A Numerical Method and its Error Estimates for the Decoupled Forward-Backward Stochastic Differential Equations

Weidong Zhao[1], Wei Zhang[1] and Lili Ju[2,3,*]

[1] *School of Mathematics, Shandong University, Jinan, Shandong 250100, P.R. China.*
[2] *Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.*
[3] *Beijing Computational Science Research Center, Beijing 100084, P.R. China.*

Abstract. In this paper, a new numerical method for solving the decoupled forward-backward stochastic differential equations (FBSDEs) is proposed based on some specially derived reference equations. We rigorously analyze errors of the proposed method under general situations. Then we present error estimates for each of the specific cases when some classical numerical schemes for solving the forward SDE are taken in the method; in particular, we prove that the proposed method is second-order accurate if used together with the order-2.0 weak Taylor scheme for the SDE. Some examples are also given to numerically demonstrate the accuracy of the proposed method and verify the theoretical results.

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1 Introduction

Let $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{0\leq t\leq T})$ be a complete, filtered probability space on which a standard d-dimensional Brownian motion $W_t=(W_t^1,W_t^2,\cdots,W_t^d)^*$ is defined, such that $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the natural filtration of the Brownian motion W_t and all the P-null sets are augmented to each σ-field $\mathcal{F}_t$. Here the operator $(\cdot)^*$ denotes the transpose operator for a matrix or

vector. We consider the decoupled forward-backward stochastic differential equations (FBSDEs) on $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{0\leq t\leq T})$

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s, & t\in[0,T], \\ Y_t = \varphi(X_T) + \displaystyle\int_t^T f(s,X_s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, & t\in[0,T], \end{cases} \tag{1.1}$$

with the functions $b(t,X)$: $[0,T]\times\mathbb{R}^q\to\mathbb{R}^q$, $\sigma(t,X)$: $[0,T]\times\mathbb{R}^q\to\mathbb{R}^{q\times d}$, $f(t,X,Y,Z)$: $[0,T]\times \mathbb{R}^q\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\to\mathbb{R}^m$, and $\varphi(\cdot)$: $\mathbb{R}^q\to\mathbb{R}^m$. Note that the integrals in (1.1) with respect to the d-dimensional Brownian motion W_s are the Itô type stochastic integrals. The first equation in (1.1) is the standard (forward) stochastic differential equation (SDE), and the second equation is the so-called backward stochastic differential equation (BSDE). A process (X_t,Y_t,Z_t) is called an L^2 solution of the decoupled FBSDEs (1.1) if it is $\{\mathcal{F}_t\}$-adapted and square integrable and satisfy (1.1). In the sequel, a solution means a L^2 solution. Under standard conditions on f and φ, Pardoux and Peng [25] originally proved the existence and uniqueness of solution of nonlinear BSDEs. Since then a lot of efforts have been devoted to study of FBSDEs [2–6, 8–11, 13, 14, 19–24, 30, 32] due to their natural applications in many fields including mathematical finance, partial differential equations (PDEs), stochastic PDEs, stochastic control, risk measure, game theory, and so on.

It is well-known that it is often difficult to obtain analytic solutions in the close form for the FBSDEs, even for the linear case, so that computing approximate solutions of FB-SDEs becomes highly desired. There are lots of works on numerical methods for numerically solving BSDEs. Based on the relation between the FBSDEs and their corresponding parabolic partial differential equations (PDEs) [7,10,12,17,26], some algorithms were proposed to solve FBSDEs in [5,10,11,21–24]. There are also some other numerical methods for solving BSDEs or FBSDEs, which were proposed based on directly discretizing BSDEs or FBSDEs [2,4,6,8,13,28,29,33–35]. Many existing numerical methods for the decoupled FBSDEs (1.1) are half order and one order in time such as those in [3,4,8,10,13,14,20,29]. In these methods, forward or backward trapezoidal rules were often used to approximate the integrals in (1.1), and the martingale representation was used in their error analysis. In this paper, based on properties of the Itô's integral and the nature of solution of the FBSDEs, we will propose a numerical method for solving the decoupled FBSDEs (1.1) that utilizes the trapezoidal rule and approximations of some reference equations with a newly defined standard Brownian motion. We rigorously derive error estimates for this method for general cases. Under certain regularity assumptions on the functions b, σ, f and φ, we also show that the proposed scheme can be up to second-order accurate in time.

Now let us introduce some notations which will be used in this paper:

- $|\cdot|$: the standard Euclidean norm in the Euclidean space $\mathbb{R}$, $\mathbb{R}^q$ and $\mathbb{R}^{q\times d}$.

- $L^2 = L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$: the set of all $\mathcal{F}_t$-adapted and mean-square-integrable processes valued in $\mathbb{R}^d$.

- $\mathcal{F}_s^{t,x}$ ($t \le s \le T$) be a σ-field generated by the diffusion process $\{x + X_r - X_t, t \le r \le s\}$ starting from the time-space point (t,x). When $s = T$, we use $\mathcal{F}^{t,x}$ to denote $\mathcal{F}_T^{t,x}$.

- $\mathbb{E}_s^{t,x}[X]$: the conditional mathematical expectation of the random variable X under the σ-field $\mathcal{F}_s^{t,x}$, i.e., $\mathbb{E}_s^{t,x}[X] = \mathbb{E}[X|\mathcal{F}_s^{t,x}]$. When $s = t$, we use $\mathbb{E}_t^x[X]$ to denote $\mathbb{E}[X|\mathcal{F}_t^{t,x}]$.

- $C_b^{l,k,k}$: the set of continuously differential functions $\phi : [0,T] \times R^q \times R \to R$ with uniformly bounded partial derivatives $\partial_t^{l_1} \phi$ and $\partial_x^{k_1} \partial_y^{k_2} \phi$ for $l_1 \le l$ and $k_1 + k_2 \le k$.

- $C_b^{k_3,k_4}$: the set of functions $\phi : (t,x) \in [0,T] \times R^q \to R$ with uniformly bounded partial derivatives $\partial_t^{l_3} \partial_x^{l_4} \phi$ for $l_3 \le k_3$ and $l_4 \le k_4$.

The rest of the paper is organized as follows. In Section 2, we first derive some reference equations that will be used for numerical discretization. Based on these reference equations, we propose a numerical method for solving the decoupled FBSDEs in Section 3. In Section 4 we rigorously analyze errors of the proposed method for general cases. And in following, we present specific error estimates for each of the cases when some classical numerical schemes are taken in the method for solving the forward equation; in particular, we show that the proposed method is first order accurate when the Euler scheme or the Milstein scheme is used, and is second order accurate when the order-2.0 weak Taylor scheme is used. Then in Section 5 we demonstrate through numerical experiments the accuracy of the proposed method and verify the theoretical results. Finally some conclusions are summarized in Section 6.

2 Reference equations

Let $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ be the solution of (1.1) starting from time t with $X_t = x$, that is, $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ satisfies

$$
\begin{cases}
X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r, & s \in [t,T], \\
Y_s^{t,x} = \varphi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, & s \in [t,T].
\end{cases}
\tag{2.1}
$$

For the time interval $[0,T]$, we introduce the following partition:

$$0 = t_0 < \cdots < t_N = T.$$

Let $\Delta t_n = t_{n+1} - t_n$ and $\Delta t = \max_{0 \le n \le N-1} \Delta t_n$. We also assume that the time partition have the following regularity:

$$
\frac{\max\limits_{0 \le n \le N-1} \Delta t_n}{\min\limits_{0 \le n \le N-1} \Delta t_n} \le c_0,
\tag{2.2}
$$

where $c_0 \geq 1$ is a constant.

Let $Y_{t_N}^{t_n,X^n} = \varphi(X_T^{t_n,X^n})$ and $(X_t^{t_n,X^n}, Y_t^{t_n,X^n}, Z_t^{t_n,X^n})$ be the solution of (2.1) with $X^n = x$ for $t \in [t_n,T]$. Denote $f(s, X_s^{t_n,X^n}, Y_s^{t_n,X^n}, Z_s^{t_n,X^n})$ by $f_s^{t_n,X^n}$. Then it is easy to get that

$$X_{t_{n+1}}^{t_n,X^n} = X^n + \int_{t_n}^{t_{n+1}} b(s, X_s^{t_n,X^n}) ds + \int_{t_n}^{t_{n+1}} \sigma(s, X_s^{t_n,X^n}) dW_s, \tag{2.3a}$$

$$Y_{t_n}^{t_n,X^n} = Y_{t_{n+1}}^{t_n,X^n} + \int_{t_n}^{t_{n+1}} f_s^{t_n,X^n} ds - \int_{t_n}^{t_{n+1}} Z_s^{t_n,X^n} dW_s, \tag{2.3b}$$

for $n = 0,1,\cdots,N-1$.

Take the conditional mathematical expectation $\mathbb{E}_{t_n}^{X^n}[\cdot]$ on both sides of (2.3b), we obtain

$$Y_{t_n}^{t_n,X^n} = \mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}] ds. \tag{2.4}$$

Under the filtration $\mathcal{F}_{t_n}$, the integrand $\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}]$ on the right-hand side of (2.4) is a deterministic function of time s. Thus some numerical integration methods may be used to accurately approximate the integral in (2.4). In particular, in this paper, we use the trapezoidal rule to approximate this integral as

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}] ds = \frac{1}{2}\Delta t_n f_{t_n}^{t_n,X^n} + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n}] + R_y^n, \tag{2.5}$$

where

$$R_y^n = \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}] - \frac{1}{2}f_{t_n}^{t_n,X^n} - \frac{1}{2}\mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n}] \right) ds. \tag{2.6}$$

Inserting (2.5) into (2.4), we obtain the following reference equation for solving $Y_{t_n}^{t_n,X^n}$:

$$Y_{t_n}^{t_n,X^n} = \mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n}] + \frac{1}{2}\Delta t_n f_{t_n}^{t_n,X^n} + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n}] + R_y^n. \tag{2.7}$$

In order to solve $Z_t^{t_n,X^n}$, we still need obtain another reference equation from (2.3b). To proceed, let us introduce a new Brownian motion $\Delta\tilde{W}_s$ defined by

$$\Delta\tilde{W}_s = 2\Delta W_s - \frac{3}{\Delta t_n}\int_{t_n}^{s}(r - t_n) dW_r, \tag{2.8}$$

where $\Delta W_s = W_s - W_{t_n}(t_n \leq s \leq t_{n+1})$, which is a standard Brownian motion with mean zero and variance $s - t_n$. By the definition (2.8), $\Delta\tilde{W}_s = (\Delta\tilde{W}_s^1, \Delta\tilde{W}_s^2, \cdots, \Delta\tilde{W}_s^d)^*$ is a Gaussian

process with the properties $\mathbb{E}_{t_n}^{X^n}[\Delta\tilde{W}_s]=0$, $\mathbb{E}_{t_n}^{X^n}[\Delta\tilde{W}_s^i\Delta\tilde{W}_s^j]=0$ for $i\neq j$, and

$$
\begin{aligned}
\mathbb{E}_{t_n}^{X^n}[(\Delta\tilde{W}_s^i)^2] &= \mathbb{E}_{t_n}^{X^n}\left[\left(2\Delta W_s^i-\frac{3}{\Delta t_n}\int_{t_n}^s(r-t_n)dW_r^i\right)^2\right] \\
&= 4(s-t_n)-\frac{12}{\Delta t_n}\int_{t_n}^s(r-t_n)dr+\frac{9}{\Delta t_n^2}\int_{t_n}^s(r-t_n)^2dr \\
&= 4(s-t_n)-\frac{12}{\Delta t_n}\frac{1}{2}(s-t_n)^2+\frac{9}{\Delta t_n^2}\frac{1}{3}(s-t_n)^3 \\
&= 4(s-t_n)-\frac{6(s-t_n)^2}{\Delta t_n}+\frac{3(s-t_n)^3}{\Delta t_n^2}.
\end{aligned}
$$

Then when $s=t_{n+1}$, we have $\mathbb{E}_{t_n}^{X^n}[\Delta\tilde{W}_{t_{n+1}}]=0$ and $\mathbb{E}_{t_n}^{X^n}[(\Delta\tilde{W}_{t_{n+1}})^2]=\Delta t_n$.

