mathbf{C}_{JM}(\hat{k})Y^*_{l''m''}(\hat{k})=\int_{\bigcirc} d\hat{k}\mathbf{Y}^{*[J']}_{J',M'}(\hat{k})\cdot\mathbf{Y}^{[J]}_{J,M}(\hat{k})Y^*_{l''m''}(\hat{k})\\
=&(-1)^{M+1}[(2J+1)(2J'+1)(2J+1)(2J'+1)(2l''+1)/4\pi]^{\frac{1}{2}}\\
&\times\begin{pmatrix} J & J' & l'' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} J & J' & l'' \\ -M & M' & m'' \end{pmatrix}\begin{Bmatrix} J & J' & l'' \\ J' & J & 1 \end{Bmatrix}\\
=&(-1)^{M}[(2J+1)(2J'+1)(2l''+1)/4\pi]^{\frac{1}{2}}\frac{J'(J'+1)+J(J+1)-l''(l''+1)}{2[J'(J'+1)J(J+1)]^{\frac{1}{2}}}\\
&\times\begin{pmatrix} J & J' & l'' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} J & J' & l'' \\ -M & M' & m'' \end{pmatrix}.
\end{aligned}
\tag{B.12}
$$

In the above, we have used the relation (5.6), (B.11) and the property of the Wigner 6-j symbol [12]

$$
\begin{Bmatrix} J & J' & l'' \\ J' & J & 1 \end{Bmatrix}=(-1)^{J+J'+l''+1}\frac{2[J'(J'+1)+J(J+1)-l''(l''+1)]}{[2J'(2J'+1)(2J'+2)2J(2J+1)(2J+2)]^{\frac{1}{2}}}.
\tag{B.13}
$$

B.3 Evaluation of the Gaunt coefficient G_{BC}

Using (5.4), (5.5), (5.6) and (5.49), we can write the Gaunt coefficient G_{BC} as

$$
\begin{aligned}
G_{BC}=&\int_{\bigcirc} d\hat{k}\mathbf{B}^*_{J'M'}(\hat{k})\cdot\mathbf{C}_{JM}(\hat{k})Y^*_{l''m''}(\hat{k})\\
=&-i\int_{\bigcirc} d\hat{k}\frac{\sqrt{2J'+1}}{\sqrt{J'}}\mathbf{Y}^{*[J']}_{J'+1,M'}(\hat{k})\cdot\mathbf{Y}^{[J]}_{J,M}(\hat{k})Y^*_{l''m''}(\hat{k}).
\end{aligned}
\tag{B.14}
$$

Using (B.11), the property of the Wigner 6-j symbol [12]

$$
\begin{aligned}
&\begin{Bmatrix} J & J' & l'' \\ J'+1 & J & 1 \end{Bmatrix}\\
=&(-1)^{J+J'+l''+1}\left[\frac{2(l''+J+J'+2)(-l''+J+J'+1)(l''-J+J'+1)(l''+J-J')}{2J(2J+1)(2J+2)(2J'+1)(2J'+2)(2J'+3)}\right]^{\frac{1}{2}}
\end{aligned}
\tag{B.15}
$$

and the property of the Wigner 3-j symbol

$$
\begin{pmatrix} J & J'+1 & l'' \\ 0 & 0 & 0 \end{pmatrix}=-\left[\frac{(J+J'+1+l'')(J'+l''-J)}{(J'+l''-J+1)(J+J'+l''+2)}\right]^{\frac{1}{2}}\times\begin{pmatrix} J & J' & l''-1 \\ 0 & 0 & 0 \end{pmatrix},
\tag{B.16}
$$

we obtain the evaluation of (B.14) as

$$
\begin{aligned}
G_{BC}=&-i\alpha(J,J',l'')(-1)^{M}\frac{1}{2\sqrt{J(J+1)}\sqrt{J'(J'+1)}}\\
&\times\begin{pmatrix} J & J' & l''-1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} J & J' & l'' \\ -M & M' & m'' \end{pmatrix},
\end{aligned}
\tag{B.17}
$$

where

$$\alpha(J,J',l'') = [(2J+1)(2J'+1)(2l''+1)/4\pi]^{\frac{1}{2}} \\ \times [(J+J'+l''+1)(J'+l''-J)(J+J'-l''+1)(l''+J-J')]^{\frac{1}{2}}. \tag{B.18}$$

B.4 Evaluation of the Gaunt coefficient G_{PP}

Observe that

$$\begin{aligned} G_{PP} &= \int_{\bigcirc} d\hat{k} \mathbf{P}^*_{J'M'}(\hat{k}) \cdot \mathbf{P}_{JM}(\hat{k}) Y^*_{l''m''}(\hat{k}) \\ &= \int_{\bigcirc} d\hat{k} \hat{\mathbf{e}}^*_k Y^*_{J'M'}(\hat{k}) \cdot \hat{\mathbf{e}}_k Y_{JM}(\hat{k}) Y^*_{l''m''}(\hat{k}) \\ &= \int_{\bigcirc} d\hat{k} Y^*_{J'M'}(\hat{k}) Y_{JM}(\hat{k}) Y^*_{l''m''}(\hat{k}) = G^{[0]}(J'M'|JM|l''m''). \end{aligned} \tag{B.19}$$

Note that the identity $G_{PP} = G^{[0]}(J'M'|JM|l''m'')$ provides another insight for the fact that the translation of the longitudinal waves is just as that of scalar wave. Using (B.6), we have

$$G_{PP} = (-1)^M [(2J+1)(2J'+1)(2l''+1)/4\pi]^{\frac{1}{2}} \times \begin{pmatrix} J & J' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J & J' & l'' \\ -M & M' & m'' \end{pmatrix}. \tag{B.20}$$

It is remarked that the Gaunt coefficients G_{CC}, G_{BC} and G_{PP} can be generally written as

$$(-1)^M f(J,J',l'') \begin{pmatrix} J & J' & l'' \\ -M & M' & m'' \end{pmatrix}. \tag{B.21}$$

Since $f(J,J',l'')$ depends only on 3 parameters, and

$$\begin{pmatrix} J & J' & l'' \\ -M & M' & m'' \end{pmatrix}$$

depends on 6 parameters, the computation cost of (B.21) is mainly determined by the evaluation of the Wigner 3-j symbol. For the fast evaluation of the Wigner 3-j symbol, one can refer to [38]. The evaluation of vector translation coefficients based on the recurrence relations was presented in [39].

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