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Symmetric Energy-Conserved Splitting FDTD Scheme for the Maxwell's Equations

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Abstract. In this paper, a new symmetric energy-conserved splitting FDTD scheme (symmetric EC-S-FDTD) for Maxwell's equations is proposed. The new algorithm inherits the same properties of our previous EC-S-FDTDI and EC-S-FDTDII algorithms: energy-conservation, unconditional stability and computational efficiency. It keeps the same computational complexity as the EC-S-FDTDI scheme and is of second-order accuracy in both time and space as the EC-S-FDTDII scheme. The convergence and error estimate of the symmetric EC-S-FDTD scheme are proved rigorously by the energy method and are confirmed by numerical experiments.

AMS subject classifications: 65N10, 65N15, 78M20

Key words: Maxwell's equation, ADI method, FDTD, energy-conserved, second-order accuracy, symmetric scheme.

1 Introduction

For solving multidimensional partial differential equations, specially for parabolic problems, the alternating direction implicit methods (ADI) and the fractional step methods (FS) are very attractive and popular (see, e.g., [6, 8, 26, 27, 29]; and more recent works [5,7,18,20], etc). In computations of Maxwell's equations, many works related to the ADI technique have been studied for reducing the complexities and the large computational costs. For example, Holland in [16] discussed the ADI method combining with Yee's

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scheme for the two-dimensional problems. However, the proposed scheme was difficult to obtain the unconditional stability property for three-dimensional Maxwell's equations. Zheng et al. in [33] first proposed an unconditionally stable ADI-FDTD scheme for the three-dimensional Maxwell's equations with an isotropic and lossless medium. The accuracy and dispersion of this scheme was further studied in [14, 32]. Meanwhile, Namiki [23] proposed a kind ADI-FDTD scheme for the Maxwell's equations in two-dimensions. The unconditional stability of the scheme was analyzed in [23,31]. Recently, combining the splitting technique with the staggered Yee's grid, Gao et al. in [11,12] proposed the splitting finite-difference time-domain methods for Maxwell's equations: the S-FDTDI and S-FDTDII schemes for the two-dimensional problems and the S-FDTD and IS-FDTD schemes for three-dimensional problems. All the schemes are efficient and easy to be implemented.

On the other aspect, to keep the original physical features of problems is of great importance in constructing numerical schemes for the long time computations. In the propagation of electromagnetic wave in lossless medium without sources, it is well known that the density of the electromagnetic energy of the wave is constant at different times. The previous ADI or splitting schemes are unconditionally stable and effective for high dimensional problems but often break the property of energy conservation of Maxwell's equations. More recently, in [4] we developed two energy-conserved splitting finite-difference time-domain schemes (EC-S-FDTDI and EC-S-FDTDII), which have important properties: i) Energy-conservation; ii) Unconditional stability; iii) Efficient computation at each time step; iv) Dissipation-free.

Based on the staggered Yee's grid, by applying the splitting technique, the proposed energy-conserved splitting finite-difference time-domain scheme (EC-S-FDTDI) in [4] consists of two stages at each time step and therefore is simple in computational complexity. However, it is only first-order accurate in time. The EC-S-FDTDII scheme in [4] is a three stages scheme, which keeps all the above mentioned properties i) - iv) as the EC-S-FDTDI scheme and is of second-order accuracy in time. The EC-S-FDTDII scheme improves the accuracy of the EC-S-FDTDI scheme but it contains three stages at each time step, i.e., three tri-diagonal systems are to be solved at each time step. By analyzing these two schemes, we note that the EC-S-FDTDI is just symmetric in space but not in time, which may explain its first-order convergence in the time direction. Thus, we propose to modify the EC-S-FDTDI scheme by distinguishing the time steps between the even time step and the odd time step so that the derived scheme is symmetric in the time direction. The new scheme is called the symmetric EC-S-FDTD scheme, which has the same computational complexity as the original EC-S-FDTDI scheme. We prove that the symmetric EC-S-FDTD scheme is energy-conserved, unconditionally stable and dissipative-free. Furthermore, it is shown that the scheme is of second-order accuracy in both time and space. These properties are confirmed by numerical experiments as well.

The remaining of the paper is organized as follows. In Section 2, the conservation properties of Maxwell's equations are introduced and the new symmetric EC-S-FDTD scheme is proposed for the two-dimensional case. In Section 3, the energy conservations, stability and convergence of the symmetric EC-S-FDTD scheme will be established by using the energy method. The divergence-free property is also analyzed. Numerical experiments for problems with the constant and piecewise constant coefficients are presented in Section 4.

2 Maxwell's equations and energy-conserved splitting FDTD schemes

Let $\vec{E}$ be the electric field, $\vec{H}$ the magnetic filed, $\vec{D}$ the electric displacement, and $\vec{B}$ the magnetic flux density. The Maxwell's equations in the differential form are given as (see, e.g., [1,9]):

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t},\quad \nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}, \tag{2.1}$$

$$\nabla\cdot\vec{B}=0,\quad \nabla\cdot\vec{D}=\rho, \tag{2.2}$$

where ρ is the charge density, $\vec{J}$ is the current density. The electric and magnetic field variables are related through the constitutive relations as

$$\vec{D}=\epsilon\vec{E},\quad \vec{B}=\mu\vec{H},\quad \vec{J}=\sigma\vec{E}, \tag{2.3}$$

where ϵ is the electric permittivity, μ is the magnetic permeability, and σ is the electric conductivity. We note that the current density $\vec{J}$ typically includes different types of contributions to the current, e.g., eddy current $\sigma\vec{E}$ and impressed currents. In this work, we regard this current $\vec{J}$ as a pure eddy current $\sigma\vec{E}$.

Consider a two-dimensional transverse electric (TE) polarization case in a *lossless* medium without sources. We thus get $\rho=0,\vec{J}=0$, $\vec{E}=(E_x(x,y,t),E_y(x,y,t),0)$ and $\vec{H}=(0,0,H_z(x,y,t))$. Therefore, the Maxwell's equations (2.1)-(2.2) become:

$$\frac{\partial E_x}{\partial t}=\frac{1}{\epsilon}\frac{\partial H_z}{\partial y}, \tag{2.4}$$

$$\frac{\partial E_y}{\partial t}=-\frac{1}{\epsilon}\frac{\partial H_z}{\partial x}, \tag{2.5}$$

$$\frac{\partial H_z}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right), \tag{2.6}$$

where $\vec{E}=(E_x(x,y,t),E_y(x,y,t))$ and $H_z=H_z(x,y,t)$ denote the electric field and the magnetic field respectively. For simplicity, we consider the perfectly electric conducting (PEC) boundary condition on the boundary $\partial\Omega$ of the rectangle domain $\Omega=[0,a]\times[0,b]$:

$$(\vec{E},0)\times(\vec{n},0)=0,\quad \text{on}\quad (0,T]\times\partial\Omega, \tag{2.7}$$

where $\vec{n}$ is the outward normal vector on $\partial\Omega$. The PEC condition (2.7) can be recast as

$$E_x(x,0,t)=E_x(x,b,t)=E_y(0,y,t)=E_y(a,y,t)=0, \quad \text{on} \quad (0,T]\times\partial\Omega. \tag{2.8}$$

To solve the system, the initial conditions are needed:

$$\vec{E}(x,y,0)=\vec{E}_0(x,y)=(E_{x_0}(x,y),E_{y_0}(x,y)), \quad \text{and} \quad H_z(x,y,0)=H_{z_0}(x,y). \tag{2.9}$$

The problem (2.4)-(2.9) has a unique solution for suitably smooth data (see [19]).

For ease of in notations, ϵ and μ are assumed to be constant. The algorithms described in this paper can be easily extended to the case of variable coefficients. In the numerical experiments, the constant and discontinuous electric permittivity cases are considered to confirm our theoretical results.

2.1 Energy conservations in lossless medium

In a *lossless* medium without sources, $\vec{J}=0$ in the Maxwell's equations. We then have two energy conservation properties (see [4]).

Theorem 2.1 (Energy conservations)**.** *If $\vec{E}$ and $\vec{H}$ are the solutions of the Maxwell's equations* (2.1)-(2.2) *in lossless medium, and satisfy the boundary conditions:*

$$\vec{E}\times\vec{n}=0, \quad \text{or} \quad \vec{H}\times\vec{n}=0, \tag{2.10}$$

then it holds that energy conservation I

$$\int_\Omega \epsilon \left|\vec{E}(x,t)\right|^2 dx+\int_\Omega \mu\left|\vec{H}(x,t)\right|^2 dx \equiv \text{Constant}, \tag{2.11}$$

and energy conservation II

$$\int_\Omega \left(\epsilon\left|\frac{\partial\vec{E}}{\partial t}\right|^2+\mu\left|\frac{\partial\vec{H}}{\partial t}\right|^2\right)dx\equiv\text{Constant}. \tag{2.12}$$

The energy conservation I is just the well-known Poynting theorem [1, 9], which means that the total electromagnetic energy in losses medium without sources keeps constant at any time.

Now for the electromagnetic waves in the lossless medium, there are two conservation laws, namely, (2.11) and (2.12). Therefore it is natural to ask whether the numerical schemes can keep these properties. To answer the question, in our previous paper [4], two energy-conserved splitting FDTD methods (EC-S-FDTDI and EC-S-FDTDII) are proposed.