Now let us multiply (2.3b) by $\Delta\tilde{W}_{t_{n+1}}^*$ and take the conditional mathematical expectation $\mathbb{E}_{t_n}^{X^n}[\cdot]$ on both sides of the derived equation, then we obtain

$$
0=\mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n}\Delta\tilde{W}_{t_{n+1}}^*]+\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}\Delta\tilde{W}_{t_{n+1}}^*]ds-\mathbb{E}_{t_n}^{X^n}\left[\int_{t_n}^{t_{n+1}}Z_s^{t_n,X^n}dW_s\cdot\Delta\tilde{W}_{t_{n+1}}^*\right]. \quad (2.9)
$$

For the first integral term on the right-hand side of (2.9), we easily get the identity

$$
\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}\Delta\tilde{W}_{t_{n+1}}^*]ds=\Delta t_n\mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n}\Delta\tilde{W}_{t_{n+1}}^*]+R_1^n, \quad (2.10)
$$

where

$$
R_1^n=\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n}\Delta\tilde{W}_{t_{n+1}}^*]ds-\Delta t_n\mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n}\Delta\tilde{W}_{t_{n+1}}^*]. \quad (2.11)
$$

For the second integral term on the right-hand side of (2.9), we have

$$
\begin{aligned}
-\mathbb{E}_{t_n}^{X^n}\left[\int_{t_n}^{t_{n+1}}Z_s^{t_n,X^n}dW_s\cdot\Delta\tilde{W}_{t_{n+1}}^*\right] &= -Z_{t_n}^{t_n,X^n}\mathbb{E}_{t_n}^{X^n}[\Delta W_{t_{n+1}}\Delta\tilde{W}_{t_{n+1}}^*]+R_2^n \\
&= -\frac{1}{2}\Delta t_n Z_{t_n}^{t_n,X^n}+R_2^n, \quad (2.12)
\end{aligned}
$$

where

$$
R_2^n=\frac{1}{2}\Delta t_n Z_{t_n}^{t_n,X^n}-\mathbb{E}_{t_n}^{X^n}\left[\int_{t_n}^{t_{n+1}}Z_s^{t_n,X^n}dW_s\cdot\Delta\tilde{W}_{t_{n+1}}^*\right]. \quad (2.13)
$$

From Eqs. (2.9), (2.10) and (2.12), we obtain the following reference equation for solving $Z_{t_n}^{t_n,X^n}$:

$$
\frac{1}{2}\Delta t_n Z_{t_n}^{t_n,X^n}=\mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n}\Delta\tilde{W}_{t_{n+1}}^*]+\Delta t_n\mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n}\Delta\tilde{W}_{t_{n+1}}^*]+R_z^n, \quad (2.14)
$$

where

$$
R_z^n=R_1^n+R_2^n. \quad (2.15)
$$

Based on the two reference equations (2.7) and (2.14), we will propose a new numerical scheme for solving (Y_t,Z_t) of the decoupled FBSDEs (2.1) in the following section.

3   A numerical scheme for solving the decoupled FBSDEs

Now we propose a new numerical scheme for solving the decoupled FBSDEs (2.1). Let (X^n, Y^n, Z^n) denote an approximation to the analytic solution (X_t, Y_t, Z_t) of (2.1) at time $t = t_n$, $n = N, N-1, \cdots, 0$. To simplify the presentation, we let $f^n = f(t_n, X^n, Y^n, Z^n)$ for $n = N, N-1, \cdots, 0$. Based on (2.7) and (2.14), we propose a numerical scheme, for solving the FBSDEs (2.1) as:

Scheme 3.1. (*Numerical Method for the Decoupled FBSDEs*) Given random variables X_0, Y^N and Z^N. Let $\Delta \tilde{W}_{t_{n+1}} (0 \leq n \leq N-1)$ be defined by (2.8) with $s = t_{n+1}$. For $n = N-1, N-2, \cdots, 0$, solve random variables Y^n and Z^n by

$$Y^n = \mathbb{E}_{t_n}^{X^n}[Y^{n+1}] + \frac{1}{2}\Delta t_n f^n + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X^n}[f^{n+1}], \tag{3.1a}$$

$$\frac{1}{2}\Delta t_n Z^n = \mathbb{E}_{t_n}^{X^n}[Y^{n+1}\Delta \tilde{W}_{t_{n+1}}^*] + \Delta t_n \mathbb{E}_{t_n}^{X^n}[f^{n+1}\Delta \tilde{W}_{t_{n+1}}^*], \tag{3.1b}$$

with

$$X^{n+1} = X^n + \phi(t_n, X^n, \Delta t_n, \Delta W_{t_{n+1}}, \xi^{n+1}). \tag{3.2}$$

Note that (3.1a) and (3.1b) use the trapezoidal rule for approximating time integrals. We also would like to point out that (3.2) represents any classical numerical schemes (see [16]) for solving the forward SDE (2.3a) in which ξ^{n+1} is a Gaussian random variable related to $\Delta W_{t_{n+1}}$, and X^{n+1} is needed in calculations of (3.1a) and (3.1b).

Remark 3.1. Several facts on Scheme 3.1 are given below:

1. The accuracy of Scheme 3.1 depends not only on the accuracy of the discretizations (3.1a) and (3.1b) for solving the BSDE in (2.1) but also on the accuracy of the discretization (3.2) for solving the SDE in (2.1).

2. The computation of $\Delta W_{t_{n+1}}$ and $\Delta \tilde{W}_{t_{n+1}}$ can be simulated by

$$\begin{cases} \Delta W_{t_{n+1}} = \sqrt{\Delta t_n} N(0,1), \\ \Delta \tilde{W}_{t_{n+1}} = \frac{\sqrt{\Delta t_n}}{2}\left(N(0,1) \pm \sqrt{3}\tilde{N}(0,1)\right), \end{cases} \tag{3.3}$$

   where $N(0,1)$ and $\tilde{N}(0,1)$ are two independent random variables with normal distribution.

3. Scheme 3.1 is an implicit for solving Y^n, but is always explicit for solving Z^n. When $f = f(t,x,y,z)$ is Lipschitz continuous with respect to y, there exists unique solution (X^n, Y^n, Z^n) of Scheme 3.1 for small time partition step Δt.

4 Error estimates

Let us denote by $\tilde{Y}_{t_{n+1}}^{t_n,X^n}$ and $\tilde{Z}_{t_{n+1}}^{t_n,X^n}$ the approximate values of $Y_{t_{n+1}}^{t_n,X^n}$ and $Z_{t_{n+1}}^{t_n,X^n}$ at the time-space (t_{n+1}, X^{n+1}), respectively, where X^{n+1} is the approximate solution of $X_{t_{n+1}}^{t_n,X^n}$ calculated by (3.2), that is $\tilde{Y}_{t_{n+1}}^{t_n,X^n} = Y_{t_{n+1}}^{t_{n+1},X^{n+1}}$ and $\tilde{Z}_{t_{n+1}}^{t_n,X^n} = Z_{t_{n+1}}^{t_{n+1},X^{n+1}}$. To simplify the presentation, in the sequel, we let

$$\tilde{f}_{t_{n+1}}^{t_n,X^n} = f(t_{n+1}, X^{n+1}, \tilde{Y}_{t_{n+1}}^{t_n,X^n}, \tilde{Z}_{t_{n+1}}^{t_n,X^n}),$$

and denote $Y_{t_n}^{t_n,X^n} - Y^n$ by e_y^n, $Z_{t_n}^{t_n,X^n} - Z^n$ by e_z^n, $\tilde{Y}_{t_{n+1}}^{t_n,X^n} - Y^{n+1}$ by e_y^{n+1}, $\tilde{Z}_{t_{n+1}}^{t_n,X^n} - Z^{n+1}$ by e_z^{n+1}, and $\tilde{f}_{t_{n+1}}^{t_n,X^n} - f^{n+1}$ by e_f^{n+1}.

4.1 A useful theorem

We now present an important theorem that will be useful in our error estimates of Scheme 3.1.

Theorem 4.1. *Let (X_t, Y_t, Z_t), $t \in [0,T]$ and (X^n, Y^n, Z^n), $n = 0, 1, \cdots, N$, be the exact solution of the decoupled FBSDEs (1.1) and the approximate solution obtained by Scheme 3.1, respectively. Assume that the function $f(t, X, Y, Z)$ is Lipschitz continuous with respect to X, Y and Z and the Lipschitz constant is L. Let c_0 be the time partition regularity parameter defined in (2.2). Then for sufficiently small time step Δt_n, it holds that*

$$\mathbb{E}[|e_y^n|^2] + \Delta t \sum_{i=n}^{N-1} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \mathbb{E}[|e_z^i|^2]$$

$$\leq C' \left(\mathbb{E}[|e_y^N|^2] + \Delta t \mathbb{E}[|e_z^N|^2] \right)$$

$$+ \sum_{i=n}^{N-1} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \frac{C\mathbb{E}[|R_{y_1}^i|^2 + (\Delta t)^2 |R_{y_2}^i|^2 + |R_y^i|^2]}{\Delta t(1-C\Delta t)}$$

$$+ \sum_{i=n}^{N-1} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \frac{C\Delta t \mathbb{E}[(\frac{1}{\Delta t_n})^2 |R_{z_1}^i|^2 + |R_{z_2}^i|^2 + (\frac{1}{\Delta t_n})^2 |R_z^i|^2]}{1-C\Delta t} \tag{4.1}$$

for $n = N-1, \cdots, 1, 0$, where C is a positive constant depending on c_0 and L, C' is also a positive constants depending on c_0, T and L, R_y^i and R_z^i are defined in (2.6) and (2.15), and

$$R_{y_1}^n = \mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n} - \tilde{Y}_{t_{n+1}}^{t_n,X^n}], \tag{4.2a}$$

$$R_{y_2}^n = \mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n} - \tilde{f}_{t_{n+1}}^{t_n,X^n}], \tag{4.2b}$$

$$R_{z_1}^n = \mathbb{E}_{t_n}^{X^n}[(Y_{t_{n+1}}^{t_n,X^n} - \tilde{Y}_{t_{n+1}}^{t_n,X^n})\Delta \tilde{W}_{t_{n+1}}^*], \tag{4.2c}$$

$$R_{z_2}^n = \mathbb{E}_{t_n}^{X^n}[(f_{t_{n+1}}^{t_n,X^n} - \tilde{f}_{t_{n+1}}^{t_n,X^n})\Delta \tilde{W}_{t_{n+1}}^*]. \tag{4.2d}$$

Proof. For each integer n such that $0 \leq n \leq N-1$, subtracting (3.1a) from (2.7) gives us

$$
\begin{aligned}
e_y^n &= \mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n} - Y^{n+1}] + \frac{1}{2}\Delta t_n(f_{t_n}^{t_n,X^n} - f^n) + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n} - f^{n+1}] + R_y^n \\
&= \mathbb{E}_{t_n}^{X^n}[Y_{t_{n+1}}^{t_n,X^n} - \tilde{Y}_{t_{n+1}}^{t_n,X^n} + \tilde{Y}_{t_{n+1}}^{t_n,X^n} - Y^{n+1}] + \frac{\Delta t_n}{2}(f_{t_n}^{t_n,X^n} - f^n) \\
&\quad + \frac{\Delta t_n}{2}\mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n} - \tilde{f}_{t_{n+1}}^{t_n,X^n} + \tilde{f}_{t_{n+1}}^{t_n,X^n} - f^{n+1}] + R_y^n \\
&= \mathbb{E}_{t_n}^{X^n}[e_y^{n+1}] + \frac{\Delta t_n}{2}e_f^n + \frac{\Delta t_n}{2}\mathbb{E}_{t_n}^{X^n}[e_f^{n+1}] + R_{y_1}^n + \frac{\Delta t_n}{2}R_{y_2}^n + R_y^n.
\end{aligned}
\tag{4.3}
$$

