COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. 5, No. 2-4, pp. 745-759

Commun. Comput. Phys.
February 2009

# Numerical Calculation of Monotonicity Properties of the Blow-Up Time of NLS

Hans Peter Stimming*

*Wolfgang Pauli Institute, c/o Institut für Mathematik, Universität Wien, Botzmanngasse 9, 1090 Wien, Austria.*

Received 9 October 2007; Accepted (in revised version) 18 December 2007

Available online 1 August 2008

**Abstract.** We investigate blow-up of the focusing nonlinear Schrödinger equation, in the critical and supercritical cases. Numerical simulations are performed to examine the dependence of the time at which blow-up occurs on properties of the data or the equation. Three cases are considered: dependence on the scale of the nonlinearity when the initial data are fixed; dependence upon the strength of a quadratic oscillation in the initial data when the equation and the initial profile are fixed; and dependence upon a damping factor when the initial data are fixed. In most of these situations, monotonicity in the evolution of the blow-up time does not occur.

**AMS subject classifications**: 35Q55, 65M70, 81Q05

**Key words**: Nonlinear Schrödinger equation, finite time blow-up, wave collapse.

## 1 Introduction

Finite time blow-up of Nonlinear Schrödinger equations (NLS) is an interesting topic that has drawn much attention. A (by far) non-exhaustive list of important papers on the topic is [10, 15–19, 23], and we refer to [21] for a nice survey of the latest results. As recalled in [21], the main three directions of research in this subject are: giving sufficient conditions to have finite time blow-up in the energy space; estimating the blow-up rate and the stability of the blow-up regimes; describing the spatial structure of the singularity formation.

The question addressed in this paper is to investigate the blow-up time and its possible relations to features of the equation or the data. This point of view differs from much of the other work on the subject of blow-up in the sense that it is not concerned only with properties of the solution "close" to the blow-up, but also needs to take into account parts

*Corresponding author. *Email address:* hans.peter.stimming@univie.ac.at (H. P. Stimming)

of the solution which actually are "far away" (either in space or in time) from the actual blow-up region.

The present paper is an extension of [8], where a detailed study of the relevant analytic results is done in addition to the numerical tests.

Consider the nonlinear Schrödinger equation

$$i\partial_t u+\Delta u=-\lambda|u|^{2\sigma}u\ ,\ (t,x)\in\mathbb{R}_+\times\mathbb{R}^n;\quad u_{|t=0}=u_0, \tag{1.1}$$

with a focusing power nonlinearity, where $\lambda,\sigma>0$. Such an equation arises in nonlinear optics as an envelope equation in the propagation of laser beams (see, e.g., [22]), and also in applications of quantum mechanics, where other terms like confining potentials and coupling to Poisson's equations are usually also a part of the model equation. It is well known that if $\sigma<\frac{2}{n-2}$, then for $u_0\in H^1(\mathbb{R}^n)$ the equation (1.1) has a unique solution in $H^1(\mathbb{R}^n)$, defined locally in time. In general this solution does not remain in $H^1(\mathbb{R}^n)$ globally in time, when $\sigma\geq\frac{2}{n}=\sigma_{crit}$, finite time blow-up may occur ($\lambda>0$ means that the nonlinearity is focusing). For proofs of these standard results we refer to the monographs [11, 22]. The $L^2$-norm or mass of $u(t,\cdot)$ is independent of time, and finite time blow-up means that there exists $T^*<\infty$ such that:

$$\|\nabla_x u(t)\|_{L^2}\to+\infty\quad\text{as } t\to T^*.$$

In this paper, we investigate by numerical experiments the dependence of the blow-up time upon, for instance, a varying coupling constant $\lambda$ when the initial datum $u_0$ is fixed. To motivate our study, we recall now some results from [12, 13]. In [12], the authors prove that if the initial datum $u_0(x)$ is replaced by $u_0(x)e^{-ib|x|^2/4}$, then the blow-up time of the corresponding new solution $u_b$ can be related explicitly to that of $u$, in the case of a critical nonlinearity, $\sigma=\frac{2}{n}$. It is a consequence of the conformal invariance. In the super-critical case $\sigma>\frac{2}{n}$, the conformal transform does not leave (1.1) invariant. It is also established that if $u$ has negative energy (in this case, there is finite time blow-up at least if $xu_0\in L^2(\mathbb{R}^n)$ [14]) then *for large* $b$, blow-up occurs sooner than for $b=0$; unlike in the conformally invariant case, one does not know whether the blow-up time is monotonous with respect to $b$. The numerical experiments we present here show that it is not monotonous with respect to $b$.

In [13], the author considers the damped cubic Schrödinger equation in space dimension two:

$$i\partial_t\psi+\Delta\psi=-i\delta\psi-|\psi|^{2\sigma}\psi\ ,\ (t,x)\in\mathbb{R}_+\times\mathbb{R}^2;\quad \psi_{|t=0}=u_0, \tag{1.2}$$

with $\sigma=1$. It is conjectured that the blow-up time is monotonous with respect to $\delta>0$. Our numerical experiments show that this guess is not satisfied. The guess is plausible when one thinks of the initial data $u_0$ as a single hump, for example a gaussian. In this case the experiments show monotonicity; however when the data $u_0$ is made of, say, two humps, the intuition goes wrong.

In addition to the linear damping term in (1.2), often also a cubic or quintic nonlinear damping term is considered, for example in BEC modeling. A study of this case is beyond the scope of this paper. We note that in this case, blow-up is usually prevented (see [22]) and refer to [1,2,5] for further numerical studies of the damped NLS including the case of nonlinear damping.

