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# CONVERGENCE ANALYSIS OF FINITE ELEMENT SOLUTION OF ONE-DIMENSIONAL SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS ON EQUIDISTRIBUTING MESHES

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**Abstract.** In this paper convergence on equidistributing meshes is investigated. Equidistributing meshes, or more generally approximate equidistributing meshes, are constructed through the well-known equidistribution principle and a so-called adaptation (or monitor) function which is defined based on estimates on interpolation error for polynomial preserving operators. Detailed convergence analysis is given for finite element solution of singularly perturbed two-point boundary value problems without turning points. Illustrative numerical results are given for a convection-diffusion problem and a reaction-diffusion problem.

**Key Words.** Mesh adaptation, equidistribution, error analysis, finite element method.

## 1. Introduction

The concept of equidistribution has been used for long for adaptive mesh generation. It is first used by Burchard [7] and then by a number of researchers (cf. the early works [2, 13, 15, 24, 27]) for the error analysis of best spline approximations with variable knots. An algorithm, now known as de Boor's algorithm, is introduced by de Boor [14] for computing equidistributed meshes. Russell and Christiansen [31] give an early review on mesh selection strategies based on equidistribution, and one of such strategies is implemented in the general-purpose code COLSYS by Ascher, Christiansen, and Russell [1]. The equidistribution principle has also been playing an important role in multi-dimensional adaptive mesh generation. The concept can naturally be incorporated into the variational mesh generation framework, and a number of methods have been developed along this line, e.g., see [6, 8, 9, 16, 17, 19, 21].

Convergence analysis for the numerical solution of partial differential equations (PDEs) using equidistributing meshes can be traced back to works in the seventies of the last century. For example, Pereyra and Sewell [27] give an asymptotical bound for the truncation error when finite differences are used for solving two-point boundary value problems on an equidistributing mesh. Babuška and Rheinboldt [2] obtain a posteriori error estimates for finite element solutions for one dimensional

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problems in an asymptotic form for the size of elements going to zero. They show that a mesh is asymptotically optimal if all error indicators on subintervals are equal (and thus the mesh is equidistributing). Recent progress has been made on mathematically more rigorous convergence analysis. Notably, Qiu and Sloan [28] and Qiu, Sloan, and Tang [29] investigate the uniform convergence of upwind finite difference approximations to a singularly perturbed problem. Beckett and Mackenzie study the convergence of finite difference approximations to convection-diffusion problems without turning points and reaction-diffusion problems in [3, 4, 23] and finite element approximations to reaction-diffusion problems in [5]. The meshes considered in these works are either chosen a prior or determined through the equidistribution relation based on the explicit expression of the exact solution. The stability and convergence of the finite element solution to one-dimensional convection-dominated problems are studied by Chen and Xu [10] for some a prior chosen meshes. The convergence analysis to a fully discrete problem where meshes are determined completely by the computed solution is recently presented by Kopteva and Stynes [22] for the upwind finite difference discretization of a quasi-linear one-dimensional convection-diffusion problem without turning points.

The objective of this paper is to study the convergence of finite element solution to one-dimensional singularly perturbed PDEs on equidistributing meshes. We attempt to develop a general theory for use in convergence analysis. Our approach is different from those used in the existing works. Specifically, following [18, 20] we first investigate interpolation error for polynomial preserving operators on a general mesh. Several mesh quality measures are defined, and estimates for interpolation error are obtained in terms of these mesh quality measures. An equididistributing mesh which satisfies the equidistribution principle, or more generally, an approximate equididistributing mesh which satisfies the equidistribution principle only approximately, is characterized as a mesh with a bounded overall quality measure (see (16)) as it is refined. The interpolation error estimates are then used to analyze the convergence of the (standard) finite element solution to singularly perturbed PDEs on (approximate) equidistributing meshes. The analysis is carried out for two separate cases, the convection-diffusion case and the reaction-diffusion one. To our best knowledge, this is the first work on the convergence of standard finite element solution of one-dimensional convection-diffusion problems on equidistributing meshes while an analysis for one-dimensional reaction-diffusion problems is given by Beckett and Mackenzie [5]. It is emphasized that unlike the existing approaches, our analysis does not use an a prior chosen mesh nor requires the mesh be given through the equidistribution principle with an analytical expression of the exact solution. What we require is that the mesh satisfies the equidistribution relation approximately, i.e., (16) or (39) and (40). Numerical results presented in Section 4 show that such a mesh can be obtained using De Boor's algorithm [14].

An outline of the paper is as follows. In section 2, we study approximation properties of polynomial preserving operators on a general mesh and define several mesh quality measures. In Section 3, the results of Section 2 are applied to the convergence analysis of the finite element solution to singularly perturbed boundary value problems without turning points. Convection-diffusion and reaction-diffusion equations are covered in Subsections 3.1 and 3.2, respectively. Numerical results are presented in Section 4. Finally, Section 5 contains the conclusions.

Throughout the paper, we use $C$ as a generic constant which may take different values at different occurrences.

## 2. Error estimates for polynomial preserving operators

In this section we first study approximation properties of polynomial preserving operators on a general mesh and then define the mesh quality measures according to the obtained results. Our approach is similar to that used in [18] where mesh quality measures are defined for a more general situation in multi-dimensions.

**2.1. Interpolation error estimates.** We start with introducing some notation. For a general open set $D$ of $\Re$, we denote the norm and semi-norm of the Sobolev space $H^m(D)$ by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$, respectively. The scaled semi-$H^m$ norm will also be used,

$$\langle v\rangle_{m,D} = \left(\frac{1}{|D|}\int_D |v^{(m)}|^2 dx\right)^{1/2} \qquad \forall v \in H^m(D),$$

where $|D|$ is the length of $D$. Note that $\langle v\rangle_{m,D}$ represents the $L^2$ average of $v^{(m)}$. Denote by $P_k(D)$ the space of polynomials of degree no more than $k$ defined on $D$.

Consider a general mesh

$$x_0 = a < x_1 < ... < x_N = b \tag{1}$$

on the interval $\Omega \equiv (a, b)$. Let

$$h_j = x_j - x_{j-1}, \qquad h = \max_j h_j, \qquad I_j = (x_{j-1}, x_j).$$

For a given integer $k \geq 0$, we consider a polynomial preserving operator $\Pi_k$ defined on $H^{k+1}(\Omega)$: for $j = 1, ..., N$,

$$\Pi_k|_{I_j} v = v \qquad \forall v \in P_k(I_j),$$

where $\Pi_k|_{I_j}$ is the restriction of $\Pi_k$ on $I_j$. We assume that $\Pi_k$ is linear and continuous and has the following approximation property: for an integer $m$: $0 \leq m \leq k+1$,

$$\left(\sum_{j=1}^N |v - \Pi_k v|^2_{m,I_j}\right)^{\frac{1}{2}} \leq C\left(\sum_{j=1}^N h_j^{2q+1}\langle V_{k+1}\rangle^2_{I_j}\right)^{\frac{1}{2}} \qquad \forall v \in H^{k+1}(\Omega), \tag{2}$$

where $q = k + 1 - m$, $C$ is a constant independent of $v$ and $N$, and $V_{k+1}$ is a majorizing function for $v^{(k+1)}$, viz.,

$$|v^{(k+1)}(x)| \leq C_1 V_{k+1}(x) \qquad \forall x \in \Omega \tag{3}$$

for some positive constant $C_1$. Obviously, $|v^{(k+1)}|$ is a natural choice for $V_{k+1}$. But it is emphasized that our development also works for other choices of the majorizing function. The approximation property (2) is the key assumption to our analysis.