Then we have

$$
\begin{aligned}
|e_y^n| &\leq |\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]| + \frac{\Delta t_n}{2}|e_f^n| + \frac{\Delta t_n}{2}|\mathbb{E}_{t_n}^{X^n}[e_f^{n+1}]| + |R_{y_1}^n| + \frac{\Delta t_n}{2}|R_{y_2}^n| + |R_y^n| \\
&\leq |\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]| + \frac{\Delta t_n}{2}L(|e_y^n| + |e_z^n|) + \frac{\Delta t_n}{2}L\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}| + |e_z^{n+1}|] \\
&\quad + |R_{y_1}^n| + \frac{\Delta t_n}{2}|R_{y_2}^n| + |R_y^n|.
\end{aligned}
\tag{4.4}
$$

Using the inequalities

$$
(a+b)^2 \leq (1+\gamma\Delta t)a^2 + \left(1 + \frac{1}{\gamma\Delta t}\right)b^2, \qquad \left(\sum_{n=1}^m a_n\right)^2 \leq m\sum_{n=1}^m a_n^2
$$

for any positive real number γ and positive integer m, we deduce

$$
\begin{aligned}
|e_y^n|^2 &\leq (1+\gamma\Delta t)|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2 + \left(1 + \frac{1}{\gamma\Delta t}\right)\left\{\frac{\Delta t_n}{2}L(|e_y^n| + |e_z^n|)\right. \\
&\quad \left. + \frac{\Delta t_n}{2}L\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}| + |e_z^{n+1}|] + |R_{y_1}^n| + \frac{\Delta t_n}{2}|R_{y_2}^n| + |R_y^n|\right\}^2 \\
&\leq (1+\gamma\Delta t)|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2 + 5\left(1 + \frac{1}{\gamma\Delta t}\right)\left\{\left(\frac{\Delta t_n L}{2}\right)^2(|e_y^n| + |e_z^n|)^2\right. \\
&\quad \left. + \left(\frac{\Delta t_n L}{2}\right)^2(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}| + |e_z^{n+1}|])^2 + |R_{y_1}^n|^2 + \left(\frac{\Delta t_n}{2}\right)^2|R_{y_2}^n|^2 + |R_y^n|^2\right\} \\
&\leq (1+\gamma\Delta t)|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2 + \left\{\frac{5\Delta t^2 L^2}{2}(|e_y^n|^2 + |e_z^n|^2)\right. \\
&\quad \left. + \frac{5\Delta t^2 L^2}{2}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2 + |e_z^{n+1}|^2] + 5|R_{y_1}^n|^2 + \frac{5\Delta t^2}{4}|R_{y_2}^n|^2 + 5|R_y^n|^2\right\} \\
&\quad + \frac{1}{\gamma}\left\{\frac{5\Delta t L^2}{2}(|e_y^n|^2 + |e_z^n|^2) + \frac{5\Delta t L^2}{2}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2 + |e_z^{n+1}|^2]\right\} \\
&\quad + \frac{1}{\gamma\Delta t}\left\{5|R_{y_1}^n|^2 + \frac{5\Delta t^2}{4}|R_{y_2}^n|^2 + 5|R_y^n|^2\right\}.
\end{aligned}
\tag{4.5}
$$

By (2.14) and (3.1b) we have

$$\frac{\Delta t_n}{2}e_z^n = \mathbb{E}_{t_n}^{X^n}[(Y_{t_{n+1}}^{t_n,X^n}-Y^{n+1})\Delta\tilde{W}_{t_{n+1}}^*] + \Delta t_n\mathbb{E}_{t_n}^{X^n}[(f_{t_{n+1}}^{t_n,X^n}-f^{n+1})\Delta\tilde{W}_{t_{n+1}}^*] + R_z^n. \tag{4.6}$$

Inserting

$$\mathbb{E}_{t_n}^{X^n}[(Y_{t_{n+1}}^{t_n,X^n}-Y^{n+1})\Delta\tilde{W}_{t_{n+1}}^*] = \mathbb{E}_{t_n}^{X^n}[(Y_{t_{n+1}}^{t_n,X^n}-\tilde{Y}_{t_{n+1}}^{t_n,X^n})\Delta\tilde{W}_{t_{n+1}}^*] + \mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{t_{n+1}}^*]$$
$$= R_{z_1}^n + \mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{t_{n+1}}^*].$$

and

$$\mathbb{E}_{t_n}^{X^n}[(f_{t_{n+1}}^{t_n,X^n}-f^{n+1})\Delta\tilde{W}_{t_{n+1}}^*] = \mathbb{E}_{t_n}^{X^n}[(f_{t_{n+1}}^{t_n,X^n}-\tilde{f}_{t_{n+1}}^{t_n,X^n}+\tilde{f}_{t_{n+1}}^{t_n,X^n}-f^{n+1})\Delta\tilde{W}_{t_{n+1}}^*]$$
$$= R_{z_2}^n + \mathbb{E}_{t_n}^{X^n}[e_f^{n+1}\Delta\tilde{W}_{t_{n+1}}^*]$$

into (4.6), we obtain

$$e_z^n = \frac{2}{\Delta t_n}\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{t_{n+1}}^*] + 2\mathbb{E}_{t_n}^{X^n}[e_f^{n+1}\Delta\tilde{W}_{t_{n+1}}^*] + \frac{2}{\Delta t_n}R_{z_1}^n + 2R_{z_2}^n + \frac{2}{\Delta t_n}R_z^n,$$

and consequently we have the estimate

$$|e_z^n| \le \frac{2}{\Delta t_n}|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{t_{n+1}}^*]| + 2|\mathbb{E}_{t_n}^{X^n}[e_f^{n+1}\Delta\tilde{W}_{t_{n+1}}^*]| + \frac{2}{\Delta t_n}|R_{z_1}^n| + 2|R_{z_2}^n| + \frac{2}{\Delta t_n}|R_z^n|. \tag{4.7}$$

By using Holder's inequality and the inequality $(a+b)^2 \le (1+\varepsilon)a^2 + (1+\frac{1}{\varepsilon})b^2$ for any positive real number ε, we obtain the following inequality from (4.7)

$$|e_z^n|^2 \le (1+\varepsilon)\left(\frac{2}{\Delta t_n}\right)^2|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{t_{n+1}}^*]|^2 + \left(1+\frac{1}{\varepsilon}\right)\left\{2|\mathbb{E}_{t_n}^{X^n}[e_f^{n+1}\Delta\tilde{W}_{t_{n+1}}^*]|\right.$$
$$\left. + \frac{2}{\Delta t_n}|R_{z_1}^n| + 2|R_{z_2}^n| + \frac{2}{\Delta t_n}|R_z^n|\right\}^2$$
$$\le (1+\varepsilon)\left(\frac{2}{\Delta t_n}\right)^2|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{t_{n+1}}^*]|^2 + 16\left(1+\frac{1}{\varepsilon}\right)\left\{\mathbb{E}_{t_n}^{X^n}[|e_f^{n+1}|^2]\mathbb{E}_{t_n}^{X^n}[|\Delta\tilde{W}_{t_{n+1}}^*|^2]\right.$$
$$\left. + \left(\frac{1}{\Delta t_n}\right)^2|R_{z_1}^n|^2 + |R_{z_2}^n|^2 + \left(\frac{1}{\Delta t_n}\right)^2|R_z^n|^2\right\}. \tag{4.8}$$

Furthermore, applying

$$\mathbb{E}_{t_n}^{X^n}[|\Delta\tilde{W}_{t_{n+1}}|^2] = \Delta t_n,$$
$$\mathbb{E}_{t_n}^{X^n}[|e_f^{n+1}|^2] \le \mathbb{E}_{t_n}^{X^n}[|L(|e_y^{n+1}|+|e_z^{n+1}|)|^2] \le 2L^2\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2+|e_z^{n+1}|^2],$$

and

$$|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{t_{n+1}}^*]|^2 = |\mathbb{E}_{t_n}^{X^n}[(e_y^{n+1}-\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}])\Delta\tilde{W}_{t_{n+1}}^*]|^2$$
$$\le \mathbb{E}_{t_n}^{X^n}[|\Delta\tilde{W}_{t_{n+1}}^*|^2]\mathbb{E}_{t_n}^{X^n}[(e_y^{n+1}-\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}])^2]$$
$$= \Delta t_n(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2]-|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2)$$

into (4.8), we obtain

$$|e_z^n|^2 \le (1+\varepsilon)\frac{4}{\Delta t_n}(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2] - |\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2)$$

$$+32\left(1+\frac{1}{\varepsilon}\right)L^2\Delta t_n\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2 + |e_z^{n+1}|^2]$$

$$+16(1+\frac{1}{\varepsilon})\left\{\left(\frac{1}{\Delta t_n}\right)^2|R_{z_1}^2| + |R_{z_2}^n| + \left(\frac{1}{\Delta t_n}\right)^2|R_z^n|^2\right\}. \tag{4.9}$$

Dividing both sides of the inequality (4.9) by $(1+\varepsilon)\frac{4}{\Delta t}$, we obtain

$$\frac{\Delta t}{4(1+\varepsilon)}|e_z^n|^2 \le \frac{\Delta t}{\Delta t_n}(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2] - |\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2) + \frac{8L^2}{\varepsilon}\Delta t_n\Delta t\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2$$

$$+|e_z^{n+1}|^2] + \frac{4\Delta t}{\varepsilon}\left\{\frac{1}{(\Delta t_n)^2}|R_{z_1}^n|^2 + |R_{z_2}^n|^2 + \frac{1}{(\Delta t_n)^2}|R_z^n|^2\right\}$$

$$\le c_0(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2] - |\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2) + \frac{8L^2}{\varepsilon}\Delta t^2\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2$$

$$+|e_z^{n+1}|^2] + \frac{4\Delta t}{\varepsilon}\left\{\frac{1}{(\Delta t_n)^2}|R_{z_1}^n|^2 + |R_{z_2}^n|^2 + \frac{1}{(\Delta t_n)^2}|R_z^n|^2\right\}. \tag{4.10}$$

Then multiply the inequality (4.5) by c_0 and add the derived inequality to the inequality (4.10), we get

$$c_0|e_y^n|^2 + \frac{\Delta t}{4(1+\varepsilon)}|e_z^n|^2 \le c_0(1+\gamma\Delta t)|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2 + 5c_0\left\{\frac{L^2\Delta t^2}{2}(|e_y^n|^2 + |e_z^n|^2)\right.$$

$$+\frac{L^2\Delta t^2}{2}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2 + |e_z^{n+1}|^2] + |R_{y_1}^n|^2 + \frac{\Delta t^2}{4}|R_{y_2}^n|^2 + |R_y^n|^2\right\}$$

$$+\frac{c_0}{\gamma}\left\{\frac{5L^2\Delta t}{2}(|e_y^n|^2 + |e_z^n|^2) + \frac{5L^2\Delta t}{2}\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2 + |e_z^{n+1}|^2]\right\}$$

$$+\frac{c_0}{\gamma\Delta t}\left\{5|R_{y_1}^n|^2 + \frac{5}{4}\Delta t^2|R_{y_2}^n|^2 + 5|R_y^n|^2\right\}$$

$$+c_0(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2] - |\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]|^2) + \frac{8L^2}{\varepsilon}\Delta t^2\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2 + |e_z^{n+1}|^2]$$

$$+\frac{4\Delta t}{\varepsilon}\left\{\frac{1}{(\Delta t_n)^2}|R_{z_1}^n|^2 + |R_{z_2}^n|^2 + \frac{1}{(\Delta t_n)^2}|R_z^n|^2\right\}$$

$$\le c_0\left(1+\left(\gamma+\frac{5L^2}{2\gamma}+\frac{5L^2\Delta t}{2}+\frac{8L^2\Delta t}{c_0\varepsilon}\right)\Delta t\right)\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|^2]$$

$$+\left(\frac{5c_0}{2\gamma}+\left(\frac{5c_0}{2}+\frac{8}{\varepsilon}\right)\Delta t\right)L^2\Delta t\mathbb{E}_{t_n}^{X^n}[|e_z^{n+1}|^2]$$

$$+\left(\frac{5c_0}{2\gamma}+\frac{5c_0\Delta t}{2}\right)L^2\Delta t(|e_y^n| + |e_z^n|^2)$$

