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FINITE VOLUME ELEMENT METHOD FOR SECOND ORDER HYPERBOLIC EQUATIONS

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Abstract. We discuss a priori error estimates for a semidiscrete piecewise linear finite volume element (FVE) approximation to a second order wave equation in a two-dimensional convex polygonal domain. Since the domain is convex polygonal, a special attention has been paid to the limited regularity of the exact solution. Optimal error estimates in L^2, H^1 norms and quasioptimal estimates in L^∞ norm are discussed without quadrature and also with numerical quadrature. Numerical results confirm the theoretical order of convergence.

Key Words. finite element, finite volume element, second order hyperbolic equation, semidiscrete method, numerical quadrature, Ritz projection, optimal error estimates.

1. Introduction

In this paper, we are interested in the finite volume element method (FVEM) for the following second order linear hyperbolic initial boundary value problem :
Given $f(x,t)$, $g(x)$ and $w(x)$, and $t \in (0,T]$ for $x \in \Omega$, find $u = u(x,t)$ such that

$$
\begin{aligned}
u_{tt} - \nabla.(A(x)\nabla u) &= f(x,t) \quad \forall x \in \Omega,\ 0 < t \le T, \\
u(x,t) &= 0 \quad\quad\quad \forall x \in \partial\Omega,\ 0 < t \le T, \\
u(x,0) &= g(x) \quad\quad \forall x \in \Omega, \\
u_t(x,0) &= w(x) \quad\quad \forall x \in \Omega,
\end{aligned} \tag{1.1}
$$

where Ω is a bounded, convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $A(x) = (a_{ij}(x))_{i,j=1}^2$ is a real-valued and uniformly positive definite matrix in Ω. It is assumed that the functions $f,\ g,\ w$ have enough regularity and they satisfy appropriate compatibility conditions so that the boundary value problem (1.1) has a unique solution satisfying the regularity results as demanded by our subsequent error analysis.
The FVEM employs a finite element partition of the domain $\overline{\Omega} = \Omega \cup \partial\Omega$. It may be considered as a Petrov-Galerkin finite element method in which the trial space is C^0- piecewise linear on the finite element partition of $\overline{\Omega}$ and the test space is piecewise constant over the control volume to be defined in Section 2. The FVEM has been studied by Bank and Rose [3], Cai [4], Chatzipantelidis [6], R. Li et al. [13], Ewing et al. [10], etc. for elliptic problems and by Chou et al. [5], Chatzipantelidis et al. [7] and Sinha et al. [18] for parabolic problems. For elliptic problems, the authors [13] have obtained optimal order H^1 and L^2 error estimates of the following form

$$\|u - u_h\|_0 \le Ch^2 \|u\|_{W^{3,p}(\Omega)},\ p > 1,$$

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where u is the exact solution and u_h is the FV approximation of u. Note that the regularity on the exact solution seems to be too high compared to that for the finite element methods. On the other hand, it may be difficult to have $u \in H^3$ if Ω is a convex polygonal domain. The authors [10] have derived optimal L^2 error estimate assuming that the exact solution $u \in H^2$ and the source term $f \in H^1$. They have also provided an example that if $f \in L^2$, then FVE solution may not have optimal error estimates in L^2 norm. In [7], the authors have extended the analysis of [10] to parabolic problems in a convex polygonal domain. They have also considered the effect of quadrature for the L^2 inner product and derived a priori error estimates. Ewing et al. [11] have discussed a priori error estimates for the parabolic integro-differential equations using FVEM. In the present paper, we have extended the results to include the second order hyperbolic equation. Special attention has been paid for the data with minimal regularity assumption. Moreover, the effect of quadrature is also discussed.

Let us relate our work with the literature for the second order hyperbolic equations. R. Li et al. [13] have proved the optimal order of convergence in H^1- norm without quadrature using elliptic projection, but the regularity of the exact solution seems to be high compared to our results. The finite element analysis for the second order hyperbolic equations without quadrature was discussed by Baker [1] and with quadrature by Baker and Dougalis [2] and Dupont [9]. Baker and Dougalis [2] have proved that the finite element solution for hyperbolic equation has optimal order convergence in $L^\infty(L^2)$ for the semidiscrete scheme, provided $g \in H^5 \cap H^1_0$ and $w \in H^4 \cap H^1_0$. In [15], Rauch has also discussed the convergence of the Galerkin approximation to a second order wave equation by using piecewise linear polynomials and proved optimal $L^\infty(L^2)$ estimate with $g \in H^3 \cap H^1_0$ and $w = 0$ which are the minimal regularity conditions for the second order wave equation. Pani et al. [14] and Sinha [17] have also studied the effect of numerical quadrature in finite element method for parabolic and hyperbolic integro-differential equations with the assumption that $g \in H^3 \cap H^1_0$ and $w \in H^2 \cap H^1_0$. In this paper, we have derived optimal $L^\infty(L^2)$ estimate even with quadrature when $g \in H^3 \cap H^1_0$ and $w \in H^2 \cap H^1_0$.