The assumption (2) holds for most commonly-used polynomial preserving operators. For example, consider a Taylor interpolation operator defined by

$$(\Pi_k v)|_{I_j}(x) = \sum_{i=0}^k v^{(i)}(x_{j-1})(x - x_{j-1})^i \qquad \forall x \in I_j, \quad j = 1, ..., N$$

for $v \in H^{k+1}(\Omega)$. Using Taylor's theorem, one can readily show that (2) holds for all integers $0 \leq m \leq k+1$.

Another important example is the interpolation operator associated with an affine family of finite elements on $\Omega$. Let $\hat{I}$ be the reference finite element, $\hat{P}_{\hat{I}}$ be the space of finite element approximations on $\hat{I}$, and $s$ the greatest order of

derivatives occurring in the definition of $\hat{P}_{\hat{I}}$. It is known (e.g., see [12]) that (2) holds for integers $k \geq 0$ and $0 \leq m \leq k+1$ satisfying

$$H^{k+1}(\hat{I}) \hookrightarrow C^s(\hat{I}) \qquad \text{and} \qquad P_k(\hat{I}) \subset \hat{P}_{\hat{I}} \hookrightarrow H^m(\hat{I}), \tag{4}$$

where $\hookrightarrow$ denotes the inclusion with continuous injection and $C^s(\hat{I})$ is the space of functions $s$ times continuously differentiable on $\hat{I}$. Particularly, for an affine family of Lagrange finite elements, (4) and therefore (2) are satisfied for all integers $k \geq 0$ and $0 \leq m \leq k+1$.

**2.2. The equidistribution principle and mesh quality measures.** We now define the quality measure for mesh adaptation using the approach of [18]. The definition is based on the well-known equidistribution principle [7] which evenly distributes a so-called adaptation (or monitor) function over all the subintervals between the mesh nodes. To be able to define a strictly positive adaptation function, we regularize the right-hand side term of (2) with a to-be-determined positive parameter $\alpha_h$ (which is referred to as the intensity parameter), i.e.,

$$\begin{aligned} \left(\sum_{j=1}^{N} |v - \Pi_k v|^2_{m,I_j}\right)^{\frac{1}{2}} &\leq C \left(\sum_{j=1}^{N} h_j^{2q+1} \left(\alpha_h + \langle V_{k+1}\rangle^2_{I_j}\right)\right)^{\frac{1}{2}} \\ &= C\sqrt{\alpha_h} \left(\sum_{j=1}^{N} h_j^{2q+1} \left(1 + \frac{1}{\alpha_h}\langle V_{k+1}\rangle^2_{I_j}\right)\right)^{\frac{1}{2}}. \end{aligned} \tag{5}$$

Following [18], we define the adaptation function and the intensity parameter $\alpha_h$ as

$$\rho_j = \left(1 + \frac{1}{\alpha_h}\langle V_{k+1}\rangle^2_{I_j}\right)^{\frac{\gamma}{2}} \qquad j = 1, ..., N, \tag{6}$$

$$\alpha_h = \left[\frac{1}{b-a}\sum_{j=1}^{N} h_j \langle V_{k+1}\rangle^{\gamma}_{I_j}\right]^{\frac{2}{\gamma}} \tag{7}$$

for some number $\gamma \in (0, 2]$. As shown in [20], the optimal value (which yields the smallest error bound; cf. Theorem 2.1 below) is $\gamma = 2/(2q+1)$ when the error is measured in the $H^m$ semi-norm. We consider a general value of $\gamma$ so that we can deal with more complicated norms occurring in the convergence analysis of finite element approximations to differential equations; see the next section. For this adaptation function, the equidistribution principle reads as

$$\rho_j h_j = \frac{\sigma_h}{N} \qquad j = 1, ..., N, \tag{8}$$

where

$$\sigma_h = \sum_{j=1}^{N} h_j \rho_j. \tag{9}$$

With $\alpha_h$ being defined as in (7), it can be shown that $\sigma_h \le 2(b-a)$. Indeed, from Jensen's inequality it follows

$$\begin{aligned}
\sigma_h &= \sum_{j=1}^{N} h_j \left(1 + \alpha_h^{-1} \langle V_{k+1} \rangle_{I_j}^2\right)^{\frac{\gamma}{2}} \\
&\le \sum_{j=1}^{N} h_j \left(1 + \alpha_h^{-\frac{\gamma}{2}} \langle V_{k+1} \rangle_{I_j}^{\gamma}\right) \\
&= (b-a) + \alpha_h^{-\frac{\gamma}{2}} \sum_{j=1}^{N} h_j \langle V_{k+1} \rangle_{I_j}^{\gamma} \\
&= 2(b-a). \qquad (10)
\end{aligned}$$

In practice, it is more realistic to assume that a computed mesh satisfying (8) only approximately. To measure the effect of this approximation, we define the adaptation quality measure as

$$Q_{adp}(I_j) = \frac{N h_j \rho_j}{\sigma_h} \qquad j = 1, ..., N. \qquad (11)$$

It is not difficult to see that $\max_j Q_{adp}(I_j) = 1$ implies that the mesh satisfies (8) exactly. The larger $\max_j Q_{adp}(I_j)$, the farther away the mesh is from satisfying (8).

We now rewrite the bound on interpolation error in terms of $Q_{adp}$. From (5), (6), (10), and (11), we have

$$\begin{aligned}
\sum_{j=1}^{N} |v - \Pi_k v|_{m,I_j}^2 &\le C\alpha_h \sum_{j=1}^{N} h_j h_j^{2q} \rho_j^{\frac{2}{\gamma}} \\
&= C\alpha_h \sum_{j=1}^{N} h_j \rho_j \left(Q_{adp}(I_j) \sigma_h \rho_j^{-1} N^{-1}\right)^{2q} \rho_j^{\frac{2}{\gamma}-1} \\
&= C\alpha_h \sigma_h^{2q} N^{-2q} \sum_{j=1}^{N} h_j \rho_j Q_{adp}^{2q}(I_j) \rho_j^{\frac{2}{\gamma}-1-2q}. \\
&\le C\alpha_h N^{-2q} Q_{mesh,m}^2 \max_j \rho_j^{\frac{2}{\gamma}-1-2q}, \qquad (12)
\end{aligned}$$

where $\sigma_h \le 2(b-a)$ has been used in the last step and $Q_{mesh,m}$, the overall mesh quality measure, is defined as

$$Q_{mesh,m} = \left(\frac{1}{(\sum_{j=1}^{N} h_j \rho_j)} \sum_{j=1}^{N} h_j \rho_j Q_{adp}^{2q}(I_j)\right)^{\frac{1}{2}} = \left(\frac{1}{\sigma_h} \sum_{j=1}^{N} h_j \rho_j Q_{adp}^{2q}(I_j)\right)^{\frac{1}{2}}. \qquad (13)$$

The letter $m$ in the subscript is used to indicate the dependence of the definition on $m$. It is remarked that when $2/\gamma - 1 - 2q \le 0$ or $\gamma \ge 2/(2q+1)$, we have $\rho_j^{2/\gamma-1-2q} \le 1$ and (12) reduces to

$$\sum_{j=1}^{N} |v - \Pi_k v|_{m,I_j}^2 \le C\alpha_h N^{-2q} Q_{mesh,m}^2.$$

These results are summarized in the following theorem.