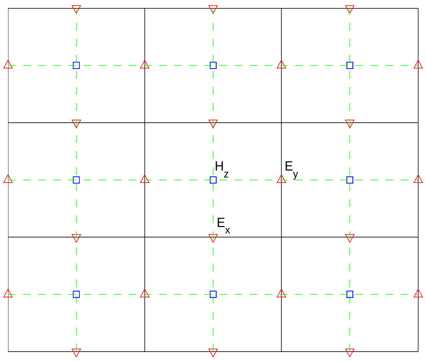

Figure 1: Staggered grid,'□' for $H_z{}^n_{i+\frac{1}{2},j+\frac{1}{2}}$, '△' for $E_y{}^n_{i,j+\frac{1}{2}}$, '∇' for $E_x{}^n_{i+\frac{1}{2},j}$.

2.2 Energy-conserved splitting FDTD schemes

The staggered Yee's grid is used in the FDTD methods (see Fig. 1). Let Δx and Δy be the mesh sizes along the x and y directions respectively, and Δt the time step size. For $i=0,1,\cdots,I$, $j=0,1,\cdots,J$ and $n=0,1,\cdots,N$, define $(x_i,y_j,t^n)=(i\Delta x,j\Delta y,n\Delta t)$, $x_{i+\frac{1}{2}}=x_i+\frac{1}{2}\Delta x$, $y_{j+\frac{1}{2}}=y_j+\frac{1}{2}\Delta y$ and $t^{n+\frac{1}{2}}=t^n+\frac{1}{2}\Delta t$. The grid function $U^n_{\alpha,\beta}$ is defined on the staggered grid where $\alpha=i$ or $i+\frac{1}{2}$ and $\beta=j$ or $j+\frac{1}{2}$, and $\delta_x U$, $\delta_y U$ and $\delta_u\delta_v U$ are defined as follows:

$$\delta_t U^n_{\alpha,\beta}=\frac{U^{n+\frac{1}{2}}_{\alpha,\beta}-U^{n-\frac{1}{2}}_{\alpha,\beta}}{\Delta t},\qquad \delta_x U^n_{\alpha,\beta}=\frac{U^n_{\alpha+\frac{1}{2},\beta}-U^n_{\alpha-\frac{1}{2},\beta}}{\Delta x},$$
$$\delta_y U^n_{\alpha,\beta}=\frac{U^n_{\alpha,\beta+\frac{1}{2}}-U^n_{\alpha,\beta-\frac{1}{2}}}{\Delta y},\quad \delta_u\delta_v U^m_{\alpha,\beta}=\delta_u(\delta_v U^n_{\alpha,\beta}),$$

where u and v can be taken as x- or y-direction. For the grid function $U^n_{\alpha,\beta}$, we may drop the subscript if there is no confusion.

Before we introduce the new symmetric EC-S-FDTD scheme, let us firstly review the two EC-S-FDTD algorithms in our previous paper [4].

2.2.1 The EC-S-FDTDI scheme

Stage 1: Compute E_y^{n+1} and the intermediate variable H_z^* from H_z^n and E_y^n:

$$\frac{E_y{}^{n+1}_{i,j+\frac{1}{2}}-E_y{}^n_{i,j+\frac{1}{2}}}{\Delta t}=-\frac{1}{2\epsilon}\delta_x\{H_z{}^*_{i,j+\frac{1}{2}}+H_z{}^n_{i,j+\frac{1}{2}}\}, \tag{2.13}$$
$$\frac{H_z{}^*_{i+\frac{1}{2},j+\frac{1}{2}}-H_z{}^n_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}=-\frac{1}{2\mu}\delta_x\{E_y{}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}+E_y{}^n_{i+\frac{1}{2},j+\frac{1}{2}}\}. \tag{2.14}$$

Stage 2: Compute E_x^{n+1} and H_z^{n+1} from E_x^n and H_z^*:

$$\frac{{E_x}^{n+1}_{i+\frac{1}{2},j}-{E_x}^{n}_{i+\frac{1}{2},j}}{\Delta t}=\frac{1}{2\epsilon}\delta_y\{{H_z}^{n+1}_{i+\frac{1}{2},j}+{H_z}^{*}_{i+\frac{1}{2},j}\}, \tag{2.15}$$

$$\frac{{H_z}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}-{H_z}^{*}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}=\frac{1}{2\mu}\delta_y\{{E_x}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}+{E_x}^{n}_{i+\frac{1}{2},j+\frac{1}{2}}\}. \tag{2.16}$$

2.2.2 The EC-S-FDTDII scheme

Stage 1: Compute the intermediate variables E_x^* and H_z^* from E_x^n and H_z^n:

$$\frac{{E_x}^{*}_{i+\frac{1}{2},j}-{E_x}^{n}_{i+\frac{1}{2},j}}{\Delta t}=\frac{1}{4\epsilon}\delta_y\{{H_z}^{*}_{i+\frac{1}{2},j}+{H_z}^{n}_{i+\frac{1}{2},j}\}, \tag{2.17}$$

$$\frac{{H_z}^{*}_{i+\frac{1}{2},j+\frac{1}{2}}-{H_z}^{n}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}=\frac{1}{4\mu}\delta_y\{{E_x}^{*}_{i+\frac{1}{2},j+\frac{1}{2}}+{E_x}^{n}_{i+\frac{1}{2},j+\frac{1}{2}}\}. \tag{2.18}$$

Stage 2: Compute ${E_y}^{n+1}$ and the intermediate variable ${H_z}^{**}$ from ${E_y}^{n}$ and H_z^*:

$$\frac{{E_y}^{n+1}_{i,j+\frac{1}{2}}-{E_y}^{n}_{i,j+\frac{1}{2}}}{\Delta t}=-\frac{1}{2\epsilon}\delta_x\{{H_z}^{**}_{i,j+\frac{1}{2}}+{H_z}^{*}_{i,j+\frac{1}{2}}\}, \tag{2.19}$$

$$\frac{{H_z}^{**}_{i+\frac{1}{2},j+\frac{1}{2}}-{H_z}^{*}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}=-\frac{1}{2\mu}\delta_x\{{E_y}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}+{E_y}^{n}_{i+\frac{1}{2},j+\frac{1}{2}}\}. \tag{2.20}$$

Stage 3: Compute ${E_x}^{n+1}$ and H_z^{n+1} from H_z^{**} and E_x^*:

$$\frac{{E_x}^{n+1}_{i+\frac{1}{2},j}-{E_x}^{*}_{i+\frac{1}{2},j}}{\Delta t}=\frac{1}{4\epsilon}\delta_y\{{H_z}^{n+1}_{i+\frac{1}{2},j}+{H_z}^{**}_{i+\frac{1}{2},j}\}, \tag{2.21}$$

$$\frac{{H_z}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}-{H_z}^{**}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}=\frac{1}{4\mu}\delta_y\{{E_x}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}+{E_x}^{*}_{i+\frac{1}{2},j+\frac{1}{2}}\}. \tag{2.22}$$

According to the PEC boundary conditions (2.8), the boundary conditions for symmetric EC-S-FDTD scheme (2.13)-(2.22) are:

$${E_x}^{n}_{i+\frac{1}{2},0}={E_x}^{n}_{i+\frac{1}{2},J}={E_y}^{n}_{0,j+\frac{1}{2}}={E_y}^{0}_{I,j+\frac{1}{2}}=0, \tag{2.23}$$

and the initial values $E^0_{\alpha,\beta}$ and $H^0_{\alpha,\beta}$ are

$${E_x}^{0}_{\alpha,\beta}=E_{x_0}(\alpha\Delta x,\beta\Delta y);\quad {E_y}^{0}_{\alpha,\beta}=E_{y_0}(\alpha\Delta x,\beta\Delta y);\quad {H_z}^{0}_{\alpha,\beta}=H_{z_0}(\alpha\Delta x,\beta\Delta y). \tag{2.24}$$

From the analysis of [4], we know that although the EC-S-FDTDI scheme has many advantages: several advantages i)-iv) as mentioned in the previous section, it is only of first-order accuracy in time. While the EC-S-FDTDII scheme keeps all the above properties of the EC-S-FDTDI scheme and is of second-order accuracy in time, it is a three-stages scheme, i.e., for every time step three computation processes are needed. To overcome this disadvantage, we propose a new symmetric EC-S-FDTD scheme by distinguishing the time steps between the even time step and the odd time step.

2.2.3 Symmetric EC-S-FDTD scheme

1. At every odd time step, i.e., from t^{2k} to $t^{2k}+\Delta t$, use E_x^{2k}, E_y^{2k} and H_z^{2k} to compute E_x^{2k+1}, E_y^{2k+1} and H_z^{2k+1} by the EC-S-FDTDI scheme.
Stage 1.1: use H_z^{2k} and E_y^{2k} to compute E_y^{2k+1} and temperate variable H_z^* :

$$\frac{E_y{}_{i,j+\frac{1}{2}}^{2k+1}-E_y{}_{i,j+\frac{1}{2}}^{2k}}{\Delta t}=-\frac{1}{2\epsilon}\delta_x\{H_z{}_{i,j+\frac{1}{2}}^{*}+H_z{}_{i,j+\frac{1}{2}}^{2k}\}, \tag{2.25}$$

$$\frac{H_z{}_{i+\frac{1}{2},j+\frac{1}{2}}^{*}-H_z{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k}}{\Delta t}=-\frac{1}{2\mu}\delta_x\{E_y{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}+E_y{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k}\}. \tag{2.26}$$

Stage 1.2: use E_x^{2k} and H_z^* to compute E_x^{2k+1} and H_z^{2k+1} :

$$\frac{E_x{}_{i+\frac{1}{2},j}^{2k+1}-E_x{}_{i+\frac{1}{2},j}^{2k}}{\Delta t}=\frac{1}{2\epsilon}\delta_y\{H_z{}_{i+\frac{1}{2},j}^{2k+1}+H_z{}_{i+\frac{1}{2},j}^{*}\}, \tag{2.27}$$

$$\frac{H_z{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}-H_z{}_{i+\frac{1}{2},j+\frac{1}{2}}^{*}}{\Delta t}=\frac{1}{2\mu}\delta_y\{E_x{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}+E_x{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k}\}. \tag{2.28}$$