Note that introducing $u(t,x)=e^{\delta t}\psi(t,x)$, Eq. (1.2) is equivalent to:

$$i\partial_t u+\Delta u=-e^{-2\delta\sigma t}|u|^2 u\ ,\ (t,x)\in\mathbb{R}_+\times\mathbb{R}^2;\quad u_{|t=0}=u_0. \tag{1.3}$$

This transform yields an equation of the form (1.1), with a time-dependent coupling "constant", $\lambda=e^{-2\delta\sigma t}$. This function of time is monotonous and decreasing.

This leads us to the question, when $\lambda$ is constant: is the blow-up time monotonous with respect to $\lambda$? Numerics show that both in the critical ($\sigma=\frac{2}{n}$) and in the supercritical case ($\sigma>\frac{2}{n}$), one should not expect the blow-up time to be monotonous with respect to $\lambda$.

In the rest of this paper, in Section 2, we recall some known analytical results on blow-up time of NLS, and the numerical tests are presented in Section 3.

## 2 Review of theoretical results

In this section, we give a short overview of some analytical results that provide bounds, from above and/or from below, for blow-up time, which are a more quantitative motivation for the numerical tests done for this study. The proofs together with a more detailed discussion of these results are given in [8]. It seems that the explicit dependence of the existence time upon some parameters had not been investigated before, except in [12].

We recall the fact that for small initial data, the solution to (1.1) does not blow up. Indeed, for small data, the conservations of mass and energy yield an a priori bound on the $H^1$-norm of the solution, thus ruling out finite time blow-up. This is why in the following proposition, $\lambda$ is "large" for fixed $u_0$.

**Proposition 2.1** (Dependence with respect to the coupling constant)**.** Let $\lambda>0$, $\sigma\geq\frac{2}{n}$ with $\sigma<\frac{2}{n-2}$ if $n\geq 3$, and $u_0\in\Sigma=H^1\cap\{f\ \big|\ xf\in L^2\}$. Assume that $u$ blows up in finite time $T^*>0$.

- We have
$$T^*\geq C\langle\lambda\rangle^{-\frac{2\sigma}{2-(n-2)\sigma}},$$
for some constant $C$ independent of $\lambda$, where $\langle\lambda\rangle=\sqrt{1+\lambda^2}$.
- If in addition $E<0$, then $T^*\leq C'\langle\lambda\rangle^{-1/2}$, for some constant $C'$ independent of $\lambda$.

The constants $C$ and $C'$ are independent of $\lambda$, but depend on the other parameters, $u_0$, $n$ and $\sigma$. We always have

$$\frac{2\sigma}{2-(n-2)\sigma}>\frac{1}{2},$$

so the above two bounds go to zero with different rates when $\lambda\to+\infty$. For the proof of this proposition, see [8].

Without even trying to see if any of these bounds is sharp, we ask the following question:

**Question 1.** For $\sigma \geq \frac{2}{n}$ and a fixed initial datum $u_0 \in \Sigma$, is the blow-up time for $u$ solution to (1.1) monotonous with respect to $\lambda$?

The results of our numerical simulations show that the answer should be no.

We now consider equation (1.2), respectively (1.3). A direct application of standard existence results gives that for $\delta$ sufficiently large, $u$ is defined globally in time, in the future. We ask the following question :

**Question 2.** For $\sigma \geq \frac{2}{n}$ and a fixed $u_0 \in \Sigma$, is the blow-up time for $u$ solution to (1.3) monotonous with respect to $\delta > 0$?

The simulations show that the answer is no.

### Initial data with quadratic oscillations

Like in [12], we now fix the equation, and alter only the initial data, with quadratic oscillations. For $a \neq 0$, define:

$$v(t,x) = \frac{e^{i\frac{|x|^2}{4(t-a)}}}{h(t)^{n/2}} u\left(\frac{at}{a-t}, \frac{x}{h(t)}\right), \tag{2.1}$$

where $h(t) = (a-t)/a$. Then $v$ solves:

$$\begin{aligned} &i\partial_t v + \Delta v = -h(t)^{n\sigma-2}|v|^{2\sigma}v, \\ &v_{|t=0} = u_0(x)e^{-i\frac{|x|^2}{4a}}. \end{aligned} \tag{2.2}$$

In the critical case $\sigma = 2/n$, $v$ solves the same equation as $u$. The only difference is the presence of (additional) quadratic oscillations in the data.

**Proposition 2.2.** Let $u_0 \in \Sigma$ and $2/n \leq \sigma < 2/(n-2)$. Suppose that $u$ blows up at time $T^* > 0$. Let $a \in \mathbb{R}^*$.

- If $a > 0$, then $v$ blows up at
$$T_a(v) = \frac{a}{a+T^*} T^* < T^*.$$
- If $a < 0$ and $a + T < 0$, then $v$ blows up at
$$T_a(v) = \frac{a}{a+T^*} T^* > T^*.$$
- If $a < 0$ and $a + T^* \geq 0$, then $v$ is globally defined in $\Sigma$ for positive times (but blows up in the past if $a + T^* > 0$).

For the critical case $\sigma = 2/n$, this result is proved in [12] (see also [11]). For a slightly different proof, which also includes the case $2/n < \sigma < 2/(n-2)$, see [8].

In the super-critical case, a natural question is to understand the role of the function $h$. Introduce $w$ solving:

$$i\partial_t w + \Delta w = -|w|^{2\sigma} w; \quad w_{|t=0} = u_0(x) e^{-i\frac{|x|^2}{4a}}. \tag{2.3}$$

**Proposition 2.3.** Let $u_0 \in \Sigma$ and $2/n < \sigma < 2/(n-2)$. If the energy $E$ of $u$ is negative, then for $a > 0$, $w$ blows up at time $T_a(w) \leq a$.