This paper is organized as follows: In Section 3, optimal order of convergence in L^2 and H^1 norms for the semidiscrete scheme without quadrature and with the assumption that the initial functions g, w are in $H^3 \cap H^1_0$ and $H^2 \cap H^1_0$, respectively, has been derived. Moreover, quasi-optimal order of convergence in maximum norm has also been proved. The integrals occurring in the semidiscrete scheme are replaced by quadrature formulae. The effect of numerical quadrature on the estimates has been discussed in Section 4. The analysis is based on the properties of the standard Ritz projection. In both Sections 3 and 4, the error estimates are derived under the assumption that the domain is convex polygon. In order to verify the derived order of convergence, some numerical experiments are discussed in Section 5.

2. Notation and Preliminaries.

In this paper, we use the standard notation for the Sobolev spaces. Let $W^{s,p}(\Omega)$ with $1 \le p \le \infty$ consist of functions that have generalized derivatives of order s in the space $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega} = \|u\|_{s,p} = \left(\sum_{|\alpha| \le s} \|D^\alpha u\|^p_{L^p} \right)^{1/p} \quad \text{for } 1 \le p < \infty,$$

and for $p = \infty$,

$$\|u\|_{s,\infty,\Omega} = \|u\|_{s,\infty} = \sup_{|\alpha|\le s} \|D^\alpha u\|_{L^\infty}.$$

We denote by $L^p(0,T;W^{s,p}(\Omega))$, $1 \le p,\ q \le \infty,\ s \ge 0$, the space of functions $\psi(t) : [0,T] \longrightarrow W^{s,p}(\Omega)$ such that $\|\psi(t)\|_{s,p,\Omega} \in L^p(0,T)$, see [12, pp.285]. In order to simplify the notation, we denote $W^{s,2}$ by H^s and skip the index $p = 2$, Ω and $(0,T)$ whenever possible. Let $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$. Now denote the inner product and norm on $L^2(\Omega)$ as $(u,v) = \int_\Omega uv dx$ and $\|v\|_0 = \left(\int_\Omega |v|^2 dx\right)^{1/2}$. The weak formulation associated with (1.1) may be stated as:
Find $u(\cdot,t) \in H_0^1(\Omega)(0 < t \le T)$ such that

$$\begin{aligned} (u_{tt}, v) + a(u,v) &= (f,v) \quad \forall v \in H_0^1(\Omega),\ 0 < t \le T, \\ u(x,t) &= 0 \quad \forall x \in \partial\Omega,\ 0 < t \le T, \\ u(x,0) = g(x),\ u_t(x,0) &= w(x) \quad \forall x \in \Omega, \end{aligned} \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the standard inner product in $L^2(\Omega)$ and $a(\cdot,\cdot) : H_0^1 \times H_0^1 \longrightarrow \mathbb{R}$ is a bilinear form defined by

$$a(u,v) = \int_\Omega A\nabla u \cdot \nabla v dx \quad \forall u,\ v \in H_0^1.$$

Since A is symmetric and positive definite, the bilinear form $a(\cdot,\cdot)$ satisfies the following coercivity condition

$$a(v,v) \ge \alpha \|v\|_1^2 \quad \forall v \in H_0^1. \tag{2.2}$$

Through out this paper, C is a generic positive constant independent of the mesh size h. The following Lemma yields a priori estimates of the exact solution in terms of the data $f,\ g$ and w.

Lemma 2.1. *Let u be the solution to (1.1). Then the following estimates hold:*

$$\begin{aligned} ess \sup_{0<t\le T} \left(\|\frac{\partial^m u}{\partial t^m}\|_2 + \|\frac{\partial^{m+1} u}{\partial t^{m+1}}\|_1 \right) \quad \le \quad & C\Big(\sum_{k=0}^{m} \|\frac{\partial^k f}{\partial t^k}\|_{L^2(0,T;\ H^{m-k})} + \|\frac{\partial^{m+1} f}{\partial t^{m+1}}\|_{L^2(0,T;\ L^2)} \\ & + \|g\|_{H^{m+2}} + \|w\|_{H^{m+1}} \Big),\quad m = 0,\ 1,\ 2. \end{aligned}$$

Proof. Differentiate (1.1) with respect to t and multiply by u_{tt} to obtain

$$(u_{ttt}, u_{tt}) + a(u_{tt}, u_t) = (f_t, u_{tt}). \tag{2.3}$$

Since the bilinear form $a(\cdot,\cdot)$ and the L^2 inner product $(\cdot,\cdot)$ are symmetric, we have

$$\frac{1}{2}\frac{d}{dt}\left[(u_{tt}, u_{tt}) + a(u_t, u_t)\right] = (f_t, u_{tt}).$$

Integrate from 0 to t, use (2.2) and the inequality $ab \le a^2/2 + b^2/2$ to obtain

$$\|u_{tt}\|_0^2 + \|u_t\|_1^2 \le C(\alpha)\left(\|u_{tt}(0)\|_0^2 + \|u_t(0)\|_1^2 + \int_0^t \|f_t\|_0^2 ds + \int_0^t \|u_{tt}\|_0^2 ds \right).$$

Using Grownwall's Lemma and the following estimate which follows from (1.1)

$$\|u_{tt}(0)\|_0 \le C\left(\|g\|_2 + \|f(0)\|_0\right),$$

we obtain

$$\|u_{tt}\|_0^2 + \|u_t\|_1^2 \le C(\alpha)\left(\|g\|_2^2 + \|w\|_1^2 + \|f(0)\|_0^2 + \int_0^T \|f_t\|_0^2 ds \right). \tag{2.4}$$