2. At every even time step, i.e., from $t^{2k}+\Delta t$ to $t^{2k}+2\Delta t$, use E_x^{2k+1}, E_y^{2k+1} and H_z^{2k+1} to compute E_x^{2k+2}, E_y^{2k+2} and H_z^{2k+2} by the (reverse) EC-S-FDTDI scheme.
Stage 2.1: use E_x^{2k+1} and H_z^{2k+1} to compute E_x^{2k+2} and temperate variable H_z^{**} :

$$\frac{E_x{}_{i+\frac{1}{2},j}^{2k+2}-E_x{}_{i+\frac{1}{2},j}^{2k+1}}{\Delta t}=\frac{1}{2\epsilon}\delta_y\{H_z{}_{i+\frac{1}{2},j}^{**}+H_z{}_{i+\frac{1}{2},j}^{2k+1}\}, \tag{2.29}$$

$$\frac{H_z{}_{i+\frac{1}{2},j+\frac{1}{2}}^{**}-H_z{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}}{\Delta t}=\frac{1}{2\mu}\delta_y\{E_x{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}+E_x{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}\}. \tag{2.30}$$

Stage 2.2: use H_z^{**} and E_y^{2k+1} to compute E_y^{2k+2} and H_z^{2k+2}:

$$\frac{E_y{}_{i,j+\frac{1}{2}}^{2k+2}-E_y{}_{i,j+\frac{1}{2}}^{2k+1}}{\Delta t}=-\frac{1}{2\epsilon}\delta_x\{H_z{}_{i,j+\frac{1}{2}}^{2k+2}+H_z{}_{i,j+\frac{1}{2}}^{**}\}, \tag{2.31}$$

$$\frac{H_z{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}-H_z{}_{i+\frac{1}{2},j+\frac{1}{2}}^{**}}{\Delta t}=-\frac{1}{2\mu}\delta_x\{E_y{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}+E_y{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}\}. \tag{2.32}$$

Note that the total number of time steps N should satisfy $N=2m$ and $0\le k\le m-1$.

Remark 2.1. It is obvious that the symmetric EC-S-FDTD scheme is a symmetric version of the EC-S-FDTDI scheme and both two schemes have the same computational complexity. In the following sections, our further analysis and numerical experiments will show that this symmetric version is of second-order accuracy in time.

3 Stability and convergence analysis

For grid functions defined on the staggered grid:

$$U:=\{U_{i+\frac{1}{2},j}\},\quad V:=\{V_{i,j+\frac{1}{2}}\},\quad W:=\{W_{i+\frac{1}{2},j+\frac{1}{2}}\},\quad \vec{F}:=\{(U_{i+\frac{1}{2},j},V_{i,j+\frac{1}{2}})\},$$

the discrete L^2 energy norms are used:

$$\|U\|_{E_x}^2=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|U_{i+\frac{1}{2},j}\right|^2\Delta x\Delta y,\qquad \|V\|_{E_y}^2=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|V_{i,j+\frac{1}{2}}\right|^2\Delta x\Delta y,$$
$$\|W\|_{H}^2=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|W_{i+\frac{1}{2},j+\frac{1}{2}}\right|^2\Delta x\Delta y,\quad \|\vec{F}\|_{E}^2=\|U\|_{E_x}^2+\|V\|_{E_y}^2.$$

3.1 Energy conservations and unconditional stability

Theorem 3.1 (Discrete energy conservations)**.** *For integer* $n\geq 0$*, if* $\vec{E}^n:=\{(E_{x}{}_{i+\frac{1}{2},j}^{n},E_{y}{}_{i,j+\frac{1}{2}}^{n})\}$ *and* $H_z^n:=\{H_{z}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{n}\}$ *are the solutions of symmetric EC-S-FDTD* (2.25)-(2.32)*, then the following discrete energies are constants:*

$$\|\epsilon^{\frac{1}{2}}\vec{E}^{n+1}\|_E^2+\|\mu^{\frac{1}{2}}H_z^{n+1}\|_H^2=\|\epsilon^{\frac{1}{2}}\vec{E}^{n}\|_E^2+\|\mu^{\frac{1}{2}}H_z^{n}\|_H^2, \tag{3.1}$$

and

$$\begin{aligned}&\|\epsilon^{\frac{1}{2}}\left(\vec{E}^{n+2}-\vec{E}^{n}\right)/(2\Delta t)\|_E^2+\|\mu^{\frac{1}{2}}\left(H_z^{n+2}-H_z^{n}\right)/(2\Delta t)\|_H^2\\ =&\|\epsilon^{\frac{1}{2}}\left(\vec{E}^{n+1}-\vec{E}^{n-1}\right)/(2\Delta t)\|_E^2+\|\mu^{\frac{1}{2}}\left(H_z^{n+1}-H_z^{n-1}\right)/(2\Delta t)\|_H^2 \quad .\end{aligned} \tag{3.2}$$

Proof. The first energy conservation relationship (3.1) can be immediately induced from the property of the EC-S-FDTDI scheme (see Theorem 10 in [4]).

For Eq. (3.2), let us consider one time period. At the even time step $(n=2k)$, we use (2.29)-(2.32). Subtracting the $2k$th step from the $(2k+2)$-th step of the symmetric EC-S-FDTD scheme, we obtain

$$\begin{aligned}&\frac{E_{x}{}_{i+\frac{1}{2},j}^{2k+2}-E_{x}{}_{i+\frac{1}{2},j}^{2k}}{\Delta t}-\frac{E_{x}{}_{i+\frac{1}{2},j}^{2k+1}-E_{x}{}_{i+\frac{1}{2},j}^{2k-1}}{\Delta t}\\ =&\frac{1}{2\epsilon}\delta_y\left[\left(H_{z}{}_{i+\frac{1}{2},j}^{**}-H_{z}{}_{i+\frac{1}{2},j}^{**-2}\right)+\left(H_{z}{}_{i+\frac{1}{2},j}^{2k+1}-H_{z}{}_{i+\frac{1}{2},j}^{2k-1}\right)\right],\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\frac{H_{z}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{**}-H_{z}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{**-2}}{\Delta t}-\frac{H_{z}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}-H_{z}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k-1}}{\Delta t}\\ =&\frac{1}{2\mu}\delta_y\left[\left(E_{x}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}-E_{x}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k}\right)+\left(E_{x}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}-E_{x}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k-1}\right)\right],\end{aligned} \tag{3.4}$$

$$\frac{E_{y\,i,j+\frac{1}{2}}^{2k+2}-E_{y\,i,j+\frac{1}{2}}^{2k}}{\Delta t}-\frac{E_{y\,i,j+\frac{1}{2}}^{2k+1}-E_{y\,i,j+\frac{1}{2}}^{2k-1}}{\Delta t}$$
$$=\frac{-1}{2\epsilon}\delta_x\left[\left(H_{z\,i,j+\frac{1}{2}}^{2k+2}-H_{z\,i,j+\frac{1}{2}}^{2k}\right)+\left(H_{z\,i,j+\frac{1}{2}}^{**}-H_{z\,i,j+\frac{1}{2}}^{**-2}\right)\right], \tag{3.5}$$

and

$$\frac{H_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}-H_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{2k}}{\Delta t}-\frac{H_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{**}-H_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{**-2}}{\Delta t}$$
$$=\frac{-1}{2\mu}\delta_x\left[\left(E_{y\,i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}-E_{y\,i+\frac{1}{2},j+\frac{1}{2}}^{2k}\right)+\left(E_{y\,i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}-E_{y\,i+\frac{1}{2},j+\frac{1}{2}}^{2k-1}\right)\right]. \tag{3.6}$$

Again, using the same method as in Theorem 10 in [4], we can show that (3.2) is true for even n. Similarly, it is also true for odd n. □

The above theorem shows that the symmetric EC-S-FDTD scheme satisfies the Poynting theorem, i.e., (2.11), in the discrete sense. Moreover, the energy conservation (2.12) holds at every two time steps.

From the above theorem, the stability of the scheme can be immediately obtained.

Corollary 3.1 (Unconditional stability)**.** *The symmetric EC-S-FDTD scheme is unconditionally stable for the two-dimensional Maxwell's equations with the PEC boundary conditions.*

3.2 Truncation errors

The symmetric EC-S-FDTD scheme can be recast as another equivalent form:

$$\frac{E_{x\,i+\frac{1}{2},j}^{2k+2}-E_{x\,i+\frac{1}{2},j}^{2k}}{\Delta t}$$
$$=\frac{1}{2\epsilon}\delta_y\left\{H_{z\,i+\frac{1}{2},j}^{2k+1}+H_{z\,i+\frac{1}{2},j}^{2k}\right\}+\frac{1}{2\epsilon}\delta_y\left\{H_{z\,i+\frac{1}{2},j}^{2k+2}+H_{z\,i+\frac{1}{2},j}^{2k+1}\right\}+\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\left\{E_{y\,i+\frac{1}{2},j}^{2k+2}-E_{y\,i+\frac{1}{2},j}^{2k}\right\}, \tag{3.7}$$

$$\frac{E_{y\,i,j+\frac{1}{2}}^{2k+2}-E_{y\,i,j+\frac{1}{2}}^{2k}}{\Delta t}$$
$$=-\frac{1}{2\epsilon}\delta_x\left\{H_{z\,i,j+\frac{1}{2}}^{2k+1}+H_{z\,i,j+\frac{1}{2}}^{2k}\right\}-\frac{1}{2\epsilon}\delta_x\left\{H_{z\,i,j+\frac{1}{2}}^{2k+2}+H_{z\,i,j+\frac{1}{2}}^{2k+1}\right\}-\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\left\{E_{x\,i,j+\frac{1}{2}}^{2k+2}-E_{x\,i,j+\frac{1}{2}}^{2k}\right\}, \tag{3.8}$$

and

$$\frac{H_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}-H_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{2k}}{\Delta t}$$
$$=\frac{1}{2\mu}\left\{\delta_y\left(E_{x\,i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}+2E_{x\,i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}+E_{x\,i+\frac{1}{2},j+\frac{1}{2}}^{2k}\right)-\delta_x\left(E_{y\,i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}+2E_{y\,i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}+E_{y\,i+\frac{1}{2},j+\frac{1}{2}}^{2k}\right)\right\}. \tag{3.9}$$