**Theorem 2.1.** *Given an integer $k \ge 0$, let $\Pi_k$ be a polynomial preserving operator defined on a general mesh (1) and having the approximation property (2). For*

*any integer $m$: $0 \le m \le k+1$ and any number $\gamma \in (0,2)$, assume that the adaptation function $\rho$, the intensity parameter $\alpha_h$, and the overall mesh quality measure $Q_{mesh,m}$ are defined in (6), (7), and (13), respectively. Then the interpolation error is bounded by*

$$\sum_{j=1}^{N} |v - \Pi_k v|^2_{m,I_j} \le C\alpha_h N^{-2q} Q^2_{mesh,m} \max_j \rho_j^{\frac{2}{\gamma}-1-2q}, \tag{14}$$

*where $q = k+1-m$. If $\gamma \ge \gamma^* \equiv 2/(2q+1) = 2/(2(k+1-m)+1)$,*

$$\sum_{j=1}^{N} |v - \Pi_k v|^2_{m,I_j} \le C\alpha_h N^{-2q} Q^2_{mesh,m}. \tag{15}$$

It can be shown (e.g., see [20]) that the optimal value for $\gamma$ (which leads to a smallest error bound) is $\gamma = \gamma^*$ when the error is measured in the semi-norm $|\cdot|_{H^m}$.

**2.3. Properties of approximate equidistributing meshes.** A mesh satisfying the equidistribution relation (8) is commonly called the equidistributing mesh for the adaptation function (6). Naturally, we can refer to a mesh satisfying a weaker constraint,

$$Q_{mesh,m} \le C \tag{16}$$

for some constant $C$, as *an approximate equidistrbuting mesh*. Note that when $C = 1$, we have $Q_{mesh,m} = 1$ and therefore the mesh is equidistributing.

We now study some properties of approximate equdistributing meshes. From (11) and (13) it follows that, for any $j = 1, ..., N$,

$$Q^2_{mesh,m}\sigma_h = \sum_{k=1}^{N} h_k\rho_k \left(\frac{Nh_k\rho_k}{\sigma_h}\right)^{2q} \ge \left(\frac{N}{\sigma_h}\right)^{2q} (h_j\rho_j)^{2q+1}.$$

Thus,

$$h_j\rho_j \le Q^{\frac{2}{1+2q}}_{mesh,m}\sigma_h N^{-\frac{2q}{1+2q}}. \tag{17}$$

Since $\sigma_h \le 2(b-a)$ and $\rho_j \ge 1$, we have

$$h_j \le 2(b-a)Q^{\frac{2}{1+2q}}_{mesh,m}N^{-\frac{2q}{1+2q}} \qquad j = 1, ..., N. \tag{18}$$

This implies

$$\max_j h_j \to 0 \quad \text{as} \quad N \to \infty \tag{19}$$

for any approximate equididistributing mesh. Moreover, using (6) we can get from (17)

$$h_j \langle V_{k+1}\rangle^{\gamma}_{I_j} \le 2(b-a)Q^{\frac{2}{1+2q}}_{mesh,m}\alpha_h^{\frac{\gamma}{2}}N^{-\frac{2q}{1+2q}} \qquad j = 1, ..., N, \tag{20}$$

which proves to be useful in the convergence analysis of finite element approximations in the next section.

Property (19) has a significant implication that qualities such as $\alpha_h$ and $Q_{mesh,m}$ converge to their continuous counterparts as $N \to \infty$. Take $\alpha_h$ as an example. It has the continuous form as

$$\alpha \equiv \left[\frac{1}{b-a}\int_a^b |V_{k+1}|^{\gamma}\, dx\right]^{\frac{2}{\gamma}}.$$

Recalling that $\langle V_{k+1}\rangle_{I_j}$ is the $L^2$ average of $V_{k+1}$ on $I_j$, from (7) we can bound $\alpha_h$ by the lower and upper Riemann sums, i.e.,

$$\left[\frac{1}{b-a}\sum_{j=1}^{N} h_j \min_{x\in I_j} V_{k+1}^{\gamma}(x)\right]^{\frac{2}{\gamma}} \le \alpha_h \le \left[\frac{1}{b-a}\sum_{j=1}^{N} h_j \max_{x\in I_j} V_{k+1}^{\gamma}(x)\right]^{\frac{2}{\gamma}}.$$

Thus, the property (19) and the Riemann integrability of $V_{k+1}$ imply that $\alpha_h \to \alpha$ as $N \to \infty$. Thus, by (15) and taking $V_{k+1} = |v^{(k+1)}|$ we can see that, for any function $v$ satisfying $\int_a^b |v^{(k+1)}|^\gamma dx < \infty$, the $H^m$ semi-norm of the interpolation error on an approximate equidistributing mesh converges at a rate $O(N^{-q}) = O(N^{-(k+1-m)})$.

## 3. Convergence analysis for finite element approximations

In this section we study the convergence of the finite element solution of singularly perturbed problems without turning points on an approximate equidistributing mesh satisfying (16). Our tools are Theorem 2.1 and the mesh properties (18) and (20).

We consider the general boundary value problem

$$-\epsilon u'' + bu' + cu = f \qquad x \in (0,1), \tag{21}$$

$$u(0) = u(1) = 0, \tag{22}$$

where $0 < \epsilon \ll 1$ is the perturbation parameter and $b$, $c$, and $f$ are given functions of $x$ and $\epsilon$. We assume that the coefficients are sufficiently smooth so that all concerned derivatives exist and are bounded uniformly in both $\epsilon$ and $x$. We also assume that

$$c(x,\epsilon) - \frac{1}{2}b_x(x,\epsilon) \ge \beta^* > 0 \qquad \forall x \in (0,1) \text{ and } \epsilon > 0. \tag{23}$$

For the case $b(x,\epsilon) \ge b_0 > 0$, this condition is not essential because the equation (21) can be transformed into a differential equation satisfying (23) through the change of variables $u = v\exp(Kx)$ with a proper value for $K$.

Define the bilinear operator $a(\cdot,\cdot)$ in $S \equiv H_0^1(0,1) = \{v \,|\, v \in H^1(0,1) \text{ and } v(0) = v(1) = 0\}$ as

$$a(u,v) = \epsilon(u',v') + (bu',v) + (cu,v), \tag{24}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(0,1)$. For the singularly perturbed problem (21) and (22), it is common practice (e.g. see [30]) to use the $\epsilon$-dependent norm

$$\|v\|_\epsilon^2 \equiv \epsilon|v|_1^2 + \|v\|^2 \tag{25}$$

for convergence analysis. It is easy to verify the coercive property

$$a(v,v) \ge \beta\|v\|_\epsilon^2 \qquad \forall v \in S, \tag{26}$$

where $\beta = \min\{1, \beta^*\}$.