$$+5c_0\Big(1+\frac{1}{\gamma\Delta t}\Big)\Big\{|R_{y_1}^n|^2+\frac{1}{4}\Delta t^2|R_{y_2}^n|^2+|R_y^n|^2\Big\}$$

$$+\frac{4\Delta t}{\varepsilon}\Big\{\frac{1}{(\Delta t_n)^2}|R_{z_1}^n|^2+|R_{z_2}^n|^2+\frac{1}{(\Delta t_n)^2}|R_z^n|^2\Big\},$$

which can be further simplified to

$$c_0[1-C_1\Delta t]\mathbb{E}[|e_y^n|^2]+C_3\Delta t\mathbb{E}[|e_z^n|^2]$$

$$\leq c_0[1+C_2\Delta t]\mathbb{E}[|e_y^{n+1}|^2]+C_4\Delta t\mathbb{E}[|e_z^{n+1}|^2]+\frac{C_5}{\Delta t}\mathbb{E}[|R_{y_1}^n|^2+\frac{1}{4}\Delta t^2|R_{y_2}^n|^2+|R_y^n|^2]$$

$$+\frac{4\Delta t}{\varepsilon}\mathbb{E}\Big[\Big(\frac{1}{\Delta t_n}\Big)^2|R_{z_1}^n|^2+|R_{z_2}^n|^2+\Big(\frac{1}{\Delta t_n}\Big)^2|R_z^n|^2\Big], \tag{4.11}$$

where

$$C_1=\Big(\frac{5}{2\gamma}+\frac{5\Delta t}{2}\Big)L^2,\quad C_2=\Big(\gamma+\frac{5L^2}{2\gamma}+\frac{5L^2\Delta t}{2}+\frac{8L^2\Delta t}{c_0\varepsilon}\Big),$$

$$C_3=\frac{1}{4(1+\varepsilon)}-\Big(\frac{5c_0}{2\gamma}+\frac{5c_0\Delta t}{2}\Big)L^2,$$

$$C_4=\Big[\frac{5c_0}{2\gamma}+\Big(\frac{5c_0}{2}+\frac{8}{\varepsilon}\Big)\Delta t\Big]L^2,\quad C_5=5c_0\frac{1+\gamma\Delta t}{\gamma}.$$

Now we choose $\varepsilon=1$, γ large enough, and Δt_0 sufficiently small, such that if $0<\Delta t\leq\Delta t_0$ then $C_1\leq C$, $C_2\leq C$, $C_5\leq C$, $1-C\Delta t>0$, and $C_3-C_4>C^*>0$, where C and C^* are two positive constants depending on c_0 and L. Then for $0<\Delta t\leq\Delta t_0$, we obtain from (4.11)

$$c_0(1-C\Delta t)\mathbb{E}[|e_y^n|^2]+C_3\Delta t\mathbb{E}[|e_z^n|^2]$$

$$\leq c_0(1+C\Delta t)\mathbb{E}[|e_y^{n+1}|^2]+C_4\Delta t\mathbb{E}[|e_z^{n+1}|^2]$$

$$+\frac{C\mathbb{E}[|R_{y_1}^n|^2+\frac{1}{4}(\Delta t)^2|R_{y_2}^n|^2+|R_y^n|^2]}{\Delta t}$$

$$+4\Delta t\mathbb{E}\Big[\Big(\frac{1}{\Delta t_n}\Big)^2|R_{z_1}^n|^2+|R_{z_2}^n|^2+\Big(\frac{1}{\Delta t_n}\Big)^2|R_z^n|^2\Big]. \tag{4.12}$$

Dividing both sides of the inequality (4.12) by $(1-C\Delta t)$, we deduce

$$c_0\mathbb{E}[|e_y^n|^2]+C_3\Delta t\mathbb{E}[|e_z^n|^2]\leq\frac{1+C\Delta t}{1-C\Delta t}\Big(c_0\mathbb{E}[|e_y^{n+1}|^2]+C_4\Delta t\mathbb{E}[|e_z^{n+1}|^2]\Big)$$

$$+\frac{C\mathbb{E}[|R_{y_1}^n|^2+\frac{1}{4}(\Delta t)^2|R_{y_2}^n|^2+|R_y^n|^2]}{\Delta t(1-C\Delta t)}$$

$$+\frac{4\Delta t\mathbb{E}[(\frac{1}{\Delta t_n})^2|R_{z_1}^n|^2+|R_{z_2}^n|^2+(\frac{1}{\Delta t_n})^2|R_z^n|^2]}{1-C\Delta t}. \tag{4.13}$$

From the inequality (4.13), by recursively inserting e_y^i, $i = n+1, \cdots, N-1$, we deduce

$$c_0 \mathbb{E}[|e_y^n|^2] + C_3 \Delta t \sum_{i=n}^{N-1} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \mathbb{E}[|e_z^i|^2]$$

$$\leq \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{N-n} c_0 \mathbb{E}[|e_y^N|^2] + C_4 \Delta t \sum_{i=n+1}^{N} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \mathbb{E}[|e_z^i|^2]$$

$$+ \sum_{i=n}^{N-1} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \frac{C\mathbb{E}[|R_{y_1}^i|^2 + \frac{1}{4}(\Delta t)^2 |R_{y_2}^i|^2 + |R_y^i|^2]}{\Delta t(1-C\Delta t)}$$

$$+ \sum_{i=n}^{N-1} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \frac{4\Delta t \mathbb{E}[(\frac{1}{\Delta t_n})^2 |R_{z_1}^i|^2 + |R_{z_2}^i|^2 + (\frac{1}{\Delta t_n})^2 |R_z^i|^2]}{1-C\Delta t},$$

that is,

$$c_0 \mathbb{E}[|e_y^n|^2] + C^* \Delta t \sum_{i=n}^{N-1} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \mathbb{E}[|e_z^i|^2]$$

$$\leq \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{N-n} c_0 \mathbb{E}[|e_y^N|^2] + C_4 \Delta t \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{N-n} \mathbb{E}[|e_z^N|^2]$$

$$+ \sum_{i=n}^{N-1} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \frac{C\mathbb{E}[|R_{y_1}^i|^2 + (\Delta t)^2 |R_{y_2}^i|^2 + |R_y^i|^2]}{\Delta t(1-C\Delta t)}$$

$$+ \sum_{i=n}^{N-1} \left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n} \frac{C\Delta t \mathbb{E}[(\frac{1}{\Delta t_n})^2 |R_{z_1}^i|^2 + |R_{z_2}^i|^2 + (\frac{1}{\Delta t_n})^2 |R_z^i|^2]}{1-C\Delta t},$$

which leads to the inequality (4.1). The proof is completed. $\qquad\square$

Remark 4.1. Theorem 4.1 implies that Scheme 3.1 is stable, and its solution continuously depends on terminal condition, that is, for any given positive number ϵ, there exists a positive integer δ, for different terminal conditions $(\overline{Y}^N, \overline{Z}^N)$ and (Y^N, Z^N), if $\mathbb{E}[|\overline{Y}^N - Y^N|^2] < \delta$ and $\mathbb{E}[|\overline{Z}^N - Z^N|^2] < \delta$, then for $0 \leq n \leq N-1$, we have

$$\mathbb{E}[|\overline{Y}^n - Y^n|^2] + \Delta t \sum_{i=n}^{N} \mathbb{E}[|\overline{Z}^i - Z^i|^2] < \epsilon.$$

Remark 4.2. The terms R_y^n and R_z^n in (2.7) and (2.14) are the truncated error terms for solving Y_t and Z_t in the BSDE in (2.1) by the discretizations (3.1a) and (3.1b) in Scheme 3.1. The four terms $R_{y_1}^i$, $R_{y_2}^i$, $R_{z_1}^i$ and $R_{z_2}^i$ are determined by the discretization (3.2) for solving the SDE in (2.1). These four terms reflect the weak errors of the scheme for solving SDE. Under certain regularity conditions on b, σ, f and φ, as long as the estimates of R_y^n, R_z^n, $R_{y_1}^i$, $R_{y_2}^i$, $R_{z_1}^i$ and $R_{z_2}^i$ are obtained, then it is easy to get error estimates by Theorem 4.1 for Scheme 3.1.

4.2 Error estimates

In this subsection, under some regularity conditions on the functions b, σ, f and φ, we will derive estimates for the error terms R_y^n, R_z^n, $R_{y_1}^i$, $R_{y_2}^i$, $R_{z_1}^i$ and $R_{z_2}^i$, and then present the main result on error estimates of Scheme 3.1. We first need the following assumption.

Assumption 4.1. Assume X_{t_0} is $\mathcal{A}_{t_0}$-measurable with $\mathbb{E}(|X_{t_0}|^2) < \infty$. We also assume that b and σ are jointly L^2-measurable in $(t,x) \in [t_0,T] \times \mathbb{R}^d$, and are uniformly Lipschitz continuous and linear growth bounded, that is, there exists a constant $L > 0$ and $K > 0$ such that

$$|b(t,x)-b(t,y)| \leq L|x-y|, \quad |\sigma(t,x)-\sigma(t,y)| \leq L|x-y|, \tag{4.14a}$$

$$|b(t,x)|^2 \leq K(1+|x|^2), \quad\quad |\sigma(t,x)|^2 \leq K(1+|x|^2), \tag{4.14b}$$

for all $t,s \in [0,T]$ and $x,y \in \mathbb{R}^q$.

We also need regularity of the exact solution (Y_t, Z_t) of the decoupled FBSDEs (1.1). Let us introduce the following lemma.

Lemma 4.1. ([10,13,15,22,26]) *Let the functions b, σ, f and φ be uniformly Lipschitz continuous w.r.t. (X,Y,Z) and Hölder continuous of parameter $\frac{1}{2}$ w.r.t. t. We also assume φ is of class $C_b^{2+\alpha}$ for some $\alpha \in (0,1)$ and the matrix-valued function $a = \sigma\sigma^*$ is uniformly elliptic. Then it is well-known that the solution (Y_t, Z_t) of (1.1) can be represented as: $Y_t = u(t,X_t)$ and $Z_t = \nabla_x u(t,X_t)\sigma(t,X_t)$, where $u(t,x)$ is the smooth solution of the following PDEs*

$$(\partial_t + \mathcal{L}_{t,x})u(t,x) + f(t,x,u(t,x),\nabla_x u(t,x)\sigma(t,x)) = 0,$$

with the terminal condition $u(T,x) = \varphi(x)$, where $\mathcal{L}$ is the second order differential operator defined by

$$\mathcal{L}_{t,x} = \frac{1}{2}\sum_{i,j}[\sigma\sigma^*]_{ij}(t,x)\partial_{x_i x_j}^2 + \sum_i b_i(t,x)\partial_{x_i}.$$

Furthermore, for $k = 0,1,2,\cdots$, if $b,\sigma \in C_b^{1+k,2+2k}$, $f \in C_b^{1+k,2+2k,2+2k,2+2k}$ and $\varphi \in C_b^{2+2k+\alpha}$ for some $\alpha \in (0,1)$, then $u \in C_b^{1+k,2+2k}$.

The accuracy of Scheme 3.1 obviously also depends on the accuracy of (3.2) for solving the forward SDE in (2.1).

Assumption 4.2. We assume that the approximation solution X^{n+1} solved by (3.2) has the stability property: for positive integer r, there exists a constant $C \in (0,\infty)$ such that

$$\max_{0 \leq n \leq N} \mathbb{E}[|X^n|^r] \leq C(1 + \mathbb{E}[|X_0|^r]), \tag{4.15}$$

and the approximation properties: there exist positive numbers r_1, r_2, β, γ such that for any $g \in C_p^{2\beta+2}$, $n = 0, 1, 2, \cdots, N$,

$$|\mathbb{E}_{t_n}^{X^n}[g(X_{t_{n+1}}^{t_n, X^n}) - g(X^{n+1})]| \le C_g (1 + |X^n|^{2r_1})(\Delta t)^{\beta+1}, \tag{4.16a}$$

$$|\mathbb{E}_{t_n}^{X^n}[(g(X_{t_{n+1}}^{t_n, X^n}) - g(X^{n+1}))\Delta \tilde{W}_{t_{n+1}}]| \le C_g (1 + |X^n|^{2r_2})(\Delta t)^{\gamma+1}, \tag{4.16b}$$

$$|\mathbb{E}[g(X_{t_n}) - g(X^n)]| \le C_g (\Delta t)^\beta, \tag{4.16c}$$

where $C_g > 0$ is a constant which does not depend on Δt. The number $\beta + 1$ will be called the local order of the approximation.