According to [4], the truncation errors of the Crank-Nicolson scheme at each time step are of second-order accuracy in both time and space. Then comparing the Crank-Nicolson scheme with the equivalent form of the symmetric EC-S-FDTD scheme (3.7)-(3.9), the truncation errors of the equivalent form of the symmetric EC-S-FDTD scheme are just second-order perturbations of the Crank-Nicolson scheme. Note that

$$\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\{E_y(t^{2k+2},x_{i+\frac{1}{2}},y_j)-E_y(t^{2k},x_{i+\frac{1}{2}},y_j)\}$$

and

$$\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\{E_x(t^{2k+2},x_i,y_{j+\frac{1}{2}})-E_x(t^{2k},x_i,y_{j+\frac{1}{2}})\}$$

can be regarded as approximations of

$$\frac{\Delta t^2}{2\mu\epsilon}\frac{\partial^2 E_w(t^{2k+1})}{\partial x\partial y},\quad (w=x,y)$$

respectively. Thus, we obtain the following lemma.

Lemma 3.1 (Truncation errors)**.** *Suppose that the exact solutions of the Maxwell's equation E_x, E_y and H_z are sufficiently smooth:*

$$\vec{E}\in C^3\big([0,T];[C^3(\bar{\Omega})]^2\big)\quad \text{and}\quad H_z\in C^3\big([0,T];C^3(\bar{\Omega})\big).$$

Let ${\xi_x}^{2k+1}_{i+\frac{1}{2},j}$, ${\xi_y}^{2k+1}_{i,j+\frac{1}{2}}$ and ${\eta_z}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}$ be the truncation errors of (3.7)-(3.9) *at time t^{2k+1}. Then they satisfy:*

$$\max_k\{|{\xi_x}^{2k+1}|,|{\xi_y}^{2k+1}|,|{\eta_z}^{2k+1}|\}\le C\{\Delta t^2+\Delta x^2+\Delta y^2\}. \tag{3.10}$$

3.3 Convergence analysis

Now, let us provide a rigorous analysis of convergence of the symmetric EC-S-FDTD scheme.

Theorem 3.2 (Convergence analysis)**.** *Suppose that the exact solutions of the Maxwell's equation E_x, E_y and H_z are smooth enough:*

$$\vec{E}\in C^3\big([0,T];[C^3(\bar{\Omega})]^2\big)\quad \text{and}\quad H_z\in C^3\big([0,T];C^3(\bar{\Omega})\big).$$

Let ${E_x}^n$, ${E_y}^n$ and ${H_z}^n$ be the numerical solutions of the symmetric EC-S-FDTD scheme (2.25)-(2.32)*. Then, for any fixed time interval $T>0$, there exists a constant $C_{\mu\epsilon}$ independent of Δt, Δx and Δy such that for $n\ge 0$,*

$$\begin{aligned}&\max_{0\le n\le N}\{\|\epsilon^{\frac{1}{2}}[E(t^n)-E^n]\|_E^2+\|\mu^{\frac{1}{2}}[H_z(t^n)-H_z^n]\|_H^2\}\\ \le\, & e^T\Big(\|\epsilon^{\frac{1}{2}}(\vec{E}(t^0)-\vec{E}^0)\|_E^2+\|\mu^{\frac{1}{2}}(H_z(t^0)-H_z^0)\|_H^2\Big)+C_{\mu\epsilon}e^T(\Delta t^2+\Delta x^2+\Delta y^2)^2.\end{aligned} \tag{3.11}$$

Proof. Firstly, let us define the error functions on the staggered grid:

$$\mathcal{E}_{w}{}^{n}_{\alpha,\beta}=E_w(x_\alpha,x_\beta,t^n)-E_{w}{}^{n}_{\alpha,\beta},\mathcal{H}_{z}{}^{n}_{\alpha,\beta}=H_z(x_\alpha,x_\beta,t^n)-H_{z}{}^{n}_{\alpha,\beta},$$

where $E_w(x_\alpha,y_\beta,t^n)$ $(w=x,y)$ and $H_z(x_\alpha,y_\beta,t^n)$ denote the values of exact solutions E_w and H_z on the point (x_α,y_β,t^n). Then, we focus on the following error equations:

$$\frac{\mathcal{E}_{y}{}^{2k+1}_{i,j+\frac{1}{2}}-\mathcal{E}_{y}{}^{2k}_{i,j+\frac{1}{2}}}{\Delta t}=-\frac{1}{2\epsilon}\delta_x\{\mathcal{H}_{z}{}^{*}_{i,j+\frac{1}{2}}+\mathcal{H}_{z}{}^{2k}_{i,j+\frac{1}{2}}\}+e_{1}{}_{i,j+\frac{1}{2}}, \tag{3.12a}$$

$$\frac{\mathcal{H}_{z}{}^{*}_{i+\frac{1}{2},j+\frac{1}{2}}-\mathcal{H}_{z}{}^{2k}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}=-\frac{1}{2\mu}\delta_x\{\mathcal{E}_{y}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}+\mathcal{E}_{y}{}^{2k}_{i+\frac{1}{2},j+\frac{1}{2}}\}+e_{2}{}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{3.12b}$$

$$\frac{\mathcal{E}_{x}{}^{2k+1}_{i+\frac{1}{2},j}-\mathcal{E}_{x}{}^{2k}_{i+\frac{1}{2},j}}{\Delta t}=\frac{1}{2\epsilon}\delta_y\{\mathcal{H}_{z}{}^{2k+1}_{i+\frac{1}{2},j}+\mathcal{H}_{z}{}^{*}_{i+\frac{1}{2},j}\}+e_{3}{}_{i+\frac{1}{2},j}, \tag{3.12c}$$

$$\frac{\mathcal{H}_{z}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}-\mathcal{H}_{z}{}^{*}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}=\frac{1}{2\mu}\delta_y\{\mathcal{E}_{x}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}+\mathcal{E}_{x}{}^{2k}_{i+\frac{1}{2},j+\frac{1}{2}}\}+e_{4}{}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{3.12d}$$

$$\frac{\mathcal{E}_{x}{}^{2k+2}_{i+\frac{1}{2},j}-\mathcal{E}_{x}{}^{2k+1}_{i+\frac{1}{2},j}}{\Delta t}=\frac{1}{2\epsilon}\delta_y\{\mathcal{H}_{z}{}^{**}_{i+\frac{1}{2},j}+\mathcal{H}_{z}{}^{2k+1}_{i+\frac{1}{2},j}\}+e_{5}{}_{i+\frac{1}{2},j}, \tag{3.12e}$$

$$\frac{\mathcal{H}_{z}{}^{**}_{i+\frac{1}{2},j+\frac{1}{2}}-\mathcal{H}_{z}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}=\frac{1}{2\mu}\delta_y\{\mathcal{E}_{x}{}^{2k+2}_{i+\frac{1}{2},j+\frac{1}{2}}+\mathcal{E}_{x}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}\}+e_{6}{}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{3.12f}$$

$$\frac{\mathcal{E}_{y}{}^{2k+2}_{i,j+\frac{1}{2}}-\mathcal{E}_{y}{}^{2k+1}_{i,j+\frac{1}{2}}}{\Delta t}=-\frac{1}{2\epsilon}\delta_x\{\mathcal{H}_{z}{}^{2k+2}_{i,j+\frac{1}{2}}+\mathcal{H}_{z}{}^{**}_{i,j+\frac{1}{2}}\}+e_{7}{}_{i,j+\frac{1}{2}}, \tag{3.12g}$$

$$\frac{\mathcal{H}_{z}{}^{2k+2}_{i+\frac{1}{2},j+\frac{1}{2}}-\mathcal{H}_{z}{}^{**}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t}=-\frac{1}{2\mu}\delta_x\{\mathcal{E}_{y}{}^{2k+2}_{i+\frac{1}{2},j+\frac{1}{2}}+\mathcal{E}_{y}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}\}+e_{8}{}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{3.12h}$$

where $e_{1}{}_{i,j+\frac{1}{2}},e_{2}{}_{i+\frac{1}{2},j+\frac{1}{2}},e_{3}{}_{i+\frac{1}{2},j}$, etc. are the truncation errors.