This result is proven in [12], and relies on the pseudo-conformal law for $w$.

To understand the influence of the quadratic oscillations on the blow-up time, we have to compare the blow-up time of $u$ and that of $w$. In the critical case, the blow-up time depends explicitly on the magnitude of the quadratic oscillations *via* Proposition 2.2, since $v \equiv w$ by conformal invariance. In the super-critical case, we ask:

**Question 3.** For $\sigma > \frac{2}{n}$ and a fixed $u_0 \in \Sigma$, is the blow-up time for $w$ solving (2.3) monotonous with respect to $a$?

This issue is addressed numerically in Section 3.3: we first compare the numerics with the analytical results in the conformally invariant case, then perform tests in the supercritical case which indicate that the answer to the question above should be no.

# 3 Numerical tests

We perform numerical tests by using a direct discretization method for Eq. (1.1), respectively (1.3). We employ two different numerical methods: the Time-Splitting Spectral method (TSSP), and the Relaxation method (RS).

The TSSP is based on an operator splitting method, the split-step method. The flow of the nonlinear equation (1.1) (or (1.3)) is decomposed into a linear (free Schrödinger) part, and a nonlinear part. A spectral method is employed to compute the flow of the free Schrödinger equation. The nonlinear flow, which is the flow of an ODE, can be computed exactly, so its integration is straightforward. The TSSP has proven to be an efficient and reliable method for NLS type equations. See for example [3,4] for a study of the NLS in the semi-classical limit case, and [9,20] for a more general numerical study. As a recent example for the extensibility of this method beyond the standard NLS case, we cite the work [6] in which the TSSP is successfully applied to a coupled system of NLS for optical interactions.

For the 2-d calculation, a parallel version of the TSSP scheme is used on the parallel cluster machine "Schrödinger III" at the University of Vienna.

The Relaxation method (RS) is a discretization of finite difference type [7]. It is based on central-difference approximation shifted by a half time-step. All tests are done with both methods.

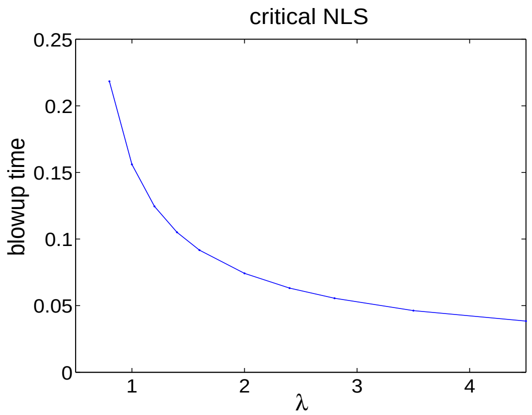



Figure 1: Blow-up time with varying $\lambda$, single Gaussian data (Test 1).

To determine whether blow-up is occurring or not, we calculate the two terms in the energy, kinetic and potential energy, and look for an increase of at least four orders of magnitude in both of them. The first time at which this is occurring is assumed to be the blow-up time.

## 3.1 Dependence on $\lambda$

### 3.1.1 Test in one space dimension

First we consider (1.1) for $\sigma=\frac{2}{n}$, that is the critical case. We study the dependence of the blow-up time on the constant $\lambda$ for a series of different data.

**Test 1.** For the one dimensional case $n=1$, the first kind of data we study is

$$u_0(x)=Ce^{-x^2}e^{-i\log(e^x+e^{-x})}. \tag{3.1}$$

The constant $C$ is equal to 1.75.We take $Np=2^{12}=4096$ mesh points, and several time step sizes, with $\Delta t=2.5\cdot 10^{-6}$ as smallest. The discretization domain is $[-8,8]$. Fig. 1 shows the blow-up time in relation to a changing $\lambda$. It can be observed that the blow-up time is decreasing monotonously with $\lambda$, as predicted by the heuristics for the case of a single Gaussian profile.

**Test 2.** The next kind of data we study is

$$u_0(x)=C\left(e^{-x^2}-0.9e^{-3x^2}\right)e^{-i\log(e^x+e^{-x})}. \tag{3.2}$$

The constant $C$ is equal to 4 which leads to $\|u_0\|_{L^2}^2=3.907$. The difference of two Gaussian profiles results in two humps in the modulus of $u_0$. The phase term has a focusing effect and its focus point does not agree with the centers of the humps. The finest discretization parameters used are $Np=2^{14}=16384$, $\Delta t=2.5\cdot 10^{-6}$. The discretization domain is $[-8,8]$. Fig. 2 shows the blow-up time in relation to a changing $\lambda$. It can be observed that for low

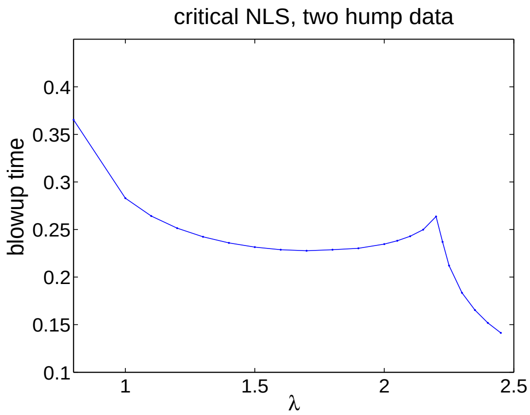



Figure 2: Blow-up time with varying $\lambda$, two hump data (Test 2).

and very high strengths of the nonlinearity $\lambda$ ($\lambda < 1.6$ and $\lambda > 2.2$), the blow-up time is monotonously decreasing with $\lambda$, while in between there is a region where monotonicity does not hold.