For a given integer $k \ge 1$, we denote by $S^h \subset S$ a finite element space of degree $k$ defined on the mesh $x_0 = 0 < x_1 < ... < x_N = 1$. We assume that members of $S_h$ are at least continuous on (0, 1), and piecewise polynomials of degree no more than $k$ form a subset of $S^h$. It is known (e.g., see [12]) that the interpolation operator associated with $S^h$ has the approximation property (2) with $q = k - m + 1$ for all integers $0 \le m \le k+1$. The finite element solution $u^h \in S^h$ to problem (21) and (22) is defined as

$$a(u^h, v) = (f, v) \qquad \forall v \in S^h. \tag{27}$$

The error equation reads as

$$a(u^h - u^*, v) = 0 \qquad \forall v \in S^h, \tag{28}$$

where $u^*$ denotes the exact solution of (21) and (22).

Let

$$b_1 = \max_{x,\epsilon} |b(x,\epsilon)|, \qquad c_1 = \max_{x,\epsilon} |c(x,\epsilon)|.$$

**Lemma 3.1.** *The error in the finite element solution of degree $k \geq 1$ to problem (21) and (22) satisfies*

$$\|u^h - u^*\|_\epsilon^2 \leq C \inf_{w \in S^h} \left( \epsilon |w - u^*|_1^2 + \frac{b_1^2}{\epsilon} \|w - u^*\|^2 + c_1^2 \|w - u^*\|^2 \right), \tag{29}$$

*where $C$ is a constant independent of $\epsilon$ and $N$.*

*Proof.* We first have

$$\begin{aligned}
& \beta \|u^h - u^*\|_\epsilon^2 \\
\overset{\text{by (26)}}{\leq} \quad & a(u^h - u^*, u^h - u^*) \\
\overset{\forall w \in S^h}{=} \quad & a(u^h - u^*, u^h - w) + a(u^h - u^*, w - u^*) \\
\overset{\text{by (28) with } v = u^h - w}{=} \quad & a(u^h - u^*, w - u^*) \\
\leq \quad & \epsilon |((u^h - u^*)', (w - u^*)')| + |(b(u^h - u^*)', w - u^*)| \\
& + |(c(u^h - u^*), w - u^*)|.
\end{aligned} \tag{30}$$

The inequality

$$ab \leq \frac{1}{2} a^2 + \frac{1}{2} b^2$$

will frequently be used in estimating the terms on the right-hand side of the last inequality of (30). Let $\zeta$ be a positive number. For the first term, Schwarz's inequality gives rise to

$$\begin{aligned}
\epsilon |((u^h - u^*)', (w - u^*)')| &\leq (\epsilon\zeta)^{1/2} \|(u^h - u^*)'\| \cdot (\frac{\epsilon}{\zeta})^{1/2} \|(w - u^*)'\| \\
&\leq \frac{\epsilon\zeta}{2} |u^h - u^*|_1^2 + \frac{\epsilon}{2\zeta} |w - u^*|_1^2.
\end{aligned} \tag{31}$$

For the second term, it follows

$$\begin{aligned}
|(b(u^h - u^*)', w - u^*)| &\leq (\epsilon\zeta)^{1/2} \|(u^h - u^*)'\| \cdot \left( \frac{b_1^2}{\epsilon\zeta} \right)^{1/2} \|w - u^*\| \\
&\leq \frac{\epsilon\zeta}{2} |u^h - u^*|_1^2 + \frac{b_1^2}{2\epsilon\zeta} \|w - u^*\|^2.
\end{aligned} \tag{32}$$

The last term can be estimated as

$$|(c(u^h - u^*), w - u^*)| \leq \zeta \|u^h - u^*\|^2 + \frac{c_1^2}{4\zeta} \|w - u^*\|^2. \tag{33}$$

Substituting (31), (32), and (33) into (30) gives rise to

$$\begin{aligned}
\beta \|u^h - u^*\|_\epsilon^2 \quad \leq \quad & \zeta \|u^h - u^*\|_\epsilon^2 + \frac{\epsilon}{2\zeta} |w - u^*|_1^2 \\
& + \frac{b_1^2}{2\epsilon\zeta} \|w - u^*\|^2 + \frac{c_1^2}{4\zeta} \|w - u^*\|^2.
\end{aligned}$$

Taking $\zeta = \beta/2$ yields

$$\|u^h - u^*\|_\epsilon^2 \le \frac{2\epsilon}{\beta^2}|w-u^*|_1^2 + \frac{2b_1^2}{\epsilon\beta^2}\|w-u^*\|^2 + \frac{c_1^2}{\beta^2}\|w-u^*\|^2.$$

The conclusion of the lemma, (29), is obtained by taking the infimum over $w \in S^h$ in the above inequality. □

This lemma indicates that the error in the finite element solution is dominated by the interpolation error in the $H^1$ semi-norm and the $L^2$ norm. We recall that on a uniform mesh, the interpolation error in these norms is bounded by

$$\|u^* - \Pi_k u^*\|^2 \le \frac{C}{N^{2(k+1)}}\int_0^1 |(u^*)^{(k+1)}|^2 dx, \tag{34}$$

$$|u^* - \Pi_k u^*|_1^2 \le \frac{C}{N^{2k}}\int_0^1 |(u^*)^{(k+1)}|^2 dx. \tag{35}$$

On the other hand, for an approximate equidistributing mesh for the adaptation function (6) with the intensity parameter given in (7) Theorem 2.1 shows that the optimal value for $\gamma$ is $\gamma^* = 2/(2k+1)$ for the $H^1$ semi-norm of the error ($m = 1$) and $\gamma^* = 2/(2k+3)$ for the $L^2$ norm ($m = 0$). In the current situation, the $\epsilon$-dependent norm $\|\cdot\|_\epsilon$ involves both $|\cdot|_{H^1}$ and $\|\cdot\|_{L^2}$. It is reasonable to expect that the optimal value for $\gamma$ for the $\epsilon$-norm stays between $2/(2k+3)$ and $2/(2k+1)$. Thus, we assume that $\gamma$ used in defining the adaptation function (6) is chosen between these two values, viz.,

$$\frac{2}{2k+3} \le \gamma \le \frac{2}{2k+1}. \tag{36}$$

By Theorem 2.1 we have

$$\|u^* - \Pi_k u^*\|^2 \le C\alpha_h N^{-2(k+1)} Q_{mesh,0}^2, \tag{37}$$

$$|u^* - \Pi_k u^*|_1^2 \le C\alpha_h N^{-2k} Q_{mesh,1}^2 \max_j \rho_j^p, \tag{38}$$

where $p = 2/\gamma - (2k+1)$ and $V_{k+1}$ (in the definitions of $\alpha_h$, $\rho_j$, $Q_{mesh,0}$, and $Q_{mesh,1}$) is a majorizing function for $(u^*)^{(k+1)}$.