Without loss of generality, we regard $\mathcal{E}_{x}{}^{2k+1}_{i+\frac{1}{2},j}$, $\mathcal{E}_{y}{}^{2k+1}_{i,j+\frac{1}{2}}$, $\mathcal{H}_{z}{}^{*}_{i+\frac{1}{2},j}$ and $\mathcal{H}_{z}{}^{**}_{i+\frac{1}{2},j}$ as the temperate variables from the $(2k+1)$-th step to the $(2k+2)$-th step. Thus we get:

$$\begin{aligned}
&e_{1}{}_{i,j+\frac{1}{2}}=\frac{1}{2}\xi_{y}{}^{2k+1}_{i,j+\frac{1}{2}}, \quad e_{2}{}_{i+\frac{1}{2},j+\frac{1}{2}}=0, \quad e_{3}{}_{i+\frac{1}{2},j}=\frac{1}{2}\xi_{x}{}^{2k+1}_{i+\frac{1}{2},j}, \quad e_{4}{}_{i+\frac{1}{2},j+\frac{1}{2}}=\frac{1}{2}\eta_{z}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}},\\
&e_{5}{}_{i+\frac{1}{2},j}=\frac{1}{2}\xi_{x}{}^{2k+1}_{i+\frac{1}{2},j}, \quad e_{6}{}_{i+\frac{1}{2},j+\frac{1}{2}}=0, \quad e_{7}{}_{i,j+\frac{1}{2}}=\frac{1}{2}\xi_{y}{}^{2k+1}_{i,j+\frac{1}{2}}, \quad e_{8}{}_{i+\frac{1}{2},j+\frac{1}{2}}=\frac{1}{2}\eta_{z}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}.
\end{aligned}$$

Note that the truncation errors $\xi_{y}{}^{2k+1}_{i,j+\frac{1}{2}},\xi_{y}{}^{2k+1}_{i,j+\frac{1}{2}}$ and $\eta_{z}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}$ satisfy (3.10). From (3.12a)-

(3.12b) and using Lemma 9 in [4], we obtain:

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\Big(\epsilon({\mathcal{E}_y^{2k+1}}_{i,j+\frac{1}{2}})^2+\mu(\mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}})^2-\epsilon({\mathcal{E}_y^{2k}}_{i,j+\frac{1}{2}})^2-\mu(\mathcal{H}^{2k}_{z_{i+\frac{1}{2},j+\frac{1}{2}}})^2\Big)$$
$$=\frac{1}{2}\Delta t\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\Big(\epsilon\xi^{2k+1}_{y_{i,j+\frac{1}{2}}}({\mathcal{E}_y^{2k+1}}_{i,j+\frac{1}{2}}+{\mathcal{E}_y^{2k}}_{i,j+\frac{1}{2}})+\mu\eta^{2k+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}(\mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{H}^{2k}_{z_{i+\frac{1}{2},j+\frac{1}{2}}})\Big).$$

By the Schwartz inequality, we have

$$(1-\frac{\Delta t}{4})\Big(\epsilon\|\mathcal{E}_y^{2k+1}\|^2_{E_y}+\mu\|\mathcal{H}_z^*\|^2_H\Big)$$
$$\le C_1\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+(1+\frac{\Delta t}{4})(\epsilon\|\mathcal{E}_y^{2k}\|^2_{E_y}+\mu\|\mathcal{H}_z^{2k}\|^2_H). \tag{3.13}$$

Similarly, it follows from (3.12c) to (3.12h) that

$$(1-\frac{\Delta t}{4})\Big(\epsilon\|\mathcal{E}_x^{2k+1}\|^2_{E_x}+\mu\|\mathcal{H}_z^{2k+1}\|^2_H\Big)$$
$$\le C_1\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+(1+\frac{\Delta t}{4})(\epsilon\|\mathcal{E}_x^{2k}\|^2_{E_x}+\mu\|\mathcal{H}_z^{*}\|^2_H), \tag{3.14}$$
$$(1-\frac{\Delta t}{4})\Big(\epsilon\|\mathcal{E}_x^{2k+2}\|^2_{E_x}+\mu\|\mathcal{H}_z^{**}\|^2_H\Big)$$
$$\le C_1\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+(1+\frac{\Delta t}{4})(\epsilon\|\mathcal{E}_x^{2k+1}\|^2_{E_x}+\mu\|\mathcal{H}_z^{2k+1}\|^2_H), \tag{3.15}$$

and

$$(1-\frac{\Delta t}{4})\Big(\epsilon\|\mathcal{E}_y^{2k+2}\|^2_{E_y}+\mu\|\mathcal{H}_z^{2k+2}\|^2_H\Big)$$
$$\le C_1\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+(1+\frac{\Delta t}{4})(\epsilon\|\mathcal{E}_y^{2k+1}\|^2_{E_y}+\mu\|\mathcal{H}_z^{**}\|^2_H). \tag{3.16}$$

Multiplying both sides of (3.13) and (3.15) with $(1-\Delta t/4)$, both sides of (3.14) and (3.16) with $(1+\Delta t/4)$, and adding (3.13) to (3.14) and (3.15) to (3.16) respectively, we can get

$$\epsilon\|\mathcal{E}_y^{2k+1}\|^2_{E_y}+\frac{1-\frac{\Delta t}{4}}{1+\frac{\Delta t}{4}}\Big(\epsilon\|\mathcal{E}_x^{2k+1}\|^2_{E_x}+\mu\|\mathcal{H}_z^{2k+1}\|^2_H\Big)$$
$$\le C_2\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+\epsilon\|\mathcal{E}_x^{2k}\|^2_{E_x}+\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}\Big(\epsilon\|\mathcal{E}_y^{2k}\|^2_{E_y}+\mu\|\mathcal{H}_z^{2k}\|^2_H\Big), \tag{3.17}$$
$$\epsilon\|\mathcal{E}_x^{2k+2}\|^2_{E_x}+\frac{1-\frac{\Delta t}{4}}{1+\frac{\Delta t}{4}}\Big(\epsilon\|\mathcal{E}_y^{2k+2}\|^2_{E_y}+\mu\|\mathcal{H}_z^{2k+2}\|^2_H\Big)$$
$$\le C_3\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+\epsilon\|\mathcal{E}_y^{2k+1}\|^2_{E_y}+\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}\Big(\epsilon\|\mathcal{E}_x^{2k+1}\|^2_{E_x}+\mu\|\mathcal{H}_z^{2k+1}\|^2_H\Big). \tag{3.18}$$

Again multiplying both sides of (3.18) with $(1-\frac{\Delta t}{4})^2(1+\frac{\Delta t}{4})^{-2}$, we obtain

$$\begin{aligned}&\frac{(1-\frac{\Delta t}{4})^2}{(1+\frac{\Delta t}{4})^2}\epsilon\|\mathcal{E}_x^{2k+2}\|_{E_x}^2+\frac{(1-\frac{\Delta t}{4})^3}{(1+\frac{\Delta t}{4})^3}\Big(\epsilon\|\mathcal{E}_y^{2k+2}\|_{E_y}^2+\mu\|\mathcal{H}_z^{2k+2}\|_H^2\Big)\\ \leq& C_4\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+\frac{(1-\frac{\Delta t}{4})^2}{(1+\frac{\Delta t}{4})^2}\epsilon\|\mathcal{E}_y^{2k+1}\|_{E_y}^2+\frac{1-\frac{\Delta t}{4}}{1+\frac{\Delta t}{4}}\Big(\epsilon\|\mathcal{E}_x^{2k+1}\|_{E_x}^2+\mu\|\mathcal{H}_z^{2k+1}\|_H^2\Big)\\ \leq& C_4\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+\epsilon\|\mathcal{E}_y^{2k+1}\|_{E_y}^2+\frac{1-\frac{\Delta t}{4}}{1+\frac{\Delta t}{4}}\Big(\epsilon\|\mathcal{E}_x^{2k+1}\|_{E_x}^2+\mu\|\mathcal{H}_z^{2k+1}\|_H^2\Big).\end{aligned}\tag{3.19}$$

Adding (3.17) to (3.19) yields

$$\begin{aligned}&\frac{(1-\frac{\Delta t}{4})^2}{(1+\frac{\Delta t}{4})^2}\epsilon\|\mathcal{E}_x^{2k+2}\|_{E_x}^2+\frac{(1-\frac{\Delta t}{4})^3}{(1+\frac{\Delta t}{4})^3}\Big(\epsilon\|\mathcal{E}_y^{2k+2}\|_{E_y}^2+\mu\|\mathcal{H}_z^{2k+2}\|_H^2\Big)\\ \leq& C_5\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+\epsilon\|\mathcal{E}_x^{2k}\|_{E_x}^2+\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}\Big(\epsilon\|\mathcal{E}_y^{2k}\|_{E_y}^2+\mu\|\mathcal{H}_z^{2k}\|_H^2\Big).\end{aligned}\tag{3.20}$$

Note that $\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}>1$. Therefore,

$$\begin{aligned}&\epsilon\|\mathcal{E}_x^{2k+2}\|_{E_x}^2+\epsilon\|\mathcal{E}_y^{2k+2}\|_{E_y}^2+\mu\|\mathcal{H}_z^{2k+2}\|_H^2\\ \leq& C_6\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2+\frac{(1+\frac{\Delta t}{4})^4}{(1-\frac{\Delta t}{4})^4}\Big(\epsilon\|\mathcal{E}_x^{2k}\|_{E_x}^2+\epsilon\|\mathcal{E}_y^{2k}\|_{E_y}^2+\mu\|\mathcal{H}_z^{2k}\|_H^2\Big).\end{aligned}\tag{3.21}$$

Since $\Delta t=T/N$ and the total number of time steps N is even, i.e., $N=2m$, it can be verified that

$$\left(\frac{(1+\frac{\Delta t}{4})^4}{(1-\frac{\Delta t}{4})^4}\right)^m\leq e^T.$$

Computing the inequality (3.21) from the $(2k+2)$-th step to the initial time step recursively by every two neighboring steps, we obtain

$$\begin{aligned}&\epsilon\|\mathcal{E}_x^{2k+2}\|_{E_x}^2+\epsilon\|\mathcal{E}_y^{2k+2}\|_{E_y}^2+\mu\|\mathcal{H}_z^{2k+2}\|_H^2\\ \leq& e^T\Big(\epsilon\|\mathcal{E}_x^{0}\|_{E_x}^2+\epsilon\|\mathcal{E}_y^{0}\|_{E_y}^2+\mu\|\mathcal{H}_z^{0}\|_H^2\Big)+C_7e^T\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2.\end{aligned}\tag{3.22}$$

Note that from the inequality (3.17), the error at the odd step can be controlled by that at the even step. Then the theorem is proved. □

3.4 Convergence of the discrete divergence

From the Maxwell's equations, the electromagnetic waves must be free divergence since we assume that the electric charge density is zero: $\rho=0$, i.e.,

$$\mathrm{div}(\epsilon\vec{E})=0.\tag{3.23}$$

The following result demonstrates that the divergence-free constraint is satisfied approximately.