Heuristically, two effects play a role here: the self-focusing, which tries to focus the mass to where the most mass is already, and the (linear) phase influence, which tends to focus the mass at zero.

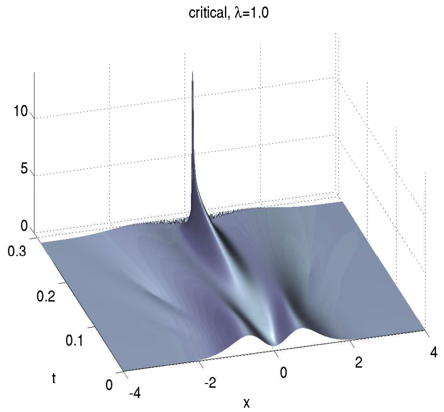
In Figs. 3 and 4, the time evolution of the modulus of $u(t,x)$ is shown for values of $\lambda$ from the three different regions of the above curve.

- The left part of Fig. 3 shows the case $\lambda = 1$. The two initial humps merge to one hump before the blow-up, which happens at a single point. In the picture of the heuristics, the phase focusing happens faster than the nonlinear focusing.
- The right part of Fig. 3 shows the case $\lambda = 2.4$. Blow-up is occurring simultaneously at two points. The nonlinear self-focusing here is faster than phase focusing.
- Fig. 4 is for $\lambda = 2.0$, which is in the non-monotonicity region. Blow-up here occurs at a single point. In this case it is not clear which of the two effects would happen at a faster time scale, nor how they would interact. Blow-up is occurring, but the blow-up time is no longer monotonous.

**Remark 3.1.** If the data (3.2) are used without the phase term, which leaves just a two-hump profile, non-monotonicity can be observed in the same way as described above. However overall blow-up times increase. Apparently the merging of the two humps can occur also by the mass dispersion tendency of the free evolution together with the focusing effect of the nonlinearity.

**Test 3.** Data with three humps: The next test uses a sum of three Gaussians and the same phase term as before, so there are three humps instead of two.

$$u_0(x) = C\left(e^{-(3x)^2} + e^{-(3(x-1))^2} + e^{-(3(x+1))^2}\right) e^{-i\log(e^x + e^{-x})}, \tag{3.3}$$



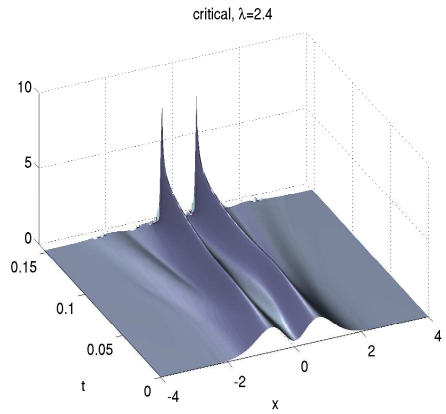




Figure 3: Time evolution with low potential energy (left) and with high potential energy (right).

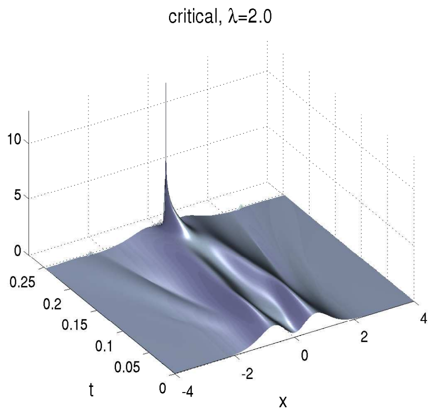



Figure 4: Time evolution in intermediate regime: non-monotonicity.

with $C=2$, and $\|u_0\|_{L^2}^2=5.09$. The discretization parameters are $Np=2^{12}$ mesh points and $\Delta t=1.5\cdot 10^{-5}$. The discretization domain is $[-8,8]$. The blow-up times with respect to changing $\lambda$ are shown in Fig. 5. The same effect as above can be observed. There are two regimes for the nonlinearity strength $\lambda$ where the blow-up time is non-monotonous.

**Test 4.** Data with two humps up, one down: In this test the data are taken to be

$$u_0(x)=C\left(e^{-(3(x-1))^2}-e^{-(3x)^2}+e^{-(3(x+1))^2}\right)e^{-i\log(e^x+e^{-x})}, \tag{3.4}$$

with $C=2$, and $\|u_0\|_{L^2}^2=4.93$. One of the three Gaussians has an opposite sign, so there is a constant phase shift in part of the data. The blow-up times are shown in the right part of Fig. 5. For $\lambda<1.6$, there is no blow-up occurring. For larger $\lambda$, non-monotonicity similar to the situation above can be observed. Observe that the slope of the curve is rather steep, for $2.7<\lambda<2.8$, which shows a highly nonlinear phenomenon.