In fact, many numerical schemes for forward stochastic differential equations, such as the Euler scheme, the Milstein scheme, and the Itô-Taylor type weak or strong schemes of order 1.5 or 2 (see [16]) have the approximation properties (4.16a), (4.16b) and (4.16c) with $\beta = 1$ or $\beta = 2$, and the stability property (4.15). Under Assumption 4.1, if $\mathbb{E}(|X_0|^{2m}) < \infty$ for some integer $m \ge 1$, the solution of (2.3a) also has the estimate

$$\mathbb{E}_{t_n}^{X^n}(|X_s^{t_n, X^n}|^{2m}) \le (1 + \mathbb{E}_{t_n}^{X^n}(|X^n|^{2m}))e^{C(s-t_n)}, \tag{4.17}$$

for any $s \in [t_n, T]$, where C is a positive constant depending only on the constants K, L and m.

In the following lemmas, we will present estimations for R_y^i, R_z^i, $R_{y_1}^n$, $R_{y_2}^n$, $R_{z_1}^n$ and $R_{z_2}^n$ under certain regularity conditions on b, σ, f and φ. For the sake of presentation simplicity, we only consider the case $q = d = 1$, and the results obtained also hold true for general positive integers q and d. In the sequel, we also let L^0 and L^1 be two differential operators defined by

$$L^1 = \sigma \partial_x, \qquad L^0 = \partial_t + b\partial_x + \frac{1}{2}\sigma^2 \partial_{xx}^2.$$

Lemma 4.2. *If $f(t,x,y,z) \in C_b^{2,4,4,4}$, $b(t,x), \sigma(t,x) \in C_b^{2,4}$, $\varphi \in C_b^{4+\alpha}, \alpha \in (0,1)$ and $|b(t,x)|^2 \le K(1+|x|^2), |\sigma(t,x)|^2 \le K(1+|x|^2)$, then for sufficiently small time step Δt_n, we have that for any $0 \le n \le N-1$,*

$$\mathbb{E}[|R_y^n|^2] \le C(1 + \mathbb{E}[|X^n|^8])(\Delta t)^6, \tag{4.18}$$

where C is a positive constant depending only on T, K, and upper bounds of the derivatives of b, σ, f and φ.

Proof. Lemma 4.1 tells us that the solution (Y_t, Z_t) of FBSDEs (2.1) can be represented as $Y_t = u(t, X_t)$ and $Z_t = \nabla_x u(t, X_t)\sigma(t, X_t)$, and if $f(t,x,y,z) \in C_b^{2,4,4,4}$, $b(t,x), \sigma(t,x) \in C_b^{2,4}, \varphi \in C_b^{4+\alpha}, \alpha \in (0,1)$, then $f(t, X_t, u(t, X_t), \nabla_x u(t, X_t)\sigma(t, X_t)) = F(t, X_t) \in C_b^{2,4}$. We denote $F(t, X_t^{t_n, X^n})$ by $F_t^{t_n, X^n}$. From the reference equation (2.5), we have

$$R_y^n = \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^{X^n}[F_s^{t_n, X^n}] - \frac{1}{2}F_{t_n}^{t_n, X^n} - \frac{1}{2}\mathbb{E}_{t_n}^{X^n}[F_{t_{n+1}}^{t_n, X^n}] \right) ds. \tag{4.19}$$

By the Itô-Taylor expansion, we deduce

$$
\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[F_s^{t_n,X^n}]ds
$$

$$
= \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[F_{t_n}^{t_n,X^n} + \int_{t_n}^{s} L^0 F_r^{t_n,X^n}dr + \int_{t_n}^{s} L^1 F_r^{t_n,X^n}dW_r]ds
$$

$$
= F_{t_n}^{t_n,X^n} \int_{t_n}^{t_{n+1}} ds + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^{X^n}[L^0 F_r^{t_n,X^n}]drds
$$

$$
= \Delta t_n F_{t_n}^{t_n,X^n} + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^{X^n}[L^0 F_{t_n}^{t_n,X^n} + \int_{t_n}^{r} L^0 L^0 F_z^{t_n,X^n}dz]drds
$$

$$
= \Delta t_n F_{t_n}^{t_n,X^n} + \frac{1}{2}(\Delta t_n)^2 L^0 F_{t_n}^{t_n,X^n} + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} \mathbb{E}_{t_n}^{X^n}[L^0 L^0 F_z^{t_n,X^n}]dzdrds, \qquad (4.20)
$$

and similarly

$$
\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[F_{t_{n+1}}^{t_n,X^n}]ds = \Delta t_n F_{t_n}^{t_n,X^n} + (\Delta t_n)^2 L^0 F_{t_n}^{t_n,X^n}
$$

$$
+ \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{r} \mathbb{E}_{t_n}^{X^n}[L^0 L^0 F_z^{t_n,X^n}]dzdrds. \qquad (4.21)
$$

By (4.19), (4.20) and (4.21), we obtain

$$
R_y^n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} \mathbb{E}_{t_n}^{X^n}[L^0 L^0 F_z^{t_n,X^n}]dzdrds - \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{r} \mathbb{E}_{t_n}^{X^n}[L^0 L^0 F_z^{t_n,X^n}]dzdrds. \quad (4.22)
$$

Square both sides of the above equation and take mathematical expectation, we get the inequality

$$
\mathbb{E}[|R_y^n|^2] \le 2\mathbb{E}\left[\left| \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} \mathbb{E}_{t_n}^{X^n}[L^0 L^0 F_z^{t_n,X^n}]dzdrds \right|^2 \right]
$$

$$
+ 2\mathbb{E}\left[\frac{1}{4} \left| \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{r} \mathbb{E}_{t_n}^{X^n}[L^0 L^0 F_z^{t_n,X^n}]dzdrds \right|^2 \right]. \qquad (4.23)
$$

From the definition of the operator L^0, we have

$$
L^0 L^0 F = F_{tt}'' + bF_{tx}'' + \frac{1}{2}\sigma^2 F_{txx}''' + b_t' F_x' + bF_{xt}'' + bb_x' F_x' + b^2 F_{xx}'' + \frac{1}{2}\sigma^2 b_{xx}'' F_x'
$$

$$
+ \frac{1}{2}\sigma^2 b F_{xxx}''' + \sigma \sigma_t' F_{xx}'' + \frac{1}{2}\sigma^2 F_{xxt}''' + b\sigma\sigma_x' F_{xx}'' + \frac{1}{2}b\sigma^2 F_{xxx}'''
$$

$$
+ \frac{1}{2}\sigma^2 (\sigma_x')^2 + \frac{1}{2}\sigma^3 \sigma_{xx}'' F_{xx}'' + \sigma^3 \sigma_x' F_{xxx}''' + \frac{1}{4}\sigma^4 F_{xxxx}^{(4)}.
$$

Thus by applying the Cauchy-Schwarz inequality to (4.23) and using (4.17), we have

$$
\begin{aligned}
\mathbb{E}[|R_y^n|^2] &\leq \frac{1}{3}(\Delta t_n)^3 \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} \mathbb{E}[|L^0 L^0 F_z^{t_n,X^n}|^2] dz dr ds \\
&\quad + \frac{1}{4}(\Delta t_n)^3 \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{r} \mathbb{E}[|L^0 L^0 F_z^{t_n,X^n}|^2] dz dr ds \\
&\leq \frac{1}{3}(\Delta t_n)^3 \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} \mathbb{E}[C(1+|X_z^{t_n,X^n}|^8)] dz dr ds \\
&\quad + \frac{1}{4}(\Delta t_n)^3 \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{r} \mathbb{E}[C(1+|X_z^{t_n,X^n}|^8)] dz dr ds \\
&\leq C(1+\mathbb{E}[|X^n|^8])(\Delta t_n)^6.
\end{aligned}
$$

The proof is complete. $\qquad\square$

Lemma 4.3. *Assume that the conditions of Lemma 4.2 hold, then for sufficiently small time step Δt_n, we have that for any $0 \leq n \leq N-1$,*

$$
\left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[f_s^{t_n,X^n} \Delta \tilde{W}_{t_{n+1}}] ds - \Delta t_n \mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n} \Delta \tilde{W}_{t_{n+1}}] \right|^2 \leq C(1+\mathbb{E}_{t_n}^{X^n}[|X^n|^8])(\Delta t_n)^6, \quad (4.24)
$$

where C is a positive constant depending only on T, K, and the upper bounds of the derivatives of b, σ, f and φ.

Proof. Under the condition of the lemma, it is easy to verify that $F(t,X_t) \in C_b^{2,4}$. By the Itô-Taylor expansion, we have

$$
\begin{aligned}
&\left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[F_s^{t_n,X^n} \Delta \tilde{W}_{t_{n+1}}] ds - \Delta t_n \mathbb{E}_{t_n}^{X^n}[F_{t_{n+1}}^{t_n,X^n} \Delta \tilde{W}_{t_{n+1}}] \right| \\
&= \left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}\left[\left(\int_{t_n}^{s} L^0 F_z^{t_n,X^n} dz + \int_{t_n}^{s} L^1 F_z^{t_n,X^n} dW_z \right) \Delta \tilde{W}_{t_{n+1}} \right] ds \right. \\
&\quad \left. - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}\left[\left(\int_{t_n}^{t_{n+1}} L^0 F_z^{t_n,X^n} dz + \int_{t_n}^{t_{n+1}} L^1 F_z^{t_n,X^n} dW_z \right) \Delta \tilde{W}_{t_{n+1}} \right] ds \right| \\
&\leq A_1 + A_2, \quad (4.25)
\end{aligned}
$$

where

$$
A_1 = \left| \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^{X^n}\left[\int_{t_n}^{s} L^0 F_z^{t_n,X^n} dz \Delta \tilde{W}_{t_{n+1}} \right] - \mathbb{E}_{t_n}^{X^n}\left[\int_{t_n}^{t_{n+1}} L^0 F_z^{t_n,X^n} dz \Delta \tilde{W}_{t_{n+1}} \right] \right) ds \right|,
$$

$$
A_2 = \left| \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^{X^n}\left[\int_{t_n}^{s} L^1 F_z^{t_n,X^n} dW_z \Delta \tilde{W}_{t_{n+1}} - \int_{t_n}^{t_{n+1}} L^1 F_z^{t_n,X^n} dW_z \Delta \tilde{W}_{t_{n+1}} \right] \right) ds \right|.
$$

Again by the Itô-Taylor expansion, we deduce

$$
\begin{aligned}
A_1 = &\left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}\left[\int_{t_n}^{s} \left(\int_{t_n}^{z} L^0 L^0 F_r^{t_n,X^n} dr + \int_{t_n}^{z} L^1 L^0 F_r^{t_n,X^n} dW_r \right) dz \Delta \tilde{W}_{t_{n+1}} \right] ds \right. \\
&\left. - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}\left[\int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{z} L^0 L^0 F_r^{t_n,X^n} dr + \int_{t_n}^{z} L^1 L^0 F_r^{t_n,X^n} dW_r \right) dz \Delta \tilde{W}_{t_{n+1}} \right] ds \right|.
\end{aligned}
$$