Theorem 3.3. *Consider the symmetric EC-S-FDTD scheme and assume that the total number of time steps N is even, $N=2m$. Then we have:*

$$\begin{aligned}&\epsilon\left(\delta_x E_{x_{i,j}}^{2k+1}+\delta_y E_{y_{i,j}}^{2k+1}+\delta_x E_{x_{i,j}}^{2k+2}+\delta_y E_{y_{i,j}}^{2k+2}\right)+\frac{\Delta t^2}{2}\delta_x\delta_y\delta_t H_{z_{i,j}}^{2k+1+\frac{1}{2}}\\ =&\epsilon\left(\delta_x E_{x_{i,j}}^{0}+\delta_y E_{y_{i,j}}^{0}+\delta_x E_{x_{i,j}}^{1}+\delta_y E_{y_{i,j}}^{1}\right)+\frac{\Delta t^2}{2}\delta_x\delta_y\delta_t H_{z_{i,j}}^{\frac{1}{2}}.\end{aligned} \tag{3.24}$$

Proof. From (2.25)-(2.32), we can get:

$$\frac{E_{x_{i+\frac12,j}}^{2k+1}-E_{x_{i+\frac12,j}}^{2k}}{\Delta t}=\frac{1}{2\epsilon}\delta_y\left\{H_{z_{i+\frac12,j}}^{2k+1}+H_{z_{i+\frac12,j}}^{2k}\right\}-\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\left\{E_{y_{i+\frac12,j}}^{2k+1}+E_{y_{i+\frac12,j}}^{2k}\right\}, \tag{3.25a}$$

$$\frac{E_{y_{i,j+\frac12}}^{2k+1}-E_{y_{i,j+\frac12}}^{2k}}{\Delta t}=-\frac{1}{2\epsilon}\delta_x\left\{H_{z_{i,j+\frac12}}^{2k+1}+H_{z_{i,j+\frac12}}^{2k}\right\}+\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\left\{E_{x_{i,j+\frac12}}^{2k+1}+E_{x_{i,j+\frac12}}^{2k}\right\}, \tag{3.25b}$$

$$\frac{H_{z_{i+\frac12,j+\frac12}}^{2k+1}-H_{z_{i+\frac12,j+\frac12}}^{2k}}{\Delta t}=\frac{1}{2\mu}\left\{\delta_y(E_{x_{i+\frac12,j+\frac12}}^{2k+1}+E_{x_{i+\frac12,j+\frac12}}^{2k})-\delta_x(E_{y_{i+\frac12,j+\frac12}}^{2k+1}+E_{y_{i+\frac12,j+\frac12}}^{2k})\right\}, \tag{3.25c}$$

$$\frac{E_{x_{i+\frac12,j}}^{2k+2}-E_{x_{i+\frac12,j}}^{2k+1}}{\Delta t}=\frac{1}{2\epsilon}\delta_y\left\{H_{z_{i+\frac12,j}}^{2k+2}+H_{z_{i+\frac12,j}}^{2k+1}\right\}+\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\left\{E_{y_{i+\frac12,j}}^{2k+2}+E_{y_{i+\frac12,j}}^{2k+1}\right\}, \tag{3.25d}$$

$$\frac{E_{y_{i,j+\frac12}}^{2k+2}-E_{y_{i,j+\frac12}}^{2k+1}}{\Delta t}=-\frac{1}{2\epsilon}\delta_x\left\{H_{z_{i,j+\frac12}}^{2k+2}+H_{z_{i,j+\frac12}}^{2k+1}\right\}-\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\left\{E_{x_{i,j+\frac12}}^{2k+2}+E_{x_{i,j+\frac12}}^{2k+1}\right\}, \tag{3.25e}$$

$$\frac{H_{z_{i+\frac12,j+\frac12}}^{2k+2}-H_{z_{i+\frac12,j+\frac12}}^{2k+1}}{\Delta t}=\frac{1}{2\mu}\left\{\delta_y(E_{x_{i+\frac12,j+\frac12}}^{2k+2}+E_{x_{i+\frac12,j+\frac12}}^{2k+1})-\delta_x(E_{y_{i+\frac12,j+\frac12}}^{2k+2}+E_{y_{i+\frac12,j+\frac12}}^{2k+1})\right\}. \tag{3.25f}$$

From Eqs. (3.25a)-(3.25f) and noting that δ_x(or δ_y) and δ_t are changeable, we get

$$\delta_t\delta_x E_{x_{i,j}}^{2k+\frac12}=\frac{1}{2\epsilon}\delta_x\delta_y\left(H_{z_{i,j}}^{2k+1}+H_{z_{i,j}}^{2k}\right)+\frac{-\Delta t}{4\mu\epsilon}\delta_x\delta_x\delta_y\left(E_{y_{i,j}}^{2k+1}+E_{y_{i,j}}^{2k}\right), \tag{3.26a}$$

$$\delta_t\delta_y E_{y_{i,j}}^{2k+\frac12}=\frac{-1}{2\epsilon}\delta_x\delta_y\left(H_{z_{i,j}}^{2k+1}+H_{z_{i,j}}^{2k}\right)+\frac{\Delta t}{4\mu\epsilon}\delta_y\delta_x\delta_y\left(E_{x_{i,j}}^{2k+1}+E_{x_{i,j}}^{2k}\right), \tag{3.26b}$$

$$\delta_t\delta_x E_{x_{i,j}}^{2k+1+\frac12}=\frac{1}{2\epsilon}\delta_x\delta_y\left(H_{z_{i,j}}^{2k+2}+H_{z_{i,j}}^{2k+1}\right)+\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_x\delta_y\left(E_{y_{i,j}}^{2k+2}+E_{y_{i,j}}^{2k+1}\right), \tag{3.26c}$$

$$\delta_t\delta_y E_{y_{i,j}}^{2k+1+\frac12}=\frac{-1}{2\epsilon}\delta_x\delta_y\left(H_{z_{i,j}}^{2k+2}+H_{z_{i,j}}^{2k+1}\right)+\frac{-\Delta t}{4\mu\epsilon}\delta_y\delta_x\delta_y\left(E_{x_{i,j}}^{2k+2}+E_{x_{i,j}}^{2k+1}\right). \tag{3.26d}$$

Summing up (3.26a)-(3.26d) yields

$$\begin{aligned}&\delta_t\delta_x E_{x_{i,j}}^{2k+\frac{1}{2}}+\delta_t\delta_y E_{y_{i,j}}^{2k+\frac{1}{2}}+\delta_t\delta_x E_{x_{i,j}}^{2k+1+\frac{1}{2}}+\delta_t\delta_y E_{y_{i,j}}^{2k+1+\frac{1}{2}}\\ =&\frac{\Delta t}{2\epsilon}\delta_x\delta_y\left\{\frac{1}{2\mu}\delta_y\left(E_{x_{i,j}}^{2k+1}+E_{x_{i,j}}^{2k}\right)-\frac{1}{2\mu}\delta_x\left(E_{y_{i,j}}^{2k+1}+E_{y_{i,j}}^{2k}\right)\right.\\ &\left.-\frac{1}{2\mu}\delta_y\left(E_{x_{i,j}}^{2k+2}+E_{x_{i,j}}^{2k+1}\right)+\frac{1}{2\mu}\delta_x\left(E_{y_{i,j}}^{2k+2}+E_{y_{i,j}}^{2k+1}\right)\right\}\\ =&\frac{\Delta t}{2\epsilon}\delta_x\delta_y\left(\delta_t H_{z_{i,j}}^{2k+\frac{1}{2}}-\delta_t H_{z_{i,j}}^{2k+1+\frac{1}{2}}\right)=-\frac{\Delta t^2}{2\epsilon}\delta_x\delta_y\delta_t\delta_t H_{z_{i,j}}^{2k+1}.\end{aligned}\tag{3.27}$$

Consequently, we have

$$\begin{aligned}&\delta_x E_{x_{i,j}}^{2k+1}+\delta_y E_{y_{i,j}}^{2k+1}+\delta_x E_{x_{i,j}}^{2k+2}+\delta_y E_{y_{i,j}}^{2k+2}+\frac{\Delta t^2}{2\epsilon}\delta_x\delta_y\delta_t H_{z_{i,j}}^{2k+1+\frac{1}{2}}\\ =&\delta_x E_{x_{i,j}}^{0}+\delta_y E_{y_{i,j}}^{0}+\delta_x E_{x_{i,j}}^{1}+\delta_y E_{y_{i,j}}^{1}+\frac{\Delta t^2}{2\epsilon}\delta_x\delta_y\delta_t H_{z_{i,j}}^{\frac{1}{2}}.\end{aligned}$$

This completes the proof of the theorem. □

4 Numerical experiments

In this section, we will show numerically the following properties: 1) Energy I conservation at every time step, energy II conservation at every two time steps and second-order approximation with respect to Energy II ; 2) Accuracy analysis; 3) Unconditionally stable for long time computation; 4) The convergence of the divergence-free constraint. Numerical experiments will consider both the constant electric permittivity case and the discontinuous coefficient case.