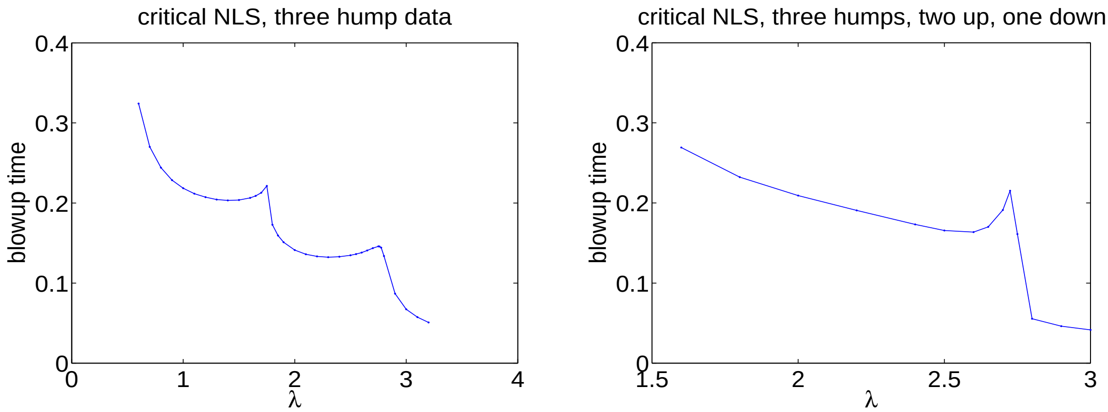



Figure 5: Blow-up time with varying $\lambda$. Left: three hump data (3.3) (Test 3), right: three hump data (3.4) (Test 4).

**Test 5.** Data with one hump up, one down: We use

$$u_0(x) = Ce^{-x^2}\tanh x e^{-i\log(e^x+e^{-x})}.$$

These data have odd parity, a property that is conserved in the time evolution and prevents a merging of the two humps. We use $C=3$, and $\|u_0\|_{L^2}^2=4.4476$, and discretizations of $Np=2^{13}$ mesh pointsand $\Delta t = 2.0\cdot 10^{-6}$. The result is shown in Fig. 6. In this case, the blow-up time is monotonous with $\lambda$.

### 3.1.2 Test in two space dimensions

**Test 6.** For the test in two space dimensions, we extend (3.2) by making the phase term radially symmetric and multiplying the one-dimensional two-hump profile by a single Gaussian in the second space dimension:

$$u_0(x,y) = C\left(e^{-x^2} - 0.9e^{-3x^2}\right)e^{-y^2}e^{-i\log 2\cosh\left(\sqrt{x^2+y^2}\right)}. \tag{3.5}$$

We choose $C=7$, and $\|u\|_{L^2}^2=15$. The smallest discretization parameters used are $Np=2^{12}$ mesh points and $\Delta t = 1\cdot 10^{-5}$ with the discretization domain $[-4,4]^2$.

The blow-up times with changing $\lambda$ are shown in Fig. 7. Non-monotonicity can be observed.

### 3.1.3 Supercritical power

**Test 7.** We tested Eq. (1.1) in one space dimension, with $\sigma=3$ and the data (3.2) ($C=3.5$, hence $\|u_0\|_{L^2}^2=2.99$). The discretization parameters are $\Delta x=0.0039$, and up to $\Delta t=1\cdot 10^{-6}$. The discretization domain is $[-8,8]$. The blow-up time with varying $\lambda$ is shown in Fig. 8. Also in the supercritical case, non-monotonicity of blow-up times can be observed.

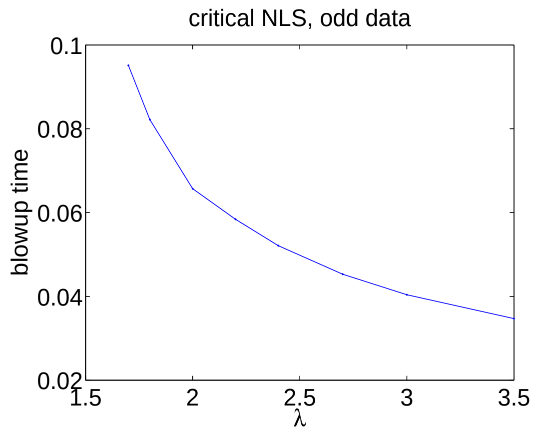



Figure 6: Blow-up time with varying $\lambda$, data with point symmetry (Test 5).

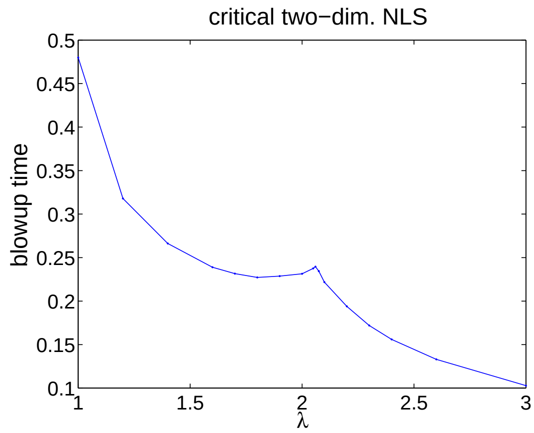



Figure 7: Blow-up time with varying $\lambda$, two dimensional case (Test 6).

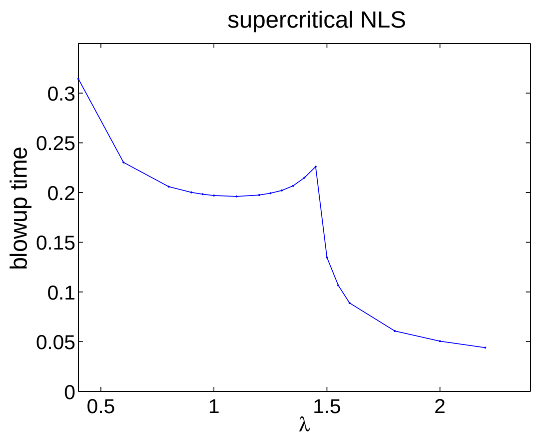



Figure 8: Blow-up time with varying constant $\lambda$ (Test 7).