To estimate $\alpha_h$ and $\max_j \rho_j^p$, we consider two separate cases, the convection-diffusion case with $b(x,\epsilon) \ge b_0 > 0$ and the reaction-diffusion one with $b(x,\epsilon) \equiv 0$ and $c(x,\epsilon) \ge c_0 > 0$. The exact solution of problem (21) and (22) behaves differently in these two cases. We assume that the approximate equidistributing mesh satisfies

$$Q_{mesh,0} = \left(\frac{1}{\sigma_h}\sum_{j=1}^N h_j\rho_j Q_{adp}^{2(k+1)}(I_j)\right)^{\frac{1}{2}} \le C_0, \tag{39}$$

$$Q_{mesh,1} = \left(\frac{1}{\sigma_h}\sum_{j=1}^N h_j\rho_j Q_{adp}^{2k}(I_j)\right)^{\frac{1}{2}} \le C_1 \tag{40}$$

for some constants $C_0$ and $C_1$.

**3.1. Convection-diffusion problems.** For this case, the exact solution of convection-diffusion problem (21) and (22) has the following property; e.g. see Roos, Stynes, and Tobiska [30] and O'Malley [25].

**Lemma 3.2.** *Suppose that*

$$b(x,\epsilon) \geq b_0 > 0, \tag{41}$$

*where $b_0$ is a constant. Then, for $i = 1, 2, ...,$*

$$|(u^*)^{(i)}| \leq C_i \left[1 + \epsilon^{-i} \exp\left(-\frac{b_0}{\epsilon}(1-x)\right)\right], \tag{42}$$

*where $C_i$ is a constant dependent only on $i$.*

It is trivial to get

$$\int_0^1 |(u^*)^{(k+1)}|^2 dx \leq C\epsilon^{-(2k+1)}.$$

Hence, by Lemma 3.1, (34), and (35) the error in the finite element approximation on a uniform mesh is bounded by

$$\|u^h - u^*\|_\epsilon \leq \frac{C}{(N\epsilon)^k}\left(1 + \frac{1}{(N\epsilon)^2}\right)^{\frac{1}{2}}. \tag{43}$$

For mesh adaptation we take the singular part of $(u^*)^{(k+1)}$, i.e.,

$$V_{k+1} = 1 + \epsilon^{-(k+1)} \exp\left(-\frac{b_0}{\epsilon}(1-x)\right). \tag{44}$$

Note that $V_{k+1}$ is monotone increasing.

We now derive an error bound on an approximate equidistributing mesh. We first estimate $\alpha_h$. The monotonicity of $V_{k+1}$ and the assumption $\gamma < 1$ (from (36)) give rise to

$$\langle V_{k+1}\rangle_{I_j}^\gamma \leq V_{k+1}^\gamma(x_j) \leq \left(1 + \epsilon^{-\gamma(k+1)} \exp\left(-\frac{\gamma b_0}{\epsilon}(1-x_j)\right)\right). \tag{45}$$

Define $x^*$ by

$$\epsilon^{-\gamma(k+1)} \exp\left(-\frac{\gamma b_0}{\epsilon}(1-x^*)\right) = 1,$$

which yields

$$1 - x^* = -\frac{k+1}{b_0}\epsilon \ln \epsilon. \tag{46}$$

Let $[x_{j^*-1}, x_{j^*})$ be the interval containing $x^*$. By (45),

$$\langle V_{k+1}\rangle_{I_j}^\gamma \leq \begin{cases} 2 & \text{for } x_j \leq x^* \\ 2\epsilon^{-\gamma(k+1)} & \text{for } x_j > x^*. \end{cases} \tag{47}$$

The definition of $\alpha_h$, (7), leads to

$$(b-a)\alpha_h^{\frac{\gamma}{2}} = \sum_{j=1}^{j^*-1} h_j \langle V_{k+1}\rangle_{I_j}^\gamma + h_{j^*}\langle V_{k+1}\rangle_{I_{j^*}}^\gamma + \sum_{j=j^*+1}^{N} h_j \langle V_{k+1}\rangle_{I_j}^\gamma.$$

Using (47) for the first and the last terms and (20) (with $m = 0$) for the second term on the right-hand side, we get

$$\begin{aligned}
(b-a)\alpha_h^{\frac{\gamma}{2}} &\leq 2\sum_{j=1}^{j^*-1} h_j + 2(b-a)Q_{mesh,0}^{\frac{2}{2k+3}}\alpha_h^{\frac{\gamma}{2}} N^{-\frac{2k+2}{2k+3}} + 2\epsilon^{-\gamma(k+1)}\sum_{j=j^*+1}^{N} h_j \\
&\leq 2(b-a) + 2(b-a)Q_{mesh,0}^{\frac{2}{2k+3}}\alpha_h^{\frac{\gamma}{2}} N^{-\frac{2k+2}{2k+3}} + 2\epsilon^{-\gamma(k+1)}(1-x^*) \\
&\leq C\left(1 + \epsilon^{1-\gamma(k+1)}|\ln\epsilon|\right) + 2(b-a)Q_{mesh,0}^{\frac{2}{2k+3}}\alpha_h^{\frac{\gamma}{2}} N^{-\frac{2k+2}{2k+3}}.
\end{aligned}$$

If $N$ is taken large enough such that

$$2Q_{mesh,0}^{\frac{2}{2k+3}}N^{-\frac{2k+2}{2k+3}} \le \frac{1}{2} \qquad \text{or} \qquad N \ge 4^{\frac{2k+3}{2k+2}}Q_{mesh,0}^{\frac{1}{k+1}},$$

we have

$$\alpha_h \le C\left(1+\epsilon^{\frac{2}{\gamma}(1-\gamma(k+1))}|\ln\epsilon|^{\frac{2}{\gamma}}\right). \tag{48}$$

We now estimate the term $\max_j \rho_j^p$ in (38). By the definition of $\rho$, we have

$$\rho_j^p \le C\alpha_h^{-\frac{p\gamma}{2}}\left(\alpha_h^{\frac{p\gamma}{2}} + \langle V_{k+1}\rangle_{I_j}^{\gamma p}\right).$$

Using (47) and (48), it follows

$$\rho_j^p \le C\alpha_h^{-\frac{p\gamma}{2}}\epsilon^{-\gamma p(k+1)}$$

and

$$\alpha_h \max_j \rho_j^p \le C\left(1+\epsilon^{(2k+1)(1-\gamma(k+1))}|\ln\epsilon|^{2k+1}\right)\epsilon^{-\gamma p(k+1)}. \tag{49}$$

Substituting (48) and (49) into (37) and (38) and using (39) and (40) yields

$$\|u^* - \Pi_k u^*\|^2 \le \frac{C}{N^{2(k+1)}}\left(1+\epsilon^{\frac{2}{\gamma}(1-\gamma(k+1))}|\ln\epsilon|^{\frac{2}{\gamma}}\right), \tag{50}$$

$$|u^* - \Pi_k u^*|_1^2 \le \frac{C}{N^{2k}}\left(1+\epsilon^{(2k+1)(1-\gamma(k+1))}|\ln\epsilon|^{2k+1}\right)\epsilon^{-\gamma p(k+1)}. \tag{51}$$

Combining these results with Lemma 3.1, we obtain the following theorem.