Notice that

$$
\left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{s} \int_{t_n}^{z} L^0 L^0 F_r^{t_n,X^n} dr dz - \int_{t_n}^{t_{n+1}} \int_{t_n}^{z} L^0 L^0 F_r^{t_n,X^n} dr dz \right) \Delta \tilde{W}_{t_{n+1}} \right] ds \right|
$$

$$
\leq \sqrt{\Delta t_n} \left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{s} \int_{t_n}^{z} L^0 L^0 F_r^{t_n,X^n} dr dz - \int_{t_n}^{t_{n+1}} \int_{t_n}^{z} L^0 L^0 F_r^{t_n,X^n} dr dz \right)^2 \right]^{\frac{1}{2}} ds \right|
$$

$$
\leq 2\Delta t_n \left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{s} \int_{t_n}^{z} L^0 L^0 F_r^{t_n,X^n} dr dz \right)^2 + \left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{z} L^0 L^0 F_r^{t_n,X^n} dr dz \right)^2 \right] ds \right|^{\frac{1}{2}}
$$

$$
\leq C (1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^8])^{\frac{1}{2}} (\Delta t_n)^{\frac{7}{2}},
$$

and by the Itô's isometry formula, we have

$$
\left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{s} \int_{t_n}^{z} L^1 L^0 F_r dW_r dz - \int_{t_n}^{t_{n+1}} \int_{t_n}^{z} L^1 L^0 F_r dW_r dz \right) \Delta \tilde{W}_{t_{n+1}} \right] ds \right|
$$

$$
\leq \left| \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{z} \mathbb{E}_{t_n}^{X^n} [2 L^1 L^0 F_r] dr dz ds \right|
$$

$$
+ \left| \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{z} \mathbb{E}_{t_n}^{X^n} \left[\frac{3}{\Delta t_n} L^1 L^0 F_r (r - t_n) \right] dr dz ds \right|
$$

$$
+ \left| \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{z} \mathbb{E}_{t_n}^{X^n} [2 L^1 L^0 F_r] dr dz ds \right|
$$

$$
+ \left| \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{z} \mathbb{E}_{t_n}^{X^n} \left[\frac{3}{\Delta t_n} L^1 L^0 F_r (r - t_n) \right] dr dz ds \right|
$$

$$
\leq C (1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^3]) (\Delta t_n)^3.
$$

The above two inequalities lead to the estimate of A_1 as

$$
A_1 \leq C (1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^8])^{\frac{1}{2}} (\Delta t_n)^{\frac{7}{2}} + C (1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^3]) (\Delta t_n)^3.
$$

Now we turn into estimate A_2. Using the identity

$$
\left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{s} dW_z \Delta \tilde{W}_{t_{n+1}} - \int_{t_n}^{t_{n+1}} dW_z \Delta \tilde{W}_{t_{n+1}} \right] ds \right|
$$

$$
= \left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{s} dW_z \left(2 \Delta W_{t_{n+1}} - \frac{3}{\Delta t_n} \int_{t_n}^{t_{n+1}} (z - t_n) dW_z \right) \right] \right.
$$

$$
\left. - \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} dW_z \left(2 \Delta W_{t_{n+1}} - \frac{3}{\Delta t_n} \int_{t_n}^{t_{n+1}} (z - t_n) dW_z \right) \right] \right|
$$

$$
= \left| \int_{t_n}^{t_{n+1}} \left(\left(\int_{t_n}^{s} 2 dz - \frac{3}{\Delta t_n} \int_{t_n}^{s} (z - t_n) dz \right) - \left(\int_{t_n}^{t_{n+1}} 2 dz - \frac{3}{\Delta t_n} \int_{t_n}^{t_{n+1}} (z - t_n) dz \right) \right) \right|
$$

$$
= 0
$$

and the Itô expansion again, we obtain

$$
\begin{aligned}
A_2 = & \left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{s} (L^1 F_{t_n}^{t_n,X^n} + \int_{t_n}^{z} L^0 L^1 F_r^{t_n,X^n} dr + \int_{t_n}^{z} L^1 L^1 F_r^{t_n,X^n} dW_r) dW_z \right) \Delta \tilde{W}_{t_{n+1}} \right] ds \right. \\
& \left. - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{t_{n+1}} (L^1 F_{t_n}^{t_n,X^n} + \int_{t_n}^{z} L^0 L^1 F_r^{t_n,X^n} dr + \int_{t_n}^{z} L^1 L^1 F_r^{t_n,X^n} dW_r) dW_z \right) \Delta \tilde{W}_{t_{n+1}} \right] ds \right| \\
= & \left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{s} \int_{t_n}^{z} L^0 L^1 F_r^{t_n,X^n} dr dW_z \Delta \tilde{W}_{t_{n+1}} \right] ds \right. \\
& \left. - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{z} L^0 L^1 F_r^{t_n,X^n} dr dW_z \Delta \tilde{W}_{t_{n+1}} \right] ds \right| \\
\leq & \, C(1 + \mathbb{E}_{t_n}^{X^n}[|X^n|^3])(\Delta t_n)^3.
\end{aligned}
$$

By (4.25) and the estimates of A_1 and A_2, we complete the proof. $\square$

Lemma 4.4. *Assume the conditions of Lemma 4.2 hold, then for sufficiently small time step Δt_n, we have that for any $0 \leq n \leq N-1$,*

$$
\left| \frac{1}{2} \Delta t_n Z_{t_n}^{t_n,X^n} - \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{t_{n+1}} Z_s^{t_n,X^n} dW_s \right) \Delta \tilde{W}_{t_{n+1}} \right] \right|^2 \leq C(1 + \mathbb{E}_{t_n}^{X^n}[|X^n|^8])(\Delta t_n)^6, \qquad (4.26)
$$

where C is a positive constant depending only on T, K, and upper bounds of the derivatives of b, σ, f and φ.

Proof. From the definition of $\Delta \tilde{W}_s$, we have

$$
\begin{aligned}
& \left| \frac{1}{2} \Delta t_n Z_{t_n}^{t_n,X^n} - \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} Z_s^{t_n,X^n} dW_s \cdot \Delta \tilde{W}_{t_{n+1}} \right] \right| \\
& = \left| \frac{1}{2} \Delta t_n Z_{t_n}^{t_n,X^n} - 2 \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[Z_s^{t_n,X^n}] ds + \frac{3}{\Delta t_n} \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[Z_s^{t_n,X^n}(s-t_n)] ds \right|.
\end{aligned}
$$

By the Itô-Taylor expansion, we obtain

$$
\begin{aligned}
& \left| \frac{1}{2} \Delta t_n Z_{t_n}^{t_n,X^n} - \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} Z_s^{t_n,X^n} dW_s \cdot \Delta \tilde{W}_{t_{n+1}} \right] \right| \\
& = \left| \frac{1}{2} Z_{t_n}^{t_n,X^n} \Delta t_n - 2 \left(\Delta t_n Z_{t_n}^{t_n,X^n} + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^{X^n}[L^0 Z_r^{t_n,X^n}] dr ds \right) \right. \\
& \qquad \left. + \frac{3}{\Delta t_n} \left(\frac{1}{2} (\Delta t_n)^2 Z_{t_n}^{t_n,X^n} + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^{X^n}[L^0 Z_r^{t_n,X^n}(s-t_n)] dr ds \right) \right| \\
& = \left| 2 \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^{X^n}[L^0 Z_r^{t_n,X^n}] dr ds - \frac{3}{\Delta t_n} \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^{X^n}[L^0 Z_r^{t_n,X^n}(s-t_n)] dr ds \right|.
\end{aligned}
$$

Use the Itô-Taylor expansion again we then get

$$
\begin{aligned}
&\left| \frac{1}{2} \Delta t_n Z_{t_n}^{t_n, X^n} - \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} Z_s^{t_n, X^n} dW_s \cdot \Delta \tilde{W}_{t_{n+1}} \right] \right| \\
&= \left| -2 \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^{X^n} \left[L^0 Z_{t_n}^{t_n, X^n} + \int_{t_n}^{r} L^0 L^0 Z_z^{t_n, X^n} dz \right] dr ds \right. \\
&\quad \left. + \frac{3}{\Delta t_n} \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^{X^n} \left[L^0 Z_{t_n}^{t_n, X^n}(s-t_n) + \int_{t_n}^{r} L^0 L^0 Z_z^{t_n, X^n}(s-t_n) dz \right] dr ds \right| \\
&= \left| \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} \left(2 \mathbb{E}_{t_n}^{X^n} [L^0 L^0 Z_z^{t_n, X^n}] - \frac{3}{\Delta t_n} \mathbb{E}_{t_n}^{X^n} [L^0 L^0 Z_z^{t_n, X^n}(s-t_n)] \right) dz dr ds \right| \\
&\leq C (1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^4]) (\Delta t_n)^3.
\end{aligned}
$$

The proof is completed. $\square$

Lemma 4.5. *Assume the conditions of Lemma 4.2 hold, then for sufficiently small time step Δt_n, we have that for any $0 \leq n \leq N-1$,*

$$
|R_z^n|^2 \leq C (1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^8]) (\Delta t)^6,
$$

where C is a positive constant depending only on T and K, and upper bounds of the derivatives of b, σ, f and φ.

Proof. From (2.14), $R_z^n = R_1^n + R_2^n$ with R_1^n and R_2^n defined in (2.10) and (2.12) respectively. It is easy to show that the lemma is the direct consequence of Lemmas 4.3 and 4.4. $\square$

Theorem 4.2. *Assume Assumption 4.2 and the conditions of Lemma 4.2 holds. Then for sufficiently small time step Δt_n, we have that for any $0 \leq n \leq N-1$,*

$$
\mathbb{E}[|e_y^n|^2] + \Delta t \sum_{i=n}^{N-1} \mathbb{E}[|e_z^i|^2] \leq C_1 (\mathbb{E}[|e_y^N|^2] + \Delta t \mathbb{E}[|e_z^N|^2]) + C_2 (\Delta t^{2\beta} + \Delta t^{2\gamma} + \Delta t^4), \tag{4.27}
$$

where C_1 is a positive constant depending on c_0, T and L, C_2 is also a positive constant depending on c_0, T, L, K, the initial value of X_t in (1.1), and the upper bounds of the derivatives of b, σ, f and φ.

Proof. From the definitions of $R_{y_1}^i$, $R_{y_2}^i$, $R_{z_1}^i$ and $R_{y_2}^i$ in Theorem 4.1, if Assumption 4.2 holds, under the conditions of Lemma 4.2 we have the following estimates

$$
\mathbb{E}[|X^i|^2] \leq C (1 + \mathbb{E}[|X_0|^2]), \tag{4.28a}
$$

$$
\mathbb{E}[|R_{y_1}^i|^2] \leq C (1 + \mathbb{E}[|X^i|^{4r_1}])(\Delta t)^{2\beta+2} \leq C (1 + \mathbb{E}[|X_0|^{4r_1}])(\Delta t)^{2\beta+2}, \tag{4.28b}
$$

$$
\mathbb{E}[|R_{y_2}^i|^2] \leq C (1 + \mathbb{E}[|X^i|^{4r_1}])(\Delta t)^{2\beta+2} \leq C (1 + \mathbb{E}[|X_0|^{4r_1}])(\Delta t)^{2\beta+2}, \tag{4.28c}
$$

$$
\mathbb{E}[|R_{z_1}^i|^2] \leq C (1 + \mathbb{E}[|X^i|^{4r_2}])(\Delta t)^{2\gamma+2} \leq C (1 + \mathbb{E}[|X_0|^{4r_2}])(\Delta t)^{2\gamma+2}, \tag{4.28d}
$$

$$
\mathbb{E}[|R_{z_2}^i|^2] \leq C (1 + \mathbb{E}[|X^i|^{4r_2}])(\Delta t)^{2\gamma+2} \leq C (1 + \mathbb{E}[|X_0|^{4r_2}])(\Delta t)^{2\gamma+2} \tag{4.28e}
$$

for $i=0,1,\cdots,N-1$. By Lemmas 4.2 and 4.5, we have that for $0\leq i\leq N-1$,

$$\mathbb{E}[|R_y^i|^2]\leq C(1+\mathbb{E}[|X_0|^8])(\Delta t)^6, \qquad \mathbb{E}[|R_z^i|^2]\leq C(1+\mathbb{E}[|X_0|^8])(\Delta t)^6. \qquad (4.29)$$

By (4.28) and (4.29), we deduce

$$\sum_{i=n}^{N-1}\left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n}\frac{C\mathbb{E}[|R_{y_1}^i|^2+(\Delta t)^2|R_{y_2}^i|^2+|R_y^i|^2]}{\Delta t(1-C\Delta t)}$$
$$\leq C(1+\mathbb{E}[|X_0|^{4r_1}]+\mathbb{E}[|X_0|^8])((\Delta t)^{2\beta}+(\Delta t)^4), \qquad (4.30)$$

and

$$\sum_{i=n}^{N-1}\left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n}\frac{C\Delta t\mathbb{E}[(\frac{1}{\Delta t_n})^2|R_{z_1}^i|^2+|R_{z_2}^i|^2+(\frac{1}{\Delta t_n})^2|R_z^i|^2]}{(1-C\Delta t)}$$
$$\leq C(1+\mathbb{E}[|X_0|^{4r_2}]+\mathbb{E}[|X_0|^8])((\Delta t)^{2\gamma}+(\Delta t)^4). \qquad (4.31)$$

Now by Theorem 4.1, the estimates (4.31) and (**??**), we complete the proof. $\square$

4.3 Classical schemes for solving SDEs

In this subsection we introduce some classical numerical schemes in the form of (3.2) that can be used in Scheme 3.1 for solving the forward SDE (2.1) and identify the validity of Assumption 4.2 for these schemes, and finally conclude the respective error estimates of the Scheme 3.1 for each of the cases.