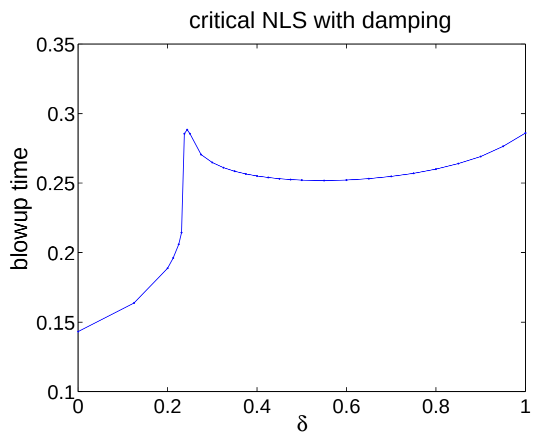



Figure 9: Blow-up time with varying damping constant $\delta$ (Test 8).

## 3.2 Damped NLS

We now turn to (1.3) in one space dimension.

**Test 8.** For (1.3) in space dimension one, we use the two-hump data (3.2). The data scale was chosen as $C=5$.The finest discretization parameters used in this test are $Np=2^{14}$ meshpoints and $\Delta t = 2.5\cdot 10^{-7}$.

The blow-up time with respect to changing $\delta$ is shown in Fig. 9. It can be seen that the blow-up time is not monotonously increasing with $\delta$. The effect is somehow more pronounced than in the case of Eq. (1.1).

## 3.3 Dependence on quadratic oscillations

We turn to Eq. (2.2) and investigate the dependence of blow-up time on the scale of quadratic oscillations. To compare the simulations to the result of Proposition 2.2, we simulate (1.1) at $\lambda = 1$ with the same data to obtain the blow-up time for this equation, and then plot the curve of $T_a$ that is predicted in Proposition 2.2 for the two regions $a > 0$ and $a < 0$, $a + T < 0$.

### 3.3.1 Critical power

We use (1.1) in space dimension one, with $\sigma = 2$ with various $u_0(x)$.

**Test 9.** Single hump: We take

$$u_0(x) = Ce^{-x^2}$$

with $C = 1.75$. The discretization parameters in this and the following test are $Np = 2^{14}$ mesh points and $\Delta t = 4 \cdot 10^{-6}$. The discretization domain is $[-8,8]$ for all cases except the two largest negative $a$ in Tests 9 and 11. Here the domain is extended up to $[-40,40]$ and $Np = 2^{14}$ for the largest negative $a$. Fig. 10 shows the blow-up time of $v$ in relation to the scale $a$ of quadratic oscillations in the data. We use both positive $a$ and negative $a$ with $a + T < 0$. Asterisks denote the blow-up times and the dashed line shows the result of Proposition 2.2 with $T$ obtained by a simulation of (1.1). It can be observed that the results agree very well.

**Test 10.** Two humps with additional phase: Here we use

$$u_0(x) = C\left(e^{-x^2} - 0.9e^{-3x^2}\right)e^{-i\log(e^x + e^{-x})}$$

which is the same as (3.2). The right part of Fig. 10 shows the blow-up times of the simulations and the results of Proposition 2.2. The results agree.

### 3.3.2 Supercritical power

We now consider Eqs. (2.2) and (2.3) with $\sigma = 3$, in space dimension one. We give results for both $w$ and $v$.

**Test 11.** Single hump: We take $u_0(x) = Ce^{-x^2}$. The discretization parameters are $Np = 2^{13}$ mesh points and $\Delta t = 1 \cdot 10^{-5}$. Fig. 11 shows the blow-up time in relation to the scale $a$ of quadratic oscillations in the data. In the left figure, the dashed line is the calculated blow-up time for $v$ and the asterisks mark the simulated blow-up times. The right figure compares the blow-up times for $v$ and $w$, where $w$ is denoted by asterisks and $v$ by dots joined by a line. We can see that $w$ blows up a bit earlier than $v$ for positive $a$. For negative $a$, it blows up a bit later, but the blow-up times are rather close to those of $v$ as in the positive $a$ case. Again the blow-up times for $v$ match the result of Proposition 2.2.

**Test 12.** Two humps placed asymmetrically: Here we use

$$u_0(x) = C\left(e^{-(3x)^2} + e^{-(3(x-1.5))^2}\right).$$

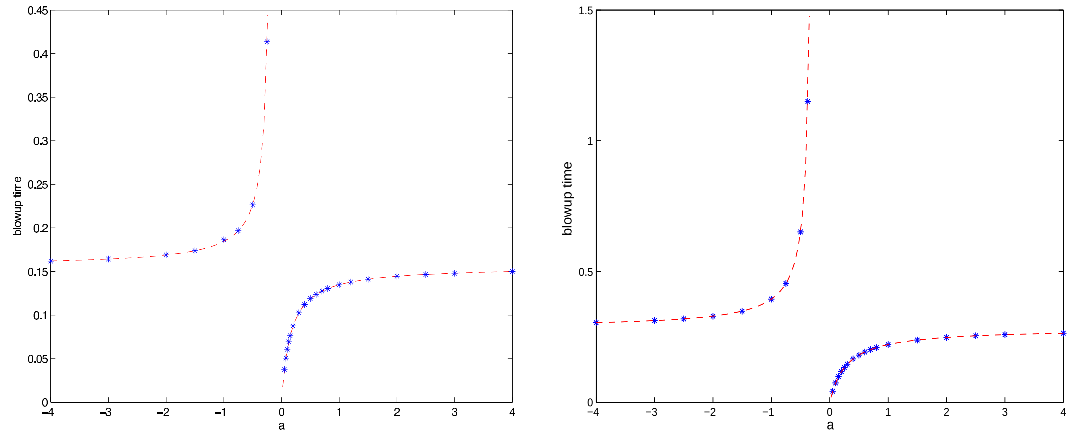



Figure 10: Blow-up time with varying $a$ in quadratic oscillations, critical power. Left: single hump data (Test 9), right: two hump data (Test 10).