**Theorem 3.1.** *Suppose that $b(x,\epsilon) \ge b_0 > 0$. Let $u^h$ be a finite element solution of degree $k$ to the problem (21) and (22).*

*(i) When a uniform mesh is used, the error in $u^h$ is bounded by*

$$\|u^h - u^*\|_\epsilon \le \frac{C}{(N\epsilon)^k}\left(1+\frac{1}{(N\epsilon)^2}\right)^{\frac{1}{2}}, \tag{52}$$

*where $C$ is a constant independent of $N$ and $\epsilon$.*

*(ii) Suppose that $\{x_j\}_{j=0}^N$ is an approximating equidistributing mesh satisfying (39) and (40), where the adaptation function $\rho$, the intensity parameter $\alpha_h$, and the majorizing function $V_{k+1}$ are given in (6), (7), and (44), respectively, together with $2/(2k+3) \le \gamma \le 2/(2k+1)$. If $N \ge 4^{\frac{2k+3}{2k+2}}Q_{mesh,0}^{\frac{1}{k+1}}$, then the error in $u^h$ is bounded as*

$$\begin{aligned}\|u^h - u^*\|_\epsilon \le{}& \frac{C}{N^k}\left[\frac{1}{N^2\epsilon}\left(1+\epsilon^{\frac{2}{\gamma}(1-\gamma(k+1))}|\ln\epsilon|^{\frac{2}{\gamma}}\right)\right.\\ &+ \left.\left(1+\epsilon^{(2k+1)(1-\gamma(k+1))}|\ln\epsilon|^{2k+1}\right)\epsilon^{-(2k+1)(1-\gamma(k+1))}\right]^{\frac{1}{2}},\end{aligned} \tag{53}$$

*where $C$ is a constant independent of $N$ and $\epsilon$.*

It is instructive to see that the error bound (53) becomes

$$\|u^h - u^*\|_\epsilon \le \frac{C}{N^k}\left[\epsilon^{-\frac{2k+1}{2k+3}} + \frac{1}{N^2\epsilon}\right]^{\frac{1}{2}}$$

for $\gamma = 2/(2k+3)$,

$$\|u^h - u^*\|_\epsilon \le \frac{C|\ln\epsilon|^{k+\frac{1}{2}}}{N^k}\left[1+\frac{|\ln\epsilon|}{N^2\epsilon}\right]^{\frac{1}{2}}$$

for $\gamma = 2/(2k+2)$, and

$$\|u^h - u^*\|_\epsilon \le \frac{C|\ln\epsilon|^{k+\frac{1}{2}}}{N^k}\left[1+\frac{1}{N^2\epsilon^2}\right]^{\frac{1}{2}}$$

for $\gamma = 2/(2k+1)$.

The advantage of using an adaptive mesh over a uniform one is clearly shown in this theorem. Indeed, (52) shows that the error bound obtained with a uniform mesh depends strongly on $\epsilon$, in the order of $\epsilon^{-k}$. On the other hand, the error bound with an approximate equidistributing mesh has much weaker $\epsilon$-dependence. For example, the dependence is of order $O\left(|\ln\epsilon|^{k+1}/(N\sqrt{\epsilon})\right)$ when $\gamma$ is taken as $2/(2k+2)$.

**3.2. Reaction-diffusion problems.** For this case,

$$b(x,\epsilon) \equiv 0 \quad \text{and} \quad c(x,\epsilon) \ge c_0 > 0, \tag{54}$$

and (29) reads as

$$\|u^h - u^*\|_\epsilon \le C \inf_{w\in S^h} \|w - u^*\|_\epsilon. \tag{55}$$

The exact solution has the following property; e.g. see [5, 26].

**Lemma 3.3.** *Suppose that (54) holds. Then, for $i = 1, 2, ...$*

$$|(u^*)^{(i)}| \le C_i\left[1+\epsilon^{-\frac{i}{2}}\left(e^{-\sqrt{\frac{c(0,0)}{\epsilon}}x} + e^{-\sqrt{\frac{c(1,0)}{\epsilon}}(1-x)}\right)\right]. \tag{56}$$

It is easy to get

$$\int_0^1 |(u^*)^{(k+1)}|^2 dx \le C\epsilon^{-\frac{2k+1}{2}}.$$

When a uniform mesh is used, combining (55) with (34) and (35) gives rise to

$$\|u^* - \Pi_k u^*\|_\epsilon \le \frac{C\epsilon^{\frac{1}{4}}}{(N\sqrt{\epsilon})^k}\left(1+\frac{1}{(N\sqrt{\epsilon})^2}\right)^{\frac{1}{2}}. \tag{57}$$

For mesh adaptation, we take

$$V_{k+1} = 1 + \epsilon^{-\frac{k+1}{2}}\left(e^{-\sqrt{\frac{c(0,0)}{\epsilon}}x} + e^{-\sqrt{\frac{c(1,0)}{\epsilon}}(1-x)}\right). \tag{58}$$

As in the preceding subsection, we have for $N \ge 8^{\frac{2k+3}{2k+2}} Q_{mesh,0}^{\frac{1}{k+1}}$

$$\alpha_h \le C\left(1+\epsilon^{\frac{1}{\gamma}(1-\gamma(k+1))}|\ln\epsilon|^{\frac{2}{\gamma}}\right), \tag{59}$$

$$\alpha_h \max_j \rho_j^p \le C\left(1+\epsilon^{\frac{1}{2}(2k+1)(1-\gamma(k+1))}|\ln\epsilon|^{2k+1}\right)\epsilon^{-\frac{1}{2}\gamma p(k+1)}. \tag{60}$$

**Theorem 3.2.** *Suppose that $b(x,\epsilon) \equiv 0$ and $c(x,\epsilon) \ge c_0 > 0$. Let $u^h$ be a finite element solution of $k$ degree to the problem (21) and (22).*

*(i) When a uniform mesh is used, the error in $u^h$ is bounded by*

$$\|u^h - u^*\|_\epsilon \le \frac{C\epsilon^{1/4}}{(N\sqrt{\epsilon})^k}\left(1+\frac{1}{(N\sqrt{\epsilon})^2}\right)^{\frac{1}{2}}, \tag{61}$$

*where $C$ is a constant independent of $N$ and $\epsilon$.*

*(ii) Suppose that $\{x_j\}_{j=0}^N$ is an approximate equidistributing mesh satisfying (39) and (40), where the adaptation function $\rho$, the intensity parameter $\alpha_h$, and the majorizing function $V_{k+1}$ are given in (6), (7), and (44), respectively, together with*

$2/(2k+3) \le \gamma \le 2/(2k+1)$. *If* $N \ge 8^{\frac{2k+3}{2k+2}} Q_{mesh,0}^{\frac{1}{k+1}}$, *then the error in* $u^h$ *is bounded by*

$$\begin{aligned} \|u^h - u^*\|_\epsilon &\le \frac{C}{N^k}\left[\frac{1}{N^2}\left(1+\epsilon^{\frac{1}{\gamma}(1-\gamma(k+1))}|\ln\epsilon|^{\frac{2}{\gamma}}\right)\right. \\ &+ \left.\left(1+\epsilon^{\frac{1}{2}(2k+1)(1-\gamma(k+1))}|\ln\epsilon|^{2k+1}\right)\epsilon^{\frac{1}{2}-\frac{1}{2}(2k+1)(1-\gamma(k+1))}\right]^{\frac{1}{2}}, \end{aligned} \tag{62}$$

*where* $C$ *is a constant independent of* $N$ *and* $\epsilon$.