4.3.1 The Euler scheme

The Euler scheme [16] for solving (2.1) is given by

$$X^{n+1}=X^n+b(t_n,X^n)\Delta t_n+\sigma(t_n,X^n)\Delta W_{t_{n+1}}. \qquad (4.32)$$

By the Itô-Taylor expansion, the exact solution X_t of (2.1) at $t=t_{n+1}$ satisfies

$$X_{t_{n+1}}=X_{t_n}+b(t_n,X_{t_n})\Delta t_n+\sigma(t_n,X_{t_n})\Delta W_{t_{n+1}}+R_x^n, \qquad (4.33)$$

where

$$R_x^n=\int_{t_n}^{t_{n+1}}\int_{t_n}^s L^0 b(r,X_r)drds+\int_{t_n}^{t_{n+1}}\int_{t_n}^s L^1 b(r,X_r)dW_r ds$$
$$+\int_{t_n}^{t_{n+1}}\int_{t_n}^s L^0\sigma(r,X_r)drdW_s+\int_{t_n}^{t_{n+1}}\int_{t_n}^s L^1\sigma(r,X_r)dW_r dW_s.$$

Thus we easily have

$$|\mathbb{E}_{t_n}^{X^n}[X_{t_{n+1}}^{t_n,X^n}-X^{n+1}]|=|\mathbb{E}_{t_n}^{X^n}[R_x^n]|\leq C(\Delta t_n)^2, \qquad (4.34)$$

and

$$\left| \mathbb{E}_{t_n}^{X^n} \left[(X_{t_{n+1}}^{t_n, X^n} - X^{n+1}) \Delta \tilde{W}_{t_{n+1}} \right] \right|$$

$$= \left| \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 b(r, X_r^{t_n, X^n}) dr ds + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 b(r, X_r^{t_n, X^n}) dW_r ds \right. \right. $$

$$\left. \left. + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 \sigma(r, X_r^{t_n, X^n}) dr dW_s + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 \sigma(r, X_r^{t_n, X^n}) dW_r dW_s \right) \Delta \tilde{W}_{t_{n+1}} \right] \right|$$

$$\leq A_1 + A_2 + A_3 + A_4, \tag{4.35}$$

where

$$A_1 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 b(r, X_r^{t_n, X^n}) dr ds \Delta \tilde{W}_{t_{n+1}} \right] \right|,$$

$$A_2 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 b(r, X_r^{t_n, X^n}) dW_r ds \Delta \tilde{W}_{t_{n+1}} \right] \right|,$$

$$A_3 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 \sigma(r, X_r^{t_n, X^n}) dr dW_s \Delta \tilde{W}_{t_{n+1}} \right] \right|,$$

$$A_4 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 \sigma(r, X_r^{t_n, X^n}) dW_r dW_s \Delta \tilde{W}_{t_{n+1}} \right] \right|.$$

For A_i $(i=1,2,3,4)$, the following estimates hold

$$A_1 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 b(r, X_r^{t_n, X^n}) dr ds \Delta \tilde{W}_{t_{n+1}} \right] \right|$$

$$\leq \left| \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 b(r, X_r^{t_n, X^n}) dr ds \right)^2 \right]^{\frac{1}{2}} \mathbb{E}_{t_n}^{X^n} [(\Delta \tilde{W}_{t_{n+1}})^2]^{\frac{1}{2}} \right|$$

$$\leq \sqrt{\Delta t_n} \left| \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dr ds \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^{X^n} [(L^0 b(r, X_r^{t_n, X^n}))^2] dr ds \right|^{\frac{1}{2}}$$

$$\leq C (1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^4])^{\frac{1}{2}} (\Delta t_n)^{\frac{5}{2}},$$

$$A_2 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 b(r, X_r^{t_n, X^n}) dW_r ds \Delta \tilde{W}_{t_{n+1}} \right] \right|$$

$$= \left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{s} L^1 b(r, X_r^{t_n, X^n}) dW_r \Delta \tilde{W}_s \right] ds \right|$$

$$\leq C (1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^2])^{\frac{1}{2}} (\Delta t_n)^2,$$

$$A_3 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 \sigma(r, X_r^{t_n, X^n}) dr dW_s \Delta \tilde{W}_{t_{n+1}} \right] \right|$$

$$\leq C (1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^2]) (\Delta t_n)^2,$$

$$A_4 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 \sigma(r, X_r^{t_n, X^n}) dW_r dW_s \Delta \tilde{W}_{t_{n+1}} \right] \right| = 0.$$

By (4.34), (4.35) and the estimates of A_i $(i=1,2,3,4)$, we obtain

$$\left|\mathbb{E}_{t_n}^{X^n}[(X_{t_{n+1}}^{t_n,X^n}-X^{n+1})\Delta\tilde{W}_{t_{n+1}}]\right|\leq C(1+\mathbb{E}_{t_n}^{X^n}[|X^n|^4])^{\frac{1}{2}}(\Delta t_n)^2. \tag{4.36}$$

Thus we conclude that Assumption 4.2 holds true for the Euler scheme with $\beta=\gamma=1$. Furthermore, if $\mathbb{E}[|Y_{t_N}-Y^N|^2]\leq C(\Delta t)^2$, $\mathbb{E}[|Z_{t_N}-Z^N|^2]\leq C(\Delta t)^2$ and the Euler scheme is used in (3.2), then by Theorem 4.2 we obtain the error estimate of Scheme 3.1 as

$$\mathbb{E}[|e_y^n|^2]+\Delta t\sum_{i=n}^{N-1}\mathbb{E}[|e_z^i|^2]\leq C(\Delta t)^2. \tag{4.37}$$

The estimate (4.37) implies Scheme 3.1 for solving the decouple FBSDEs is a first-order accurate method when the Euler scheme is used for solution of the forward SDE.

4.3.2   The Milstein scheme

The Milstein scheme [16] is given by

$$\begin{aligned}X^{n+1}=X^n&+b(t_n,X^n)\Delta t_n+\sigma(t_n,X^n)\Delta W_{t_{n+1}}\\&+\frac{1}{2}\sigma\sigma'(t_n,X^n)((\Delta W_{t_{n+1}})^2-\Delta t_n),\end{aligned} \tag{4.38}$$

which is obtained by the Itô-Taylor expansion

$$\begin{aligned}X_{t_{n+1}}=X_{t_n}&+b(t_n,X_{t_n})\Delta t_n+\sigma(t_n,X_{t_n})\Delta W_{t_{n+1}}\\&+\frac{1}{2}\sigma\sigma'(t_n,X_{t_n})((\Delta W_{t_{n+1}})^2-\Delta t_n)+R_x^n,\end{aligned} \tag{4.39}$$

where

$$\begin{aligned}R_x^n=&\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}L^0b(r,X_r^{t_n,X^n})drds+\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}L^1b(r,X_r^{t_n,X^n})dW_rds\\&+\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}L^0\sigma(r,X_r^{t_n,X^n})drdW_s+\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r}L^0L^1\sigma(z,X_z^{t_n,X^n})dzdW_rdW_s\\&+\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{r}L^1L^1\sigma(z,X_z^{t_n,X^n})dW_zdW_rdW_s.\end{aligned} \tag{4.40}$$

It is easy to check that

$$\left|\mathbb{E}_{t_n}^{X^n}[X_{t_{n+1}}^{t_n,X^n}-X^{n+1}]\right|=\left|\mathbb{E}_{t_n}^{X^n}[R_x^n]\right|\leq C(\Delta t_n)^2, \tag{4.41}$$

and

$$
\begin{aligned}
&\left| \mathbb{E}_{t_n}^{X^n} \left[(X_{t_{n+1}}^{t_n,X^n} - X^{n+1}) \Delta \tilde{W}_{t_{n+1}} \right] \right| \\
&= \left| \mathbb{E}_{t_n}^{X^n} \left[\left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 b(r, X_r^{t_n,X^n}) dr ds + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 b(r, X_r^{t_n,X^n}) dW_r ds \right. \right. \\
&\quad + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 \sigma(r, X_r^{t_n,X^n}) dr dW_s + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^0 L^1 \sigma(z, X_z^{t_n,X^n}) dz dW_r dW_s \\
&\quad \left. \left. + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^1 L^1 \sigma(z, X_z^{t_n,X^n}) dW_z dW_r dW_s \right) \Delta \tilde{W}_{t_{n+1}} \right] \right| \\
&\leq A_1 + A_2 + A_3 + B_1 + B_2,
\end{aligned}
\tag{4.42}
$$

where A_1, A_2 and A_3 are the same as those defined in Section 4.3.1, and

$$
B_1 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^0 L^1 \sigma(z, X_z^{t_n,X^n}) dz dW_r dW_s \Delta \tilde{W}_{t_{n+1}} \right] \right|, \tag{4.43a}
$$

$$
B_2 = \left| \mathbb{E}_{t_n}^{X^n} \left[\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^1 L^1 \sigma(z, X_z^{t_n,X^n}) dW_z dW_r dW_s \Delta \tilde{W}_{t_{n+1}} \right] \right|. \tag{4.43b}
$$

From the adapted properties of the solution of SDEs and of the Itô's integral, we easily get

$$
B_1 = B_2 = 0.
$$

By (4.41), (4.42), and the estimate of A_1, A_2, and A_3, we get

$$
\left| \mathbb{E}_{t_n}^{X^n} \left[(X_{t_{n+1}}^{t_n,X^n} - X^{n+1}) \Delta \tilde{W}_{t_{n+1}} \right] \right| \leq C \left(1 + \mathbb{E}_{t_n}^{X^n} [|X^n|^4] \right)^{\frac{1}{2}} (\Delta t_n)^2. \tag{4.44}
$$

Thus from (4.41) and (4.44), we conclude that Assumption 4.2 also holds true for the Milstein scheme with $\beta = \gamma = 1$. Furthermore, if $\mathbb{E}[|Y_{t_N} - Y^N|^2] \leq C(\Delta t)^2$, $\mathbb{E}[|Z_{t_N} - Z^N|^2] \leq C(\Delta t)^2$ and the Milstein scheme is used in (3.2), then by Theorem 4.2 we obtain the error estimate of Scheme 3.1 as

$$
\mathbb{E}[|e_y^n|^2] + \Delta t \sum_{i=n}^{N-1} \mathbb{E}[|e_z^i|^2] \leq C(\Delta t)^2. \tag{4.45}
$$

The estimate (4.45) implies Scheme 3.1 for solving the decouple FBSDEs is a first-order accurate method when the Milstein scheme is used for solution of the forward SDE.