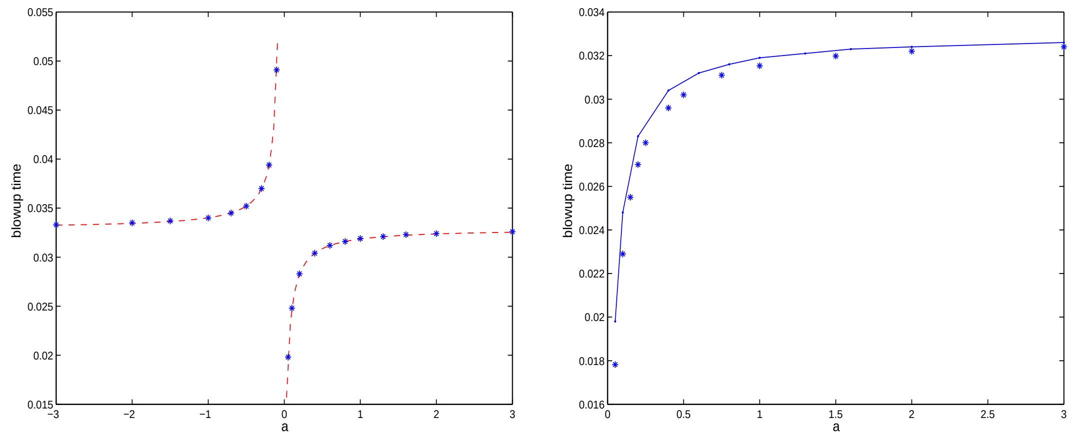



Figure 11: Blow-up time with varying $a$ in quadratic oscillations, supercritical power. Left: blow-up times for $v$. Right: comparison of $v$ and $w$, dots with line for $v$, asterisks for $w$ (Test 11).

We take $C=1.8$. The smallest discretization parameters used in this test are $Np=2^{15}$ mesh points and $\Delta t = 1.5 \cdot 10^{-6}$. The computation domain is $[-8,8]$, except for the largest value of $a$. Fig. 12 shows the blow-up time for $w$ in relation to $a$ obtained by the two numerical methods employed: the circles represent simulations done by the TSSP, the asterisks simulations by the RS. In addition a solid line displays the blow-up times for $v$. Non-monotonicity can be observed, which answers question 2 in a negative way. The blow-up time of $w$ is always smaller than that of $v$. We also see that the results of the two different schemes agree in a good way. Note that the occurrence of non-monotonicity is very sensitive to the size of the data. If we choose $C=1.7$ or 1.9 instead of $C=1.8$, monotonicity can be observed.

**Remark 3.2** (Test 12)**.** Leave out the question of quadratic oscillations, and consider $u$

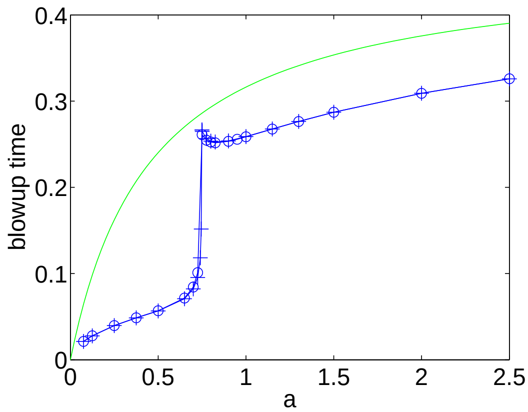



Figure 12: Blow-up time for $w$ with varying $a$ in quadratic oscillations, supercritical power, asymmetric data (asterisks and circles). Solid line: Blow-up time for $v$. (Test 12).

solving (1.1) with fixed $\lambda=1$. Its blow-up time as a function of the constant $C$ is given by:

| $C$ | 1.795 | 1.798 | 1.8 | 1.804 | 1.808 | 1.81 | 1.82 |
|---|---|---|---|---|---|---|---|
| $T^*$ | 0.528 | 0.480 | 0.462 | 0.446 | 0.507 | 0.076 | 0.048 |

Since changing $C$ in (1.1) with $\lambda=1$ is equivalent to changing $\lambda$ in (1.1) with fixed $u_0$, we see here a behavior analogous to Test 7 (Fig. 8), where similar data are used. The value $C=1.8$ is very close to (actually slightly below) the potential energy level where the blow-up changes from two point blow-up to one point blow-up, and non-monotonicity can be observed.

All tests were done with both the TSSP and the Relaxation scheme (RS). The results agree, as an example we showed the comparison of the schemes in Fig. 12. By using two numerical schemes with different discretization approaches, the possibility of observing just numerical defects introduced by a particular discretization method can be minimized, so we can conclude that the observations are indeed analytical properties.

## 4 Conclusion

In this paper, we have addressed numerically the question of the dependence of the blow-up time for solutions to nonlinear Schrödinger equations upon one specific parameter, in three cases:

- Dependence upon the coupling constant $\lambda$, for fixed $n$,$\sigma$ and $u_0$ in (1.1)
- Dependence upon the strength of the damping $\delta\geq 0$ in (1.3)
- Dependence upon the magnitude of a quadratic oscillation introduced in the initial data: only $a$ varies in the equation (2.3).