It is instructive to see that the error bound (62) becomes

$$\|u^h - u^*\|_\epsilon \le \frac{C}{N^k}\left[\epsilon^{\frac{1}{2k+3}} + \frac{1}{N^2}\right]^{\frac{1}{2}}$$

for $\gamma = 2/(2k+3)$,

$$\|u^h - u^*\|_\epsilon \le \frac{C}{N^k}\left[\sqrt{\epsilon}|\ln\epsilon|^{2k+1} + \frac{|\ln\epsilon|^{2k+2}}{N^2}\right]^{\frac{1}{2}}$$

for $\gamma = 2/(2k+2)$, and

$$\|u^h - u^*\|_\epsilon \le \frac{C}{N^k}\left[\sqrt{\epsilon}|\ln\epsilon|^{2k+1} + \frac{|\ln\epsilon|^{2k+1}}{N^2\sqrt{\epsilon}}\right]^{\frac{1}{2}}$$

for $\gamma = 2/(2k+1)$. Moreover, when $2/(2k+3) \le \gamma < 2/(2k+2)$,

$$\|u^h - u^*\|_\epsilon \le \frac{C}{N^k},$$

which shows the uniform convergence (independent of $\epsilon$) as $N \to \infty$. Here we have used the fact that for any given positive numbers $s$ and $t$, there exists a constant $\hat{C}$ such that $\epsilon^s|\ln\epsilon|^t \le \hat{C}$ for all $\epsilon \in (0,1]$.

## 4. Numerical examples

We present here some illustrative results obtained for two examples. Three methods are used. They are briefly described below.

**Method I** is the linear finite element method using a uniform mesh.

**Method II** is the linear finite element method using an adaptive mesh based on the exact function (44) or (58) with $k = 1$. To be more specific, we start with a uniform mesh. On an approximation to the equidistributing mesh, the adaptation function is calculated through (6) and (7), where $V_2$ is computed using an analytical expression and the involved integrals are approximated by the trapezoidal quadrature. De Boor's algorithm [13] is employed to find a new approximation to the equidistributing mesh. To improve the convergence of the iteration, the mesh is updated with relaxation: $0.8\, x^{\text{old}} + 0.2\, x^{\text{new}} \to x^{\text{new}}$. The process is repeated until the maximum difference between two contiguous iterates is less than $10^{-10}$ or a maximum number of iterations 2000 is reached. Finally, the finite element solution is found on the convergent mesh.

**Method III** is the linear finite element method using an adaptive mesh based on the computed solution. This method is similar to Method II, except that the iterative process involves finding both the mesh and the finite element solution. In particular, $V_2$ in (6) is replaced by an approximation of the second derivative of the computed solution, which is obtained using a derivative recovery technique (e.g., see [11, 32, 33, 34]) as the derivative

of a linear least-squares fitting polynomial based on a set of the values of the first derivative at Gaussian points in a patch of elements. The patch involves three elements for an interior element and two elements for each boundary element. Two Gaussian points are used in each element.

**Example 4.1** The first example is a convection-diffusion problem

$$-\epsilon u'' + \left(1 - \frac{\epsilon}{2}\right) u' + \frac{1}{4}\left(1 - \frac{\epsilon}{4}\right) u = e^{-\frac{x}{4}} \qquad x \in (0,1) \tag{63}$$

subject to the boundary condition (22). The exact solution is known to be

$$u(x) = e^{-\frac{x}{4}} \left( x - \frac{e^{-\frac{1-x}{\epsilon}} - e^{-\frac{1}{\epsilon}}}{1 - e^{-\frac{1}{\epsilon}}} \right).$$

For this example, $b \geq 1/2$ and $c - b_x/2 \geq 3/16$ for all $\epsilon \leq 1$ and $x \in [0,1]$.

Figs. 1 shows the computed solution on the convergent adaptive mesh and the convergence history by using Method II. One can readily see that both methods lead to correct mesh concentration, i.e., more mesh points are concentrated in the boundary layer area. The convergence history shows that the mesh quality measures, $\max_i Q_{adp}(I_i)$, $Q_{mesh,0}$, and $Q_{mesh,1}$, quickly decrease to one (in about 20 iterations). This indicates that the convergent mesh satisfies (39) and (40) (with $C_0 \approx 1$ and $C_1 \approx 1$) and thus is nearly equidistributing.

Fig. 2 shows the $\epsilon$-norm of the error (as function of the number of mesh points $N$) obtained with the three methods for two values of $\epsilon$: $10^{-2}$ and $10^{-8}$. It is clear that adaptive meshes lead to significantly more accurate results than a uniform mesh. This is especially true for small values of $\epsilon$. In the case with $\epsilon = 10^{-8}$, the convergence order of the error associated with Method I is less than one in the considered range of $N$. Moreover, the error depends severely on $\epsilon$, confirming the theoretical prediction in Theorem 3.1. On the other hand, both Method II and III show the first order convergence and mild dependence on $\epsilon$. Interestingly, Method III, a truly adaptive mesh method which utilizes approximate second order derivatives based on the computed solution during the course of adaptive mesh generation, produces results comparable to those obtained by Method II, a method being based on the analytical expression of $V_2$ (cf. (44)).

To show the effect of the choice of $\gamma$ on mesh adaptation, we depict in Fig. 3 the $\epsilon$-norm of the error with Method II and Method III for three values of $\gamma$, $2/(2k+3)$, $2/(2k+2)$, and $2/(2k+1)$. It can be seen that the three choices lead to nearly the same results for Method II whereas for Method III $\gamma = 2/(2k+3)$ yields less accurate solutions than the other choices $\gamma = 2/(2k+1)$ and $2/(2k+2)$. This noticeable difference in solution accuracy among the three choices of $\gamma$ may be due to the nature of Method III that mesh adaptation relies on the accuracy in approximating the second order derivatives from the computed solution and therefore on the accuracy in the computed finite element solution. After all, it is emphasized that Method III with choices $\gamma = 2/(2k+1)$ and $2/(2k+2)$ produces almost the same and satisfactory solutions for reasonably large $N$ ($\geq 41$).

**Example 4.2.** The second problem is a reaction-diffusion problem

$$-\epsilon u'' + u = -2\epsilon - 1 - x(1-x) \qquad x \in (0,1) \tag{64}$$

subject to the boundary condition (22). The exact solution to this problem is known to be

$$u(x) = -1 - x(1-x) + \frac{1}{1 - e^{-\frac{2}{\sqrt{\epsilon}}}} \left( e^{-\frac{1-x}{\sqrt{\epsilon}}} - e^{-\frac{1+x}{\sqrt{\epsilon}}} + e^{-\frac{x}{\sqrt{\epsilon}}} - e^{-\frac{2-x}{\sqrt{\epsilon}}} \right).$$

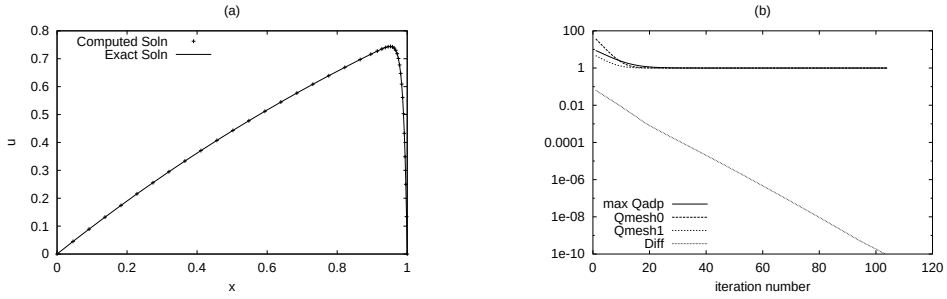


FIGURE 1. Example 4.1: Results obtained using Method II with $\epsilon = 10^{-2}$ and $N = 41$. (a): Computed and the exact solutions. (b): Mesh quality measures $\max_i Q_{adp}(I_i)$, $Q_{mesh,0}$, and $Q_{mesh,1}$, and the maximum norm of the difference between two contiguous meshes are shown against the number of iteration.