4.3.3 The order-2.0 weak Taylor scheme

The order-2.0 weak Taylor scheme [16] is given by

$$
\begin{aligned}
X^{n+1} = X^n &+ b(t_n, X^n) \Delta t_n + \sigma(t_n, X^n) \Delta W_{t_{n+1}} \\
&+ L^0 b(t_n, X^n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dr ds + L^1 b(t_n, X^n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_r ds \\
&+ L^0 \sigma(t_n, X^n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dr dW_s + L^1 \sigma(t_n, X^n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_r dW_s
\end{aligned}
\tag{4.46}
$$

which is obtained by the Itô-Taylor expansion

$$X_{t_{n+1}} = X_{t_n} + b(t_n, X_{t_n}) \Delta t_n + \sigma(t_n, X_{t_n}) \Delta W_{t_{n+1}}$$
$$+ L^0 b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dr ds + L^1 b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_r ds$$
$$+ L^0 \sigma(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dr dW_s + L^1 \sigma(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_r dW_s + R_x^n, \qquad (4.47)$$

where $R_x^n = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8$ with

$$D_1 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^0 L^0 b(z, X_z^{t_n, X^n}) dz dr ds,$$

$$D_2 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^1 L^0 b(z, X_z^{t_n, X^n}) dW_z dr ds,$$

$$D_3 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^0 L^1 b(z, X_z^{t_n, X^n}) dz dW_r ds,$$

$$D_4 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^1 L^1 b(z, X_z^{t_n, X^n}) dW_z dW_r ds,$$

$$D_5 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^0 L^0 \sigma(z, X_z^{t_n, X^n}) dz dr dW_s,$$

$$D_6 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^1 L^0 \sigma(z, X_z^{t_n, X^n}) dW_z dr dW_s,$$

$$D_7 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^0 L^1 \sigma(z, X_z^{t_n, X^n}) dz dW_r dW_s,$$

$$D_8 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{r} L^1 L^1 \sigma(z, X_z^{t_n, X^n}) dW_z dW_r dW_s.$$

It is easy to get

$$|\mathbb{E}_{t_n}^{X^n}[X_{t_{n+1}}^{t_n, X^n} - X^{n+1}]| = |\mathbb{E}_{t_n}^{X^n}[R_x^n]| \leq C(\Delta t_n)^3. \qquad (4.48)$$

Using the similar analysis as that for the Euler scheme in Section 4.3.1 and the Milstein scheme in Section 4.3.2, we can obtain

$$|\mathbb{E}_{t_n}^{X^n}[D_1 \Delta \tilde{W}_{t_{n+1}}]| \leq C(1 + \mathbb{E}_{t_n}^{X^n}[|X^n|^8])^{\frac{1}{2}} (\Delta t_n)^{\frac{7}{2}},$$
$$|\mathbb{E}_{t_n}^{X^n}[D_2 \Delta \tilde{W}_{t_{n+1}}]| \leq C(1 + \mathbb{E}_{t_n}^{X^n}[|X^n|^3])(\Delta t_n)^3,$$
$$|\mathbb{E}_{t_n}^{X^n}[D_3 \Delta \tilde{W}_{t_{n+1}}]| \leq C(1 + \mathbb{E}_{t_n}^{X^n}[|X^n|^3])(\Delta t_n)^3,$$
$$|\mathbb{E}_{t_n}^{X^n}[D_5 \Delta \tilde{W}_{t_{n+1}}]| \leq C(1 + \mathbb{E}_{t_n}^{X^n}[|X^n|^8])^{\frac{1}{2}} (\Delta t_n)^3,$$

and

$$|\mathbb{E}_{t_n}^{X^n}[D_4 \Delta \tilde{W}_{t_{n+1}}]| = 0, \quad |\mathbb{E}_{t_n}^{X^n}[D_6 \Delta \tilde{W}_{t_{n+1}}]| = 0,$$
$$|\mathbb{E}_{t_n}^{X^n}[D_7 \Delta \tilde{W}_{t_{n+1}}]| = 0, \quad |\mathbb{E}_{t_n}^{X^n}[D_8 \Delta \tilde{W}_{t_{n+1}}]| = 0.$$

Thus we also have

$$|\mathbb{E}_{t_n}^{X^n}[(X_{t_{n+1}}^{t_n,X^n}-X^{n+1})\Delta\tilde{W}_{t_{n+1}}]|\leq C(1+\mathbb{E}_{t_n}^{X^n}[|X^n|^8])^{\frac{1}{2}}(\Delta t_n)^3. \tag{4.49}$$

Now from (4.48) and (4.49), we conclude that Assumption 4.2 also holds true for the order-2.0 weak Taylor scheme with $\beta=\gamma=2$. Furthermore, if $\mathbb{E}[|Y_{t_N}-Y^N|^2]\leq C(\Delta t)^4$, $\mathbb{E}[|Z_{t_N}-Z^N|^2]\leq C(\Delta t)^4$ and the order-2.0 weak Taylor scheme is used in (3.2), then by Theorem 4.2 we obtain the error estimate of Scheme 3.1 as

$$\mathbb{E}[|e_y^n|^2]+\Delta t\sum_{i=n}^{N-1}\mathbb{E}[|e_z^i|^2]\leq C(\Delta t)^4. \tag{4.50}$$

The estimate (4.50) implies Scheme 3.1 for solving the decouple FBSDEs is a second-order accurate method when the order-2.0 weak Taylor scheme is used for solution of the forward SDE.

5 Numerical experiments

In this section, some numerical tests will be performed to demonstrate the effectiveness and accuracy of the proposed method — Scheme 3.1 for solving the decoupled FBSDEs (1.1) and verify the above theoretical results. We will show that the convergence order of Scheme 3.1 depends on the numerical method for solving the forward SDE as shown in our theoretical analysis (Theorem 4.2, (4.37), (4.45) and (4.50)) although the BSDE of (1.1) is solved by a second-order accurate scheme.

We here consider one-dimensional problems. In order to use Scheme 3.1, space partition and approximation of $\mathbb{E}_{t_n}^{X^n}[\cdot]$ at discrete space grid point x_i are needed. In our numerical experiments, with the spatial step size h, the discrete grid points are $x_i=ih$, $i=0,\pm1,\pm2,\cdots$. In the calculations of the conditional mathematical expectation $\mathbb{E}_{t_n}^{X^n}[\cdot]$, the Gauss-Hermite quadrature rule is used, and the values of the integrands of the conditional mathematical expectations at non-grid points are approximated by local cubic interpolations. Since our goal is to test the accuracy of the scheme with respect to the time step size, we set the number of the Gauss-Hermite quadrature points to be big enough so that the error contributed by spacial approximation is very small and will affect the convergence rate just very little. For simplicity, we take a uniform partition with a time step size Δt. Then the time partition number N is given by $N=\frac{T}{\Delta t}$, where T is the terminal time.

Let $|Y_0-Y^0|$ and $|Z_0-Z^0|$ represent the errors between the exact solution (Y_t,Z_t) of (2.3a) at time $t=0$ and the solution (Y^n,Z^n) of Scheme 3.1 at $n=0$. The convergence rate (CR) with respect to time step Δt is obtained by using linear least square fitting to the errors. The time step sizes used in our experiments are $N=\frac{1}{2^i}$ $(i=4,\cdots,8)$.

Example 5.1. In this example, we test our scheme for the decoupled FBSDEs which contain a linear BSDE. The considered decoupled FBSDEs are (written in the differential

form)

$$\begin{cases} dX_t = \sin(2X_t)dt + t\cos X_t dW_t, \\ -dY_t = \left(-\cos X_t - \dfrac{2\sin X_t Z_t}{t} + t^2\cos^2 X_t \left(2Y_t - \dfrac{3}{2}t\cos X_t \right) \right) dt - Z_t dW_t. \end{cases} \quad (5.1)$$

The terminal condition is chosen to be $Y_T = \sin(2X_T) + T\cos X_T$. Then the exact solution of (5.1) is

$$\begin{cases} Y_t = \sin(2X_t) + t\cos X_t, \\ Z_t = t\cos X_t(2\cos(2X_t) - t\sin X_t). \end{cases} \quad (5.2)$$

Let the Brownian motion X_t start at the time-space point $(0,1)$ (i.e. $X_0 = 1$), then the exact solution at $t = 0$ is $(Y_0, Z_0) = (\sin 2, 0)$. The errors $|Y_0 - Y^0|$ and $|Z_0 - Z^0|$ and their convergence rates are listed in Table 1, which clearly match our theoretical results (4.37), (4.45) and (4.50) very well.

Table 1: Errors and convergence rates of Scheme 3.1 in Example 5.1.

SDE Scheme	Euler		Milstein		Order-2.0 Weak Taylor	
N	$\|Y_0 - Y^0\|$	$\|Z_0 - Z^0\|$	$\|Y_0 - Y^0\|$	$\|Z_0 - Z^0\|$	$\|Y_0 - Y^0\|$	$\|Z_0 - Z^0\|$
16	3.4329E-01	5.0297E-01	1.1522E-01	8.7098E-02	3.1026E-02	6.0657E-02
32	1.6850E-01	2.3713E-01	6.6925E-02	4.7832E-02	8.0831E-03	1.5890E-02
64	8.3344E-02	1.1508E-01	3.5356E-02	2.8589E-02	2.0542E-03	4.0718E-03
128	4.1438E-02	5.6683E-02	1.8054E-02	1.5667E-02	5.1731E-04	1.0310E-03
256	2.0661E-02	2.8126E-02	9.1381E-03	8.1818E-03	1.2979E-04	2.5946E-04
CR	1.0133	1.0386	0.9203	0.9116	1.9768	1.9684

Example 5.2. In this example, we test our scheme for solving the decoupled FBSDEs which contain a nonlinear BSDE. The decouple FBSDEs are given by

$$\begin{cases} dX_t = \sin(t + X_t)dt + \dfrac{3}{10}\cos(t + X_t)dW_t, \\ -dY_t = \left(\dfrac{3}{20}Y_t Z_t - \cos(t + X_t)(1 + Y_t) \right) dt - Z_t dW_t. \end{cases} \quad (5.3)$$

We choose the terminal condition $Y_T = \sin(T + X_T)$. Then the analytic solution of (5.3) is given by

$$\begin{cases} Y_t = \sin(t + X_t), \\ Z_t = \dfrac{3}{10}\cos^2(t + X_t). \end{cases} \quad (5.4)$$

In this example, we still let $X_0 = 1$, then the exact solution (Y_t, Z_t) at the time $t = 0$ is $(Y_0, Z_0) = (\sin 1, \frac{3}{10}\cos^2 1)$. We report the errors $|Y_0 - Y^0|$ and $|Z_0 - Z^0|$ and their convergence rates in Table 2, which again are consistent with our theoretical results (4.37), (4.45) and (4.50).

Table 2: Errors and convergence rates of Scheme 3.1 in Example 5.2.

SDE Scheme	Euler		Milstein		Order-2.0 Weak Taylor	
N	$|Y_0-Y^0|$	$|Z_0-Z^0|$	$|Y_0-Y^0|$	$|Z_0-Z^0|$	$|Y_0-Y^0|$	$|Z_0-Z^0|$
16	4.5572E-02	6.4819E-03	4.5371E-02	7.5672E-03	8.5929E-04	1.7091E-03
32	2.2241E-02	2.8144E-03	2.2193E-02	3.3407E-03	2.0463E-04	4.0518E-04
64	1.0986E-02	1.3068E-03	1.0980E-02	1.5669E-03	4.9937E-05	9.8233E-05
128	5.4598E-03	6.2924E-04	5.4609E-03	7.5926E-04	1.2335E-05	2.4156E-05
256	2.7216E-03	3.0870E-04	2.7231E-03	3.7358E-04	3.0653E-06	5.9876E-06
CR	1.0158	1.0945	1.0140	1.0818	2.0314	2.0382

6 Conclusions

In this paper, we first propose a new numerical method for solving the decoupled forward-backward stochastic differential equations based on some specially derived reference equations. Then we rigorously analyze errors of the proposed method for general cases. When some classical numerical schemes are applied in the method for solving the forward equation part, we discuss specific error estimates for each of the cases. While many existing numerical methods for the decoupled FBSDEs are of only half-order accuracy in time, we in particular would like to remark that the proposed method is overall 1.0 order accurate to Y_t and Z_t when the Euler scheme or the Milstein scheme is used, and is 2.0 order accurate to Y_t and Z_t when the order-2.0 weak Taylor scheme is used. The numerical experiments are also very consistent with the theoretical results and demonstrate accuracy of the proposed scheme.

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