Using the elliptic regularity i.e., $\|u\|_2 \leq C(\|f\|_0 + \|u_{tt}\|_0)$, we obtain

$$\|u\|_2^2 + \|u_t\|_1^2 \leq C(\alpha)\left(\|g\|_2^2 + \|w\|_1^2 + \|f(0)\|_0^2 + \|f\|_0^2 + \int_0^T \|f_t\|_0^2 ds\right). \tag{2.5}$$

Also,

$$\|f(0)\|_0 \leq C\|f\|_{L^\infty(0,T;L^2)} \leq C\left(\|f\|_{L^2(0,T;L^2)} + \|f_t\|_{L^2(0,T;L^2)}\right),$$

and

$$\|f\|_0 \leq C\left(\|f(0)\|_0 + \int_0^t \|f_t\|_0 ds\right).$$

So, from (2.5), we obtain

$$\|u\|_2^2 + \|u_t\|_1^2 \leq C(\alpha)\left(\|g\|_2^2 + \|w\|_1^2 + \int_0^T (\|f\|_0^2 + \|f_t\|_0^2) ds\right). \tag{2.6}$$

In a similar way differentiate (1.1) two times with respect to t and use $\|u_t\|_2 \leq C(\|f_t\|_0 + \|u_{ttt}\|_0)$ with the following estimates

$$\|u_{ttt}(0)\|_0 \leq C(\|w\|_2 + \|f_t(0)\|_0)$$

and

$$\|u_{tt}(0)\|_1 \leq C(\|g\|_3 + \|f(0)\|_1),$$

which follow from (1.1), to obtain

$$\|u_t\|_2^2 + \|u_{tt}\|_1^2 \leq C(\alpha)\Big(\|g\|_3^2 + \|w\|_2^2 + \int_0^T (\|f\|_1^2 + \|f_t\|_0^2 + \|f_{tt}\|_0^2) ds\Big). \tag{2.7}$$

Similarly, we can prove

$$\begin{aligned} \|u_{tt}\|_2^2 + \|u_{ttt}\|_1^2 \leq\ & C(\alpha)\Big(\|g\|_4^2 + \|w\|_2^3 + \int_0^T (\|f\|_2^2 + \|f_t\|_1^2 \\ & + \|f_{tt}\|_0^2 + \|f_{ttt}\|_0^2) ds\Big). \end{aligned} \tag{2.8}$$

Combine the estimates derived in (2.6)-(2.8) to complete the rest of the proof. ■

Let T_h be a regular triangulation of the closed, convex polygonal domain $\overline{\Omega}$ into closed triangles $K_Q = K$ with barycenters Q such that $\overline{\Omega} = \cup_{K\in T_h} K$ and let $h = \max_{K\in T_h}(\text{diam}K)$. Let $N_h = \{P_i : P_i \text{ is a vertex of the element } K \in T_h \text{ and } P_i \in \overline{\Omega}\}$ and let N_h^0 be the set of interior nodes in T_h with cardinality N.

Now we introduce the dual mesh T_h^* based on T_h as follows. Let P_0 be an interior node of the triangle $K \in T_h$ and P_i $(i = 1, 2\cdots m)$ be its adjacent nodes (see FIGURE 1, $m = 6$ here). Let $M_i,\ i = 1, 2\cdots m$ denote the midpoints of $\overline{P_0P_i}$ and let $Q_i,\ i = 1, 2\cdots m$ denote the barycenters of the triangle $\triangle P_0P_iP_{i+1}$ with $P_{m+1} = P_1$. The *control volume* $K_{P_0}^*$ is obtained by joining successively $M_1,\ Q_1, \cdots, M_m,\ Q_m,\ M_1$. Let Q_i, $(i = 1, 2\cdots m)$ be the nodes of *control volume* $K_{p_i}^*$ and let N_h^* be the set of all dual nodes Q_i. For a boundary node P_1, the control volume $K_{P_1}^*$ is shown in the FIGURE 1. The union of the control volume forms a partition T_h^* of $\overline{\Omega}$.

Let the areas of the triangles $K_Q \in T_h$ and control volumes $K_{P_i}^* \in T_h^*$ be denoted by S_{K_Q} and $S_{P_i}^*$, respectively. We assume that the partitions T_h and T_h^* are quasi-uniform, i.e., there exist positive constants C_1 and C_2 independent of h such that

$$C_1 h^2 \leq S_{K_{Q_i}} \leq C_2 h^2\ \forall Q_i \in N_h^*, \tag{2.9}$$

$$C_1 h^2 \leq S_{K_{P_i}}^* \leq C_2 h^2\ \forall P_i \in N_h. \tag{2.10}$$

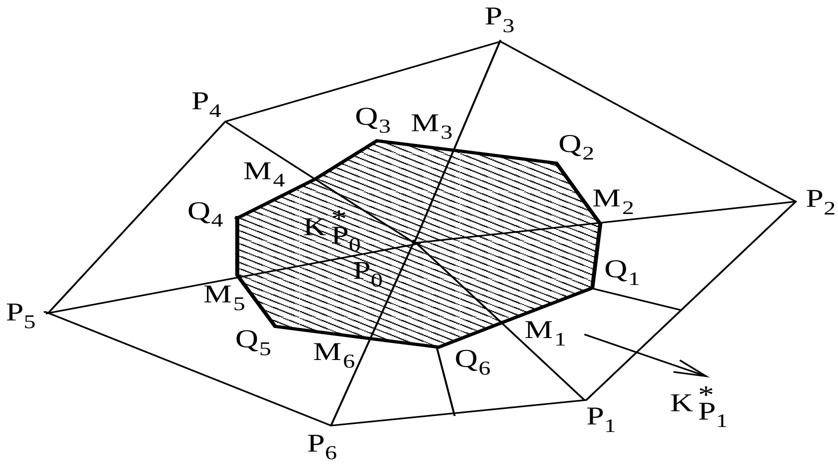

FIGURE 1

3. Finite Volume Element Method (FVEM)

For the finite volume element the standard linear finite element space on the triangulation T_h denoted by U_h is defined by

$$U_h = \{v_h \in C^0(\overline{\Omega}) \ : \ v_h|_K \text{ is linear for all } K \in T_h \text{ and } v_h|_{\partial\Omega} = 0\},$$

and the dual volume element space V_h is defined by

$$V_h = \{v_h \in L^2(\Omega) \ : \ v_h|_{K^*_{P_0}} \text{ is constant for all } K^*_{P_0} \in T^*_h \text{ and } v_h|_{\partial\Omega} = 0\}.$$

Note that, $U_h = \text{span}\{\phi_i \ : \ P_i \in N_h^0\}$ and $V_h = \text{span}\{\chi_i \ : \ P_i \in N_h^0\}$, where ϕ_i's are the standard nodal basis functions called pyramid functions which are associated with the node P_i and χ_i's are the characteristic functions corresponding to the control volume $K^*_{P_i}$ defined by

$$\chi_i(x) = \begin{cases} 1, & \text{if } x \in K^*_{P_i} \\ 0, & \text{elsewhere.} \end{cases}$$

Then, the FVE approximation $u_h : (0,T] \longrightarrow U_h$ of (1.1) is to find a solution of

$$\begin{aligned} (u_{h,tt}, v_h) + a_h(u_h, v_h) &= (f, v_h) \quad \forall v_h \in V_h, \\ u_h(0) &= g_h, \ u_{h,t}(0) = w_h, \end{aligned} \tag{3.1}$$

where the bilinear form $a_h(\cdot,\cdot) : U_h \times V_h \longrightarrow \mathbb{R}$ is defined by

$$a_h(u_h, v_h) = -\sum_{P_i \in N_h^0} v_h(P_i) \int_{\partial K^*_{P_i}} A\nabla u_h.\mathbf{n} \, ds$$

with $\mathbf{n}$ as the unit outer normal to the boundary of the control volume $K^*_{P_i}$. Here, g_h and w_h are certain approximations of g and w in U_h, respectively, to be defined later.

For our further use, let us introduce the following discrete norms

$\|v_h\|_{0,h} = \left(\sum_{K\in T_h} |v_h|^2_{0,h,K}\right)^{1/2}$ and $\|v_h\|_{1,h} = \left(\|v_h\|^2_{0,h} + |v_h|^2_{1,h}\right)^{1/2}$.

Here the seminorm $|v_h|_{1,h} = \left(\sum_{K\in T_h} |v_h|^2_{1,h,K}\right)^{1/2}$, and for $K = K_Q = \triangle P_1P_2P_3$,

$$|v_h|_{0,h,K} = \left\{\frac{1}{3}\left(v_h(P_1)^2 + v_h(P_2)^2 + v_h(P_3)^2\right)\right\}^{1/2}$$

$$|v_h|_{1,h,K} = \left\{(|\frac{\partial v_h}{\partial x}|^2 + |\frac{\partial v_h}{\partial y}|^2)S_Q\right\}^{1/2}.$$

The following Lemma gives the relation between the discrete norms and the continuous norms on the Sobolev spaces.

Lemma 3.1. [13, pp. 124] *For $v_h \in U_h$, $|\cdot|_{1,h}$ and $|\cdot|_1$ are identical; $\|\cdot\|_{0,h}$ and $\|\cdot\|_{1,h}$ are equivalent to $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively, that is, there exist positive constants $C_3, \cdots, C_6 > 0$, independent of h, such that*

$$C_3\|v_h\|_{0,h} \le \|v_h\|_0 \le C_4\|v_h\|_{0,h} \quad \forall v_h \in U_h \tag{3.2}$$

and

$$C_5||v_h||_{1,h} \le ||v_h||_1 \le C_6||v_h||_{1,h} \quad \forall v_h \in U_h. \tag{3.3}$$

Let $\Pi_h : C(\Omega) \longrightarrow U_h$ and $\Pi_h^* : C(\Omega) \longrightarrow V_h$ be the interpolation operators defined, respectively, by

$$\Pi_h u = \sum_{P_i\in N_h^0} u(P_i)\phi_i(x) \text{ and } \Pi_h^* u = \sum_{P_i\in N_h^0} u(P_i)\chi_i(x). \tag{3.4}$$

Note that for $\psi \in H^2$, Π_h has the following approximation property, (see, Ciarlet [8]):

$$\|\psi - \Pi_h\psi\|_0 \le Ch^2\|\psi\|_2. \tag{3.5}$$

Lemma 3.2. *The following statements hold true.*
(i) For $\Pi_h^* : U_h \longrightarrow V_h$ defined in (3.4),

$$(\phi_h, \Pi_h^* v_h) = (v_h, \Pi_h^*\phi_h) \quad \forall \phi_h,\ v_h \in U_h. \tag{3.6}$$

(ii) With $|||\phi_h|||_0 := (\phi_h, \Pi_h^*\phi_h)^{1/2}$, the norms $|||\cdot|||_0$ and $\|\cdot\|_0$ are equivalent on U_h, that is, there exist positive constants C_7 and C_8, independent of h, such that

$$C_7||\phi_h||_0 \le |||\phi_h|||_0 \le C_8||\phi_h||_0 \qquad \forall \phi_h \in U_h. \tag{3.7}$$

For a proof, we refer to [13, pp. 192].

4. A Priori Error Estimates

Since a direct comparison between u and u_h may not yield optimal error estimates, we now split $u - u_h = \rho + \theta$ with $\rho = u - R_hu$ and $\theta = R_hu - u_h$, where $R_h : H_0^1 \to U_h$ is the Ritz projection defined by

$$a(R_hu, \chi_h) = a(u, \chi_h) \quad \forall \chi_h \in U_h. \tag{4.1}$$

For our subsequent analysis, we need the following well known results.

Lemma 4.1. [16] *There exist positive constants C, independent of h, such that*

$$\|\rho\|_j \le Ch^{i-j}\|u\|_i \quad \forall u \in H^i \cap H_0^1,\ j = 0, 1,\ i = 1, 2 \tag{4.2}$$

$$\|\rho_t\|_j \le Ch^{i-j}\|u_t\|_i \quad \forall u \in H^i \cap H_0^1,\ j = 0, 1,\ i = 1, 2 \tag{4.3}$$

$$\|\rho_{tt}\|_j \le Ch^{i-j}\|u_{tt}\|_i \quad \forall u \in H^i \cap H_0^1,\ j = 0, 1,\ i = 1, 2 \tag{4.4}$$

and

$$\|\rho\|_{0,\infty} \leq Ch^2 \left(log\frac{1}{h}\right) \|u\|_{2,\infty}. \tag{4.5}$$

Introduce

$$\epsilon_h(f,\chi) = (f,\chi) - (f,\Pi_h^*\chi) \quad \forall \chi \in U_h,$$

and

$$\epsilon_a(\chi,\psi) = a(\chi,\psi) - a_h(\chi,\Pi_h^*\psi) \quad \forall \psi,\ \chi \in U_h.$$

The bounds for ϵ_h and ϵ_a can be given as follows:

Lemma 4.2. [6, pp. 317] *There exist positive constants C independent of h, such that for $\chi \in U_h$,*

$$|\epsilon_h(f,\chi)| \leq Ch^{i+j}\|f\|_{H^i}\|\chi\|_{H^j} \quad \forall f \in H^i,\ i,\ j = 0,\ 1. \tag{4.6}$$

$$|\epsilon_a(R_h v,\chi)| \leq Ch^{i+j}\|v\|_{H^{1+i}}\|\chi\|_{H^j} \ \ \forall v \in H^{1+i} \cap H_0^1, i,\ j = 0,\ 1, \tag{4.7}$$

and

$$|\epsilon_a(u_h,\chi)| \leq Ch\|u_h\|_{H^1}\|\chi\|_{H^1} \quad \forall u_h \in U_h. \tag{4.8}$$

Take L^2 inner product of (1.1) with $\Pi_h^*\chi$ and integrate to obtain

$$(u_{tt},\Pi_h^*\chi) + a_h(u,\Pi_h^*\chi) = (f,\Pi_h^*\chi) \qquad \forall \chi \in U_h. \tag{4.9}$$

Now using the definition of ϵ_a, (2.1) and (4.9), we arrive at

$$\begin{aligned}
\epsilon_a(\rho,\chi) &= \epsilon_a(u,\chi) - \epsilon_a(R_h u,\chi)\\
&= a(u,\chi) - a_h(u,\Pi_h^*\chi) - \epsilon_a(R_h u,\chi)\\
&= (f - u_{tt},\chi) - (f - u_{tt},\Pi_h^*\chi) - \epsilon_a(R_h u,\chi)\\
&= \epsilon_h(f - u_{tt},\chi) - \epsilon_a(R_h u,\chi).
\end{aligned} \tag{4.10}$$

4.1. Optimal L^2- error estimates.

Theorem 4.1. *Let u and u_h be the solutions of (1.1) and (3.1) respectively, and assume that $f \in L^2(H^1)$, f_t, $f_{tt} \in L^2(L^2)$, $g \in H^3 \cap H_0^1$ and $w \in H^2 \cap H_0^1$. Further, let $u_h(0) = \Pi_h g$ and $u_{h,t}(0) = \Pi_h w$, where Π_h is the interpolation operator defined in (3.4). Then, there exists a positive constant $C = C(T)$, independent of h, such that*

$$\|u(t) - u_h(t)\|_0 \leq Ch^2 \left(\|g\|_3 + \|w\|_2 + \|\frac{\partial^2 f}{\partial t^2}\|_{L^2(L^2)} + \sum_{j=0}^{1} \|\frac{\partial^j f}{\partial t^j}\|_{L^2(H^{1-j})} \right).$$

Proof. Note that $u - u_h = \rho + \theta$. Since the estimates for ρ are known from Lemma 4.1, it is enough to estimate θ.
From (1.1) and (3.1), we obtain

$$(\theta_{tt},v_h) + a_h(\rho,v_h) + a_h(\theta,v_h) = -(\rho_{tt},v_h) \quad \forall v_h \in V_h.$$

Choosing $v_h = \Pi_h^*\chi$ and using the definition of ϵ_a and (4.1), it follows that

$$(\theta_{tt},\Pi_h^*\chi) + a(\theta,\chi) = \epsilon_a(\rho,\chi) + \epsilon_a(\theta,\chi) - (\rho_{tt},\Pi_h^*\chi) \quad \forall \chi \in U_h. \tag{4.11}$$

Integrating this from 0 to t and setting $\chi = \hat{\theta}_t = \theta$, we arrive at

$$(\theta_t,\Pi_h^*\theta) + a(\hat{\theta},\theta) = \epsilon_a(\hat{\rho},\theta) + \epsilon_a(\hat{\theta},\theta) + (-\rho_t,\Pi_h^*\theta) + (w - \Pi_h w,\Pi_h^*\theta),$$

where $\hat{\theta} = \int_0^t \theta(s)ds$.
Using (3.6) and symmetry of the bilinear form $a(\cdot,\cdot)$, we find that

$$\frac{1}{2}\frac{d}{dt}\left[(\theta,\Pi_h^*\theta) + a(\hat{\theta},\hat{\theta})\right] = \epsilon_a(\hat{\rho},\theta) + \epsilon_a(\hat{\theta},\theta) + (-\rho_t,\Pi_h^*\theta) + (w - \Pi_h w,\Pi_h^*\theta).$$

Integrate from 0 to t to obtain

$$\begin{aligned}
|||\theta|||_0^2 + a(\hat{\theta},\hat{\theta}) &= |||\theta(0)|||_0^2 + 2\int_0^t \epsilon_a(\hat{\rho},\theta)ds + 2\int_0^t \epsilon_a(\hat{\theta},\theta)ds \\
&\quad +2\int_0^t (-\rho_t, \Pi_h^*\theta)ds + 2(w-\Pi_h w, \Pi_h^*\hat{\theta}) \\
&= J_1 + J_2 + J_3 + J_4 + J_5. \qquad (4.12)
\end{aligned}$$

For J_1, use (3.5), (3.7) and (4.2) to find that

$$\begin{aligned}
|J_1| \le C\|\theta(0)\|_0^2 &\le C\big(\|g - R_h g\|_0^2 + \|\Pi_h g - g\|_0^2\big) \\
&\le Ch^4\|g\|_2^2. \qquad (4.13)
\end{aligned}$$

To estimate J_2, we note that from (4.10)

$$\begin{aligned}
\epsilon_a(\hat{\rho},\hat{\theta}_t) &= \epsilon_h(\hat{f} - \hat{u}_{tt}, \hat{\theta}_t) - \epsilon_a(R_h\hat{u},\hat{\theta}_t) \\
&= \frac{d}{dt}\Big(\epsilon_h(\hat{f} - \hat{u}_{tt}, \hat{\theta}) - \epsilon_a(R_h\hat{u},\hat{\theta})\Big) - \Big(\epsilon_h(f - u_{tt}, \hat{\theta}) - \epsilon_a(R_h u,\hat{\theta})\Big),
\end{aligned}$$

and hence,

$$J_2 = 2\Big(\epsilon_h(\hat{f} - \hat{u}_{tt}, \hat{\theta}) - \epsilon_a(R_h\hat{u},\hat{\theta})\Big) - 2\int_0^t \Big(\epsilon_h(f - u_{tt}, \hat{\theta}) - \epsilon_a(R_h u,\hat{\theta})\Big)\,ds.$$

Using (4.6) and (4.7), we now obtain

$$\begin{aligned}
|J_2| &\le 2|\epsilon_h(\hat{f} - \hat{u}_{tt}, \hat{\theta})| + 2|\epsilon_a(R_h\hat{u},\hat{\theta})| \\
&\quad +2\int_0^t \Big(|\epsilon_h(f - u_{tt}, \hat{\theta})| + |\epsilon_a(R_h u,\hat{\theta})|\Big)\,ds \\
&\le Ch^2\left[\int_0^t (\|f\|_1 + \|u_{tt}\|_1 + \|u\|_2)\,ds\right]\|\hat{\theta}\|_1 \\
&\quad +Ch^2\int_0^t (\|f\|_1 + \|u_{tt}\|_1 + \|u\|_2)\,\|\hat{\theta}\|_1 ds \qquad (4.14)
\end{aligned}$$

For J_3, we apply (4.8) and the inverse inequality to find that

$$|J_3| = 2\int_0^t |\epsilon_a(\hat{\theta},\theta)|ds \le Ch\int_0^t \|\theta\|_1\|\hat{\theta}\|_1 \le C\int_0^t \|\theta\|_0\|\hat{\theta}\|_1. \qquad (4.15)$$

In order to estimate J_4, we obtain from (4.3) and the stability of Π_h^*, (i.e., $\|\Pi_h^*\theta\|_0 \le C\|\theta\|_0$) that

$$\begin{aligned}
|J_4| &\le 2\int_0^t |(\rho_t, \Pi_h^*\theta)|ds \le C\int_0^t \|\rho_t\|_0\|\theta\|_0 ds \\
&\le Ch^2\int_0^t \|u_t(s)\|_2\|\theta(s)\|_0 ds. \qquad (4.16)
\end{aligned}$$

Finally for J_5, we apply (3.5) and $\|\Pi_h^*\hat{\theta}\|_0 \le C\|\hat{\theta}\|_1$ to obtain

$$|J_5| \le \|w - \Pi_h w\|_0\|\Pi_h^*\hat{\theta}\|_0 \le Ch^2\|w\|_2\|\hat{\theta}\|_1. \qquad (4.17)$$

Substitute the estimates (4.13)-(4.17) in (4.12) and use the equivalence of the norms $|||\cdot|||$ and $\|\cdot\|_0$ from (3.7) along with the coercivity property (2.2) of $a(\cdot,\cdot)$ and

$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2,\ a,\ b \geq 0,\ \epsilon > 0$. Then a standard use of kick back arguments yields

$$\begin{aligned}\|\theta\|_0^2 + \|\hat{\theta}\|_1^2 &\leq Ch^4 \left[\|g\|_2^2 + \|w\|_2^2 + \int_0^t \left(\|f\|_1^2 + \|u\|_2^2 + \|u_{tt}\|_1^2 + \|u_t\|_2^2\right) ds\right] \\ &\quad + C\int_0^t \left(\|\theta\|_0^2 + \|\hat{\theta}\|_1^2\right) ds.\end{aligned}$$

Using Gronwall's lemma for $t \leq T$, we arrive at

$$\|\theta\|_0^2 + \|\hat{\theta}\|_1^2 \leq C(T)h^4 \left[\|g\|_2^2 + \|w\|_2^2 + \int_0^T \left(\|f\|_1^2 + \|u\|_2^2 + \|u_{tt}\|_1^2 + \|u_t\|_2^2\right) ds\right].$$

Using Lemma 2.1, we obtain

$$\begin{aligned}\|\theta\|_0^2 + \|\hat{\theta}\|_1^2 &\leq C(T)h^4 \big[\|g\|_3^2 + \|w\|_2^2 \\ &\quad + \int_0^T \left(\|f\|_1^2 + \|f_t\|_0^2 + \|f_{tt}\|_0^2\right) ds\big].\end{aligned} \tag{4.18}$$

Equation (4.2) with Lemma 2.1 leads to

$$\|\rho\|_0^2 \leq Ch^4 \left(\|g\|_2^2 + \|w\|_1^2 + \|f\|_{L^2(L^2)}^2 + \|f_t\|_{L^2(L^2)}^2\right). \tag{4.19}$$

Combine the estimates derived in (4.18) and (4.19) and use triangular inequality to complete the rest of the proof. ■

4.2. H^1- error estimate.

Theorem 4.2. *Under the assumptions of Theorem 4.1,*

$$\|u(t) - u_h(t)\|_1 \leq C(T)h \left(\|g\|_3 + \|w\|_2 + \|\frac{\partial^2 f}{\partial t^2}\|_{L^2(L^2)} + \sum_{j=0}^{1} \|\frac{\partial^j f}{\partial t^j}\|_{L^2(H^{1-j})}\right),$$

where $C(T)$ is a positive constant independent of h.

Proof. By putting $\chi = \theta_t$ in (4.11), and using (4.10), we obtain

$$\begin{aligned}(\theta_{tt}, \Pi_h^*\theta_t) + a(\theta, \theta_t) &= \epsilon_h(f, \theta_t) - \epsilon_h(u_{tt}, \theta_t) - \epsilon_a(R_h u, \theta_t) \\ &\quad + \epsilon_a(\theta, \theta_t) - (\rho_{tt}, \Pi_h^*\theta_t).\end{aligned}$$

Using (3.6) and symmetry of the bilinear form $a(\cdot,\cdot)$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\left[(\theta_t, \Pi_h^*\theta_t) + a(\theta, \theta)\right] = \epsilon_h(f - u_{tt}, \theta_t) - \epsilon_a(R_h u, \theta_t) + \epsilon_a(\theta, \theta_t) - (\rho_{tt}, \Pi_h^*\theta_t).$$

Integrating from 0 to t yields

$$\begin{aligned}|||\theta_t|||^2 + a(\theta, \theta) &= \left[|||\theta_t(0)|||^2 + a(\theta(0), \theta(0))\right] + 2\int_0^t \epsilon_h(f - u_{tt}, \theta_t) ds \\ &\quad - 2\int_0^t \epsilon_a(R_h u, \theta_t) ds + 2\int_0^t \epsilon_a(\theta, \theta_t) ds + 2\int_0^t (-\rho_{tt}, \Pi_h^*\theta_t) ds \\ &= J_1 + J_2 + J_3 + J_4 + J_5.\end{aligned} \tag{4.20}$$

For the first term on the right hand side of (4.20), we use the boundedness of $a(\cdot,\cdot)$ and (3.5), (3.7) to obtain

$$|J_1| \leq C\|\theta_t(0)\|_0^2 + \|\theta(0)\|_1^2 \leq Ch^2(\|g\|_2^2 + \|w\|_1^2). \tag{4.21}$$

For estimating J_2 and J_3, (4.6), (4.7) and inverse inequality yields

$$
\begin{aligned}
|J_2| &\leq Ch^2 \int_0^t (\|f\|_1 + \|u_{tt}\|_1) \|\theta_t\|_1 ds \\
&\leq Ch \int_0^t (\|f\|_1 + \|u_{tt}\|_1) \|\theta_t\|_0 ds, \qquad (4.22)
\end{aligned}
$$

and

$$
|J_3| \leq Ch^2 \int_0^t \|u\|_2 \|\theta_t\|_1 ds \leq Ch \int_0^t \|u\|_2 \|\theta_t\|_0 ds. \qquad (4.23)
$$

Use (4.8) and the inverse inequality to find that

$$
|J_4| \leq \int_0^t |\epsilon_a(\theta, \theta_t)| ds \leq Ch \int_0^t \|\theta\|_1 \|\theta_t\|_1 ds \leq C \int_0^t \|\theta\|_1 \|\theta_t\|_0 ds. \qquad (4.24)
$$

For J_5, apply the Hölder's inequality with L^2 stability of Π_h^* and use (4.4) to obtain

$$
|J_5| \leq Ch \int_0^t \|u_{tt}\|_1 \|\theta_t\|_0 ds. \qquad (4.25)
$$

Substituting the estimates (4.21) - (4.25) in (4.20), using coercivity of $a(\cdot,\cdot)$, equivalence of norms $|||\cdot|||$ and $\|\cdot\|$ and applying the standard kick back arguments, we obtain

$$
\|\theta_t\|_0^2 + \|\theta\|_1^2 \leq Ch^2 \left[\|w\|_1^2 + \|g\|_2^2 + \int_0^T \left(\|u_{tt}\|_1^2 + \|f\|_1^2 + \|u\|_2^2 \right) ds \right].
$$

A use of regularity result (Lemma 2.1) and triangular inequality completes the rest of the proof. ∎

Remark 4.1. *In the above analysis we are choosing an approximation for $u(0)$ and $u_t(0)$ as the interpolation operator onto U_h. One can also choose the approximation as the L^2 projection onto V_h. In this case the term $(u_t(0) - u_{h,t}(0), \Pi_h^* \theta_t)$ will be zero.*

4.3. Maximum norm estimates.

Lemma 4.3. *Assume that $f \in L^2(H^2)$, $f_t \in L^2(H^1)$, f_{tt}, $f_{ttt} \in L^2(L^2)$, $g \in H^4 \cap H_0^1$ and $w \in H^3 \cap H_0^1$. Further, let $u_h(0) = R_h g$ and $u_{h,t}(0) = \Pi_h w$, where Π_h is the interpolation operator onto U_h defined as in (3.4). Then,*

$$
\|\theta_t\|_0^2 + \|\theta\|_1^2 \leq C(T) h^4 \left(\|g\|_4^2 + \|w\|_3^2 + \|\frac{\partial^3 f}{\partial t^3}\|_{L^2(L^2)}^2 + \sum_{j=0}^{2} \|\frac{\partial^j f}{\partial t^j}\|_{L^2(H^{2-j})}^2 \right),
$$

where $C(T)$ a positive constant independent of h.

Proof. We now modify the estimates of $J_1, \cdots, J_5$ in (4.20) to obtain a superconvergence result for $\|\theta(t)\|_1$ norm. Since $u_h(0) = R_h g$, $a(\theta(0), \theta(0)) = 0$ and $u_{h,t} = \Pi_h w$, we obtain

$$
|J_1| \leq |||\theta_t(0)|||^2 \leq C \|\theta_t(0)\|_0^2 \leq Ch^4 \|w\|_2^2. \qquad (4.26)
$$

For J_2, we note that

$$
\epsilon_h(f - u_{tt}, \theta_t) = \frac{d}{dt} \epsilon_h(f - u_{tt}, \theta) - \epsilon_h(f_t - u_{ttt}, \theta),
$$

and hence, J_2 can be rewritten as

$$
J_2 = 2\epsilon_h(f - u_{tt}, \theta) - 2 \int_0^t \epsilon_h(f_t - u_{ttt}, \theta) ds.
$$

From (4.6), we now find that

$$|J_2| \leq Ch^2\Big[\left(\|f\|_1 + \|u_{tt}\|_1\right) \|\theta\|_1 + \int_0^t \left(\|f_t\|_1 + \|u_{ttt}\|_1\right) \|\theta\|_1 ds\Big]. \tag{4.27}$$

To estimate J_3, we first note that

$$\epsilon_a(R_h u, \theta_t) = \frac{d}{dt}\epsilon_a(R_h u, \theta) - \epsilon_a(R_h u_t, \theta)$$

and hence, using (4.7)

$$\begin{aligned} |J_3| &\leq 2|\epsilon_a(R_h u, \theta)| + 2\int_0^t |\epsilon_a(R_h u_t, \theta)| ds \\ &\leq Ch^2\