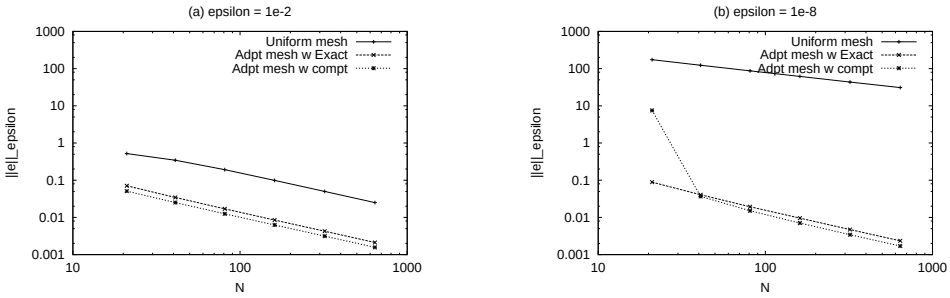


FIGURE 2. Example 4.1: The $\epsilon$-norm of the error is depicted as a function of the number of mesh points $N$ for different values of $\epsilon$. $\gamma = 2/(2k+2)$ is used in the computation of the adaptation function.

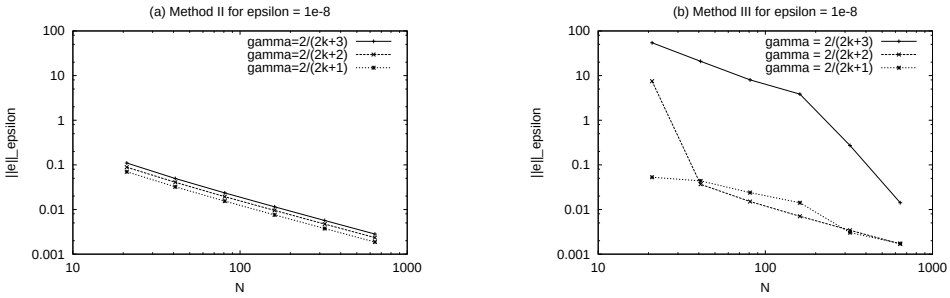


FIGURE 3. Example 4.1: The $\epsilon$-norm of the error in the FEM solution obtained using Method II (a) and Method III (b) for different values of parameter $\gamma$, $2/(2k+3)$, $2/(2k+2)$, and $2/(2k+1)$.

Typical adaptive solutions and convergence history obtained using Method III are shown in 4 and 5. Once again, one can see that mesh points are concentrated correctly in the areas of boundary layers. From the results we can make similar observations as for Example 4.1 except that the choice of $\gamma$ has an less significant

effect on the solution accuracy. This is partly because this example has less steep boundary layers and thus mesh adaptation plays a relatively less crucial role in accuracy of the numerical solution.

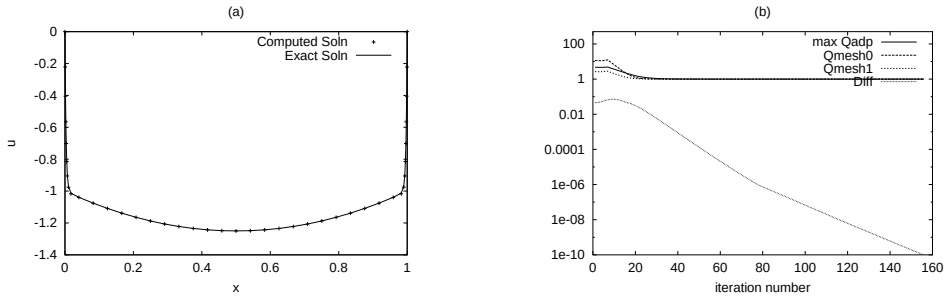


FIGURE 4. Example 4.2: Results obtained using Method III with $\epsilon = 10^{-5}$ and $N = 41$. (a): Computed and the exact solutions. (b): Mesh quality measures $\max_i Q_{adp}(I_i)$, $Q_{mesh,0}$, and $Q_{mesh,1}$, and the maximum norm of the difference between two contiguous meshes are shown against the number of iteration.

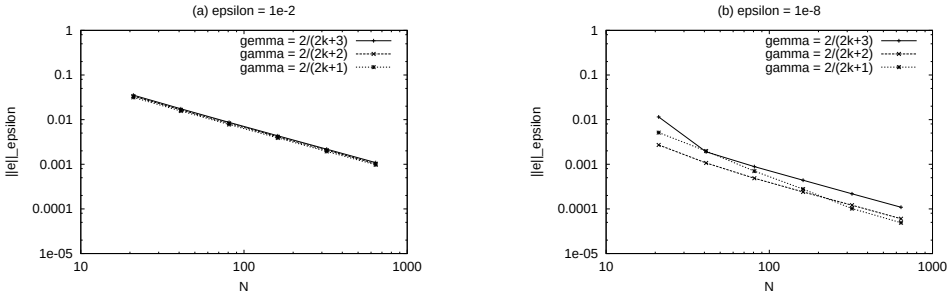


FIGURE 5. Example 4.2: The $\epsilon$-norm of the error in the FEM solution obtained using Method III for different values of $\gamma$ and $\epsilon$, $\gamma = 2/(2k+3)$, $2/(2k+2)$, $2/(2k+1)$ and $\epsilon = 10^{-2}$, $10^{-8}$.

## 5. Conclusions and Remarks

In the previous sections we have developed a convergence theory on (approximate) equidistributing meshes for polynomial preserving operators. The adaptation (or monitor) function associated with equidistribution is defined and error estimates in semi-norms of Sobolev spaces are obtained rigorously. The main results are given in Theorems 2.1.

As an application example, Theorem 2.1 is applied to the error analysis of the finite element solution of singularly perturbed boundary value problems without turning points. Error bounds are obtained for two separate cases: convection-diffusion problems (Theorem 3.1) and reaction-diffusion ones (Theorem 3.2). For the latter case, uniform convergence is obtained regardless of the size of the perturbation parameter $\epsilon$. Numerical results are presented in Section 4 for two examples to verify theoretical findings. It is shown that a truly adaptive implementation of

mesh adaptation that utilizes approximations of higher derivatives (second derivative in the examples) based on the computed solution can produce comparable solutions to those obtained with an analytical expression.

The analysis method employed in this paper does not specifically use the advantage of dimension one. It is our hope that the method and the results can be extended to multi-dimensions. Such an investigation is currently underway.

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