In the $L^2$ critical case $\sigma = \frac{2}{n}$, there is apparently no monotonicity in the first and in the second problem. In the third one, our tests agree with the analytical result: there is monotonicity of the blow-up time with respect to $a$, as recalled in Proposition 2.2.

In the $L^2$ super-critical case $\sigma > \frac{2}{n}$, the tests we performed for the first and third case highly suggest that the blow-up time is not monotonous with respect to the variation of the parameter considered.

We used two numerical methods (time-splitting spectral method and a relaxation method), for the results observed to be more convincing. We may say that they are, since the two methods yield the same results (and not only just similar results, see Fig. 12).

Note that in some cases where we observed monotonicity reversal, the slope of the blow-up time/varying parameter curve may be rather steep near the monotonicity breakup. Compare Fig. 5 (right part) near $\lambda = 2.7$ with Fig. 1. In the quadratic oscillations case, Fig. 12 shows a similar feature near $a = 0.7$.

All the numerical counter-examples to monotonicity that we found contain a somehow nontrivial profile, inasmuch as the initial datum is formed of two (or more) humps. The question of an analytical justification remains open and challenging in these cases.

## Acknowledgments

This work was supported by the Austrian Science Foundation (FWF) via the START project (Y-137-TEC) and by the WWTF (Viennese Science Fund, project MA-45).

### References

[1] W. Bao and D. Jaksch, An explicit unconditionally stable numerical method for solving damped nonlinear Schrodinger equations with a focusing nonlinearity, SIAM J. Numer. Anal., **41** (2003), no. 4, 1406–1426.

[2] W. Bao, D. Jaksch and P. A. Markowich, Three dimensional simulation of jet formation in collapsing condensates, J. Phys. B: At. Mol. Opt. Phys., **37** (2004), no. 2, 329–343.

[3] W. Bao, S. Jin, and P. A. Markowich, Numerical study of time-splitting spectral discretizations of nonlinear Schrödinger equations in the semiclassical regimes, SIAM J. Sci. Comput. **25** (2003), no. 1, 27–64.

[4] W. Bao, N.J. Mauser, and H.P. Stimming, Effective one particle quantum dynamics of electrons: a numerical study of the Schrödinger-Poisson-X$\alpha$ model, Comm. Math. Sci. **1** (2003), no. 4, 809–831.

[5] W. Bao and F. F. Sun, Efficient and stable numerical methods for the generalized and vector Zakharov System, SIAM J. Sci. Comput., **26** (2005), no. 3, 1057–1088.

[6] W. Bao and C. Zheng, A time-splitting spectral method for three-wave interactions in media with competing quadratic and cubic nonlinearities, Commun. Comput. Phys. **2** (2007), 123–140.

[7] C. Besse, A relaxation scheme for the nonlinear Schrödinger equation, SIAM J. Numer. Anal. **42** (2004), no. 3, 934–952.

[8] C. Besse, N. J. Mauser, R. Carles and H. P. Stimming, Monotonicity properties of the blow-up time for nonlinear Schrödinger equations: numerical evidence , Discr. Cont. Dyn. Syst. B **9** (2008), no. 1, 11-36.

[9] C. Besse, N. J. Mauser, and H. P. Stimming, Numerical study of the Davey-Stewartson system, Math. Model. Numer. Anal. **38** (2004), no. 6, 1035–1054.

[10] J. Bourgain and W. Wang, Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 1-2, 197–215 (1998), Dedicated to Ennio De Giorgi.

[11] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[12] T. Cazenave and F. Weissler, Rapidly decaying solutions of the nonlinear Schrödinger equation, Comm. Math. Phys. **147** (1992), 75–100.

[13] G. Fibich, Self-focusing in the damped nonlinear Schrödinger equation, SIAM J. Appl. Math. **61** (2001), no. 5, 1680–1705.

[14] R. T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, J. Math. Phys. **18** (1977), 1794–1797.

[15] F. Merle, Construction of solutions with exactly $k$ blow-up points for the Schrödinger equation with critical nonlinearity, Comm. Math. Phys. **129** (1990), no. 2, 223–240.

[16] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power, Duke Math. J. **69** (1993), no. 2, 427–454.

[17] F. Merle and P. Raphaël, On universality of blow-up profile for $L^2$ critical nonlinear Schrödinger equation, Invent. Math. **156** (2004), no. 3, 565–672.

[18] F. Merle and P. Raphaël, Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation, Comm. Math. Phys. **253** (2005), no. 3, 675–704.

[19] G. Perelman, On the formation of singularities in solutions of the critical nonlinear Schrödinger equation, Ann. Henri Poincaré **2** (2001), no. 4, 605–673.

[20] V. M. Pérez-García and X. Liu, Numerical methods for the simulation of trapped nonlinear Schrödinger systems, Appl. Math. Comput. **144** (2003), no. 2-3, 215–235.

[21] P. Raphaël, Sur la dynamique explosive des solutions de l'équation de Schrödinger non linéaire, Séminaire: Équations aux Dérivées Partielles 2004–2005, Exp. No. III, École polytech., Palaiseau.

[22] C. Sulem and P.-L. Sulem, The nonlinear Schrödinger equation, self-focusing and wave collapse, Springer-Verlag, New York, 1999.

[23] M. I. Weinstein, On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations, Comm. in Partial Diff. Eq. **11** (1986), no. 5, 545–565.