4.1 Constant electric permittivity case

Consider the magnetic field H_z in the form of $e^{\mathrm{i}(\omega t-k_x x-k_y y)}$. Then the electric field $\vec{E}$ should be

$$\frac{1}{\epsilon}\left(-\frac{k_y}{\omega},\frac{k_x}{\omega}\right)e^{\mathrm{i}(\omega t-k_x x-k_y y)},$$

and k_x and k_y satisfy the dispersion relation: $\mu\epsilon\omega^2=k_x^2+k_y^2$.

Assume the domain $\Omega=[0,\pi]\times[0,\pi]$ surrounded by a perfect conductor, which means that $E_y(0,y)=E_y(\pi,y)=0$ and $E_x(x,0)=E_x(x,\pi)=0$. So we have the following analytic solution:

$$E_x=\frac{k_y}{\epsilon\sqrt{\mu}\omega}\cos(\omega t)\cos(k_x x)\sin(k_y y),\quad E_y=-\frac{k_x}{\epsilon\sqrt{\mu}\omega}\cos(\omega t)\sin(k_x x)\cos(k_y y),\tag{4.1}$$

$$H_z=\frac{1}{\sqrt{\mu}}\sin(\omega t)\cos(k_x x)\cos(k_y y).\tag{4.2}$$

4.1.1 Energy conservation

The energies of the solution are easily to be computed; Energy I is

$$\texttt{Energy I} = \left(\int_\Omega \epsilon \left| \vec{E}(x,t) \right|^2 dxdy + \int_\Omega \mu |H_z(x,t)|^2 dxdy \right)^{\frac{1}{2}} = \frac{\pi}{2}, \tag{4.3}$$

and Energy II is:

$$\texttt{Energy II} = \left(\int_\Omega \epsilon \left| \frac{\partial \vec{E}}{\partial t}(x,t) \right|^2 dxdy + \int_\Omega \mu \left| \frac{\partial H_z}{\partial t}(x,t) \right|^2 dxdy \right)^{\frac{1}{2}} = \frac{\pi\omega}{2}. \tag{4.4}$$

Let us fix $T=\pi$, $\mu=\epsilon=1$, $\Delta t=\Delta x=\Delta y=\frac{\pi}{100}$ and change $k_x=k_y$ as 1,5 and 10. Define the relative errors as

$$\texttt{Error of Energy I} = \max_{0\le n\le N} \frac{\left| \left(\|\epsilon^{\frac{1}{2}} \vec{E}^n\|^2 + \|\mu^{\frac{1}{2}} H_z^n\|^2 \right)^{\frac{1}{2}} - \texttt{Energy I} \right|}{\texttt{Energy I}}, \tag{4.5}$$

$$\texttt{Error of Energy II} = \max_{0\le n\le N-1} \frac{\left| \left(\|\epsilon^{\frac{1}{2}} \delta_t \vec{E}^{n+\frac{1}{2}}\|^2 + \|\mu^{\frac{1}{2}} \delta_t H_z^{n+\frac{1}{2}}\|^2 \right)^{\frac{1}{2}} - \texttt{Energy II} \right|}{\texttt{Energy II}}. \tag{4.6}$$

On the other hand, to check the conservation property of Energy II, the following value is computed according to Eq. (3.2):

$$\begin{aligned} \texttt{DiffII} = \Big\{ & \max_{0\le k\le m-1} \left(\|\epsilon^{\frac{1}{2}} \left(\vec{E}^{2k+2} - \vec{E}^{2k} \right) / (2\Delta t)\|_E^2 + \|\mu^{\frac{1}{2}} \left(H_z^{2k+2} - H_z^{2k} \right) / (2\Delta t)\|_H^2 \right) - \\ & \min_{0\le k\le m-1} \left(\|\epsilon^{\frac{1}{2}} \left(\vec{E}^{2k+1} - \vec{E}^{2k-1} \right) / (2\Delta t)\|_E^2 + \|\mu^{\frac{1}{2}} \left(H_z^{2k+1} - H_z^{2k-1} \right) / (2\Delta t)\|_H^2 \right) \Big\}. \end{aligned}$$

By Theorem 3.1, the value "`DiffII`" should be very small.

Table 1 shows that Energy I of the discrete solutions perfectly equal to the exact value $\frac{\pi}{2}$ since the errors are near machine precision. Similarly, the value "`DiffII`" is also on the same scale as the machine precision.

Table 1: Relative errors of Energy I and Energy II and DiffEII of symmetric EC-S-FDTD. Parameters: $T=\pi$, $\Delta t=\Delta x=\Delta y=T/100$ and $k_x=k_y$.

$k_x=k_y$	EC-S-FDTD I		Symmetric EC-S-FDTD			EC-S-FDTD II	
	Error I	Error II	Error I	Error II	DiffEII	Error I	Error II
1	9.04*e*-15	2.26*e*-4	9.05*e*-15	2.90*e*-4	7.99*e*-14	1.27*e*-15	1.95*e*-4
5	7.63*e*-15	5.61*e*-3	7.63*e*-15	7.52*e*-3	1.79*e*-12	8.48*e*-16	4.85*e*-3
10	7.07*e*-15	2.20*e*-2	7.07*e*-15	3.12*e*-2	6.42*e*-12	5.65*e*-16	1.91*e*-2

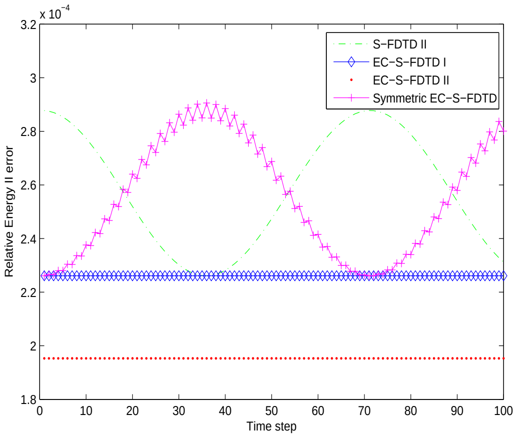

Figure 2: Relative errors of Energy II for the EC-S-FDTD schemes. Parameters: $T=\pi$ and $\Delta x=\Delta y=\Delta t=\pi/100$.

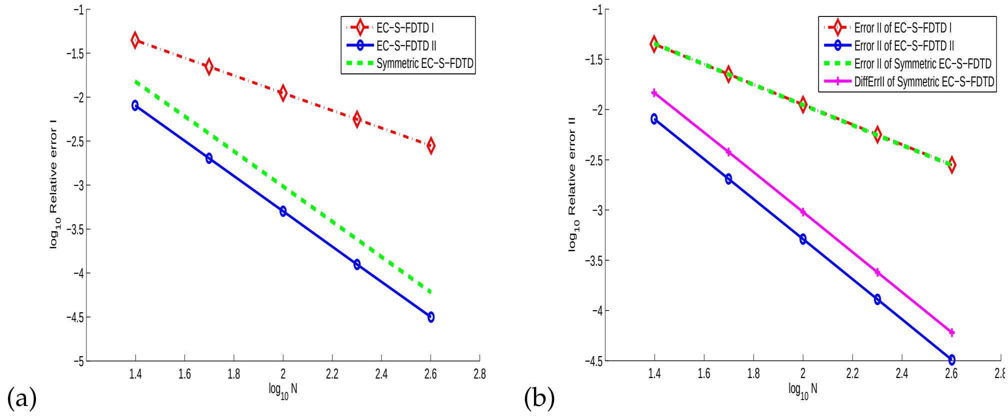

Figure 3: (a) Error I of different schemes, (b) Error II of different schemes. Parameters: $\Delta x=\Delta y=\Delta t=\pi/N$, $k_x=k_y=1$.

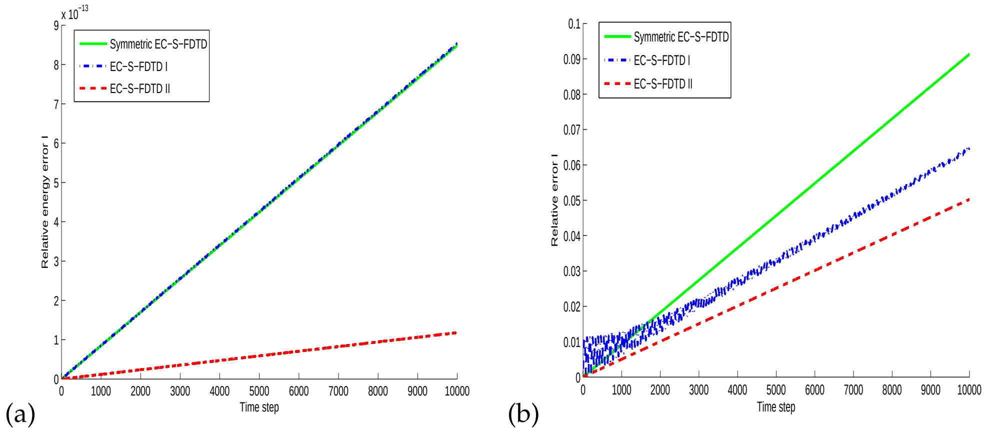

Figure 4: (a) The EnergyI error in the long time computation, (b) ErrorI in the long time computation.

Table 2: Relative errors of different schemes. Parameters: $T=\pi$ and $k_x=k_y=1,\mu=\epsilon=1$.

N	EC-S-FDTD I		Symmetric EC-S-FDTD			EC-S-FDTD II	
	Error I	Error II	Error I	Error II	DiffErrII	Error I	Error II
25	4.45e-2	4.49e-2	1.51e-2	4.54e-2	1.47e-2	8.04e-3	8.07e-3
50	2.22e-2	2.23e-2	3.86e-3	2.23e-2	3.78e-3	2.014e-3	2.04e-3
100	1.11e-2	1.11e-2	9.65e-4	1.11e-2	9.56e-4	5.04e-4	5.13e-4
200	5.60e-3	5.63e-3	2.41e-4	5.55e-3	2.40e-4	1.25e-4	1.29e-4
400	2.80e-3	2.81e-3	6.03e-5	2.81e-3	6.02e-5	3.15e-5	3.22e-5

4.1.2 Accuracy analysis

Here we set $\Delta x=\Delta y=\Delta t=\pi/N$. Table 2 compares the accuracies of the EC-S-FDTDI scheme, the symmetric EC-S-FDTD scheme and the EC-S-FDTDII scheme. First, we use the definitions of relative errors below:

$$\texttt{ErrorI}=\max_{0\le n\le N}\Big(\|\epsilon^{\frac{1}{2}}[\vec{E}(t^n)-\vec{E}^n]\|_E^2+\|\mu^{\frac{1}{2}}[H_z(t^n)-H_z^n]\|_H^2\Big)^{\frac{1}{2}}/\texttt{Energy I},$$

$$\texttt{ErrorII}=\max_{0\le n\le N-1}\Big(\|\epsilon^{\frac{1}{2}}[\delta_t\vec{E}(t^n)-\delta_t\vec{E}^n]\|_E^2+\|\mu^{\frac{1}{2}}[\delta_tH_z(t^n)-\delta_tH_z^n]\|_H^2\Big)^{\frac{1}{2}}/\texttt{Energy II},$$

where $\vec{E}^n,H_z^n$ and $\vec{E}(t^n),H_z(t^n)$ denote the numerical and analytic solutions at time level n respectively. Table 2 (see the column of "ErrorI") and Fig. 3(a) show that the relative errors are of order $\mathcal{O}(N^{-2})$, which implies that the symmetric EC-S-FDTD scheme is second-order accurate in time and space. For the approximation of the terms $\partial\vec{E}/\partial t$ and $\partial H/\partial t$, the rate of ErrorII seems to indicate that the symmetric EC-S-FDTD scheme only has first-order accuracy. Here we point out that the second-order accuracy can be recovered. Let us define a new error measurement "`DiffErrII`" as

$$\begin{aligned}&\texttt{DiffErrII}\\=&\max_k\Bigg\{\Bigg(\|\epsilon^{\frac{1}{2}}[\frac{\vec{E}(t^{2k+2})-\vec{E}(t^{2k})}{2\Delta t}-\frac{\vec{E}^{2k+2}-\vec{E}^{2k}}{2\Delta t}]\|_E^2\\&+\|\mu^{\frac{1}{2}}[\frac{H_z(t^{2k+2})-H_z(t^{2k})}{2\Delta t}-\frac{H_z^{2k+2}-H_z^{2k}}{2\Delta t}]\|_H^2\Bigg)^{\frac{1}{2}},\\&\Bigg(\|\epsilon^{\frac{1}{2}}[\frac{\vec{E}(t^{2k+1})-\vec{E}(t^{2k-1})}{2\Delta t}-\frac{\vec{E}^{2k+1}-\vec{E}^{2k-1}}{2\Delta t}]\|_E^2\\&+\|\mu^{\frac{1}{2}}[\frac{H_z(t^{2k+1})-H_z(t^{2k-1})}{2\Delta t}-\frac{H_z^{2k+1}-H_z^{2k-1}}{2\Delta t}]\|_H^2\Bigg)^{\frac{1}{2}}\Bigg\}/\texttt{Energy II}.\end{aligned}$$

Then from Table 2 and Fig. 3(b), we observe that the error "`DiffErrII`" also behaves like $\mathcal{O}(N^{-2})$, which means that the symmetric EC-S-FDTD scheme is also second-order accurate for the terms $\partial\vec{E}/\partial t$ and $\partial H/\partial t$, as also observed for the EC-S-FDTDII scheme.

4.1.3 Unconditionally stable for long-time computation

Here we set $T=100\pi$, and $\Delta x=\Delta y=\Delta t=\pi/100$, i.e., the code runs 10000 steps. Fig. 4(a) indicates that for the symmetric EC-S-FDTD, EC-S-FDTDI and EC-S-FDTDII schemes, the relative errors of energyI are controlled under 10^{-12} after 10000 time steps, and the errors grow linearly. Fig. 4(b) also suggests that when the time level increases, the errors of the solutions also grow linearly.

4.1.4 Convergence analysis of the divergence-free term

Since we wish to verify that all schemes satisfy the divergence-free property approximately, we need to compute the following values:

$$\mathrm{Div1}=\max_{\substack{1\le i,j\le N-1\\ 0\le n\le N}} \epsilon(\delta_x E^n_{x_{i,j}}+\delta_y E^n_{y_{i,j}}),$$

$$\mathrm{Div2}=\max_{0\le n\le N}\sum_{1\le i,j\le N-1}\left(\epsilon\left(\delta_x E{x}^n_{i,j}+\delta_y E{y}^n_{i,j}\right)^2\Delta x\Delta y\right)^{\frac{1}{2}}.$$

From Table 3, we can see that the numerical divergence term of the symmetric EC-S-FDTD scheme is still second-order accurate in time direction.

Table 3: Numerical divergence of different schemes. Parameters: $T=\pi$ and $k_x=k_y=1$, $\mu=\epsilon=1$.

N	EC-S-FDTD I		Symmetric EC-S-FDTD		EC-S-FDTD II	
	Div1	Div2	Div1	Div2	Div1	Div2
25	6.24e-2	9.84e-2	1.11e-2	1.75e-2	1.39e-3	2.19e-3
50	3.14e-2	4.93e-2	2.79e-3	4.38e-3	3.47e-4	5.48e-4
100	1.57e-2	2.47e-2	6.98e-4	1.10e-3	8.72e-5	1.37e-4
200	7.90e-2	1.23e-2	1.74e-4	2.74e-4	2.18e-5	3.42e-5
400	3.90e-3	6.20e-3	4.36e-5	6.85e-5	5.45e-6	8.56e-6

4.2 Discontinuous electric permittivity case

Now we consider that the electric permittivity ϵ is piece-wise constant on the domain $\bar{\Omega}=\bar{\Omega}_1\cup\bar{\Omega}_2$:

$$\epsilon=\begin{cases} 1, & \text{in } \Omega_1,\\ 4, & \text{in } \Omega_2,\end{cases}$$

where $\Omega_1\in[0,\frac{1}{2}]\times[0,1]$ and $\Omega_2\in[\frac{1}{2},1]\times[0,1]$ and the magnetic permeability $\mu=1$ in Ω. We can still construct one exact solution to check the numerical results. Take $k_y=8$ and let k_x be piece-wise constant on the domain Ω:

$$k_x=\begin{cases} 4, & \text{in } \Omega_1,\\ 16, & \text{in } \Omega_2.\end{cases}$$

Table 4: The relative errors of Energy I and Energy II when $T=1$ and ϵ is a piecewise constant.

N	EC-S-FDTD I		Symmetric EC-S-FDTD			EC-S-FDTD II	
	Error of I	Error of II	Error of I	Error of II	DiffII	Error of I	Error of II
50	6.77e-15	5.11e-2	5.11e-15	6.10e-2	5.06e-16	6.21e-15	4.56e-2
100	2.13e-14	1.33e-2	2.73e-14	1.57e-2	7.59e-16	1.71e-14	1.20e-2
200	8.94e-14	3.30e-3	1.17e-13	3.90e-3	1.26e-15	8.88e-14	3.04e-3
400	2.78e-13	7.90e-4	2.52e-13	9.60e-4	2.91e-15	2.08e-13	7.62e-4

Table 5: The relative errors of Error I and Error II when ϵ is a piecewise constant.

N	EC-S-FDTD I		Symmetric EC-S-FDTD			EC-S-FDTD II	
	Error I	Error II	Error I	Error II	DiffErrII	Error I	Error II
50	1.11	1.06	1.23	1.18	1.05	1.01	9.65e-1
100	3.10e-1	3.10e-1	3.46e-1	3.44e-1	3.37e-1	2.74e-1	2.71e-1
200	8.34e-2	8.76e-2	8.78e-2	9.30e-2	8.72e-2	6.90e-2	6.89e-2
400	2.68e-2	3.01e-2	2.20e-2	2.93e-2	2.24e-2	1.72e-2	1.73e-2

We take $\omega = \sqrt{k_x^2+k_y^2}/\sqrt{\mu\epsilon}$, which is also piece-wise constant on the domain Ω. It is easy to check that $\vec{E}$ defined by (4.1) and H_z by (4.2) are also the exact solutions of the Maxwell equations. Note that the exact solution may be discontinuous where the electric permittivity ϵ jumps.

The numerical results are similar to those of the constant coefficient case. The results of energy conservations are given in Table 4 and the error behaviors are given in Table 5. It is clear that for the piece-wise constant electric permittivity, the Energy I conservation still holds in all three schemes. But the convergence of the symmetric EC-S-FDTD changes slightly in the piecewise constant case. The convergence rate of Error I is still of second-order. However, for Error II, since the wave number is big, the relative errors are large (around 100%) when $N=50$. On the other hand, the convergence is observed when the mesh size is further decreased. This is common for high frequency waves (see, e.g., [17]).

5 Conclusions

In this paper, we developed a new symmetric energy-conserved splitting finite-difference time-domain scheme (symmetric EC-S-FDTD) for the Maxwell's equations. The new scheme has the following advantages: energy conservations, second-order accuracy in time and space, and unconditional stability in the long-time computation. These properties are theoretically proved and confirmed by numerical experiments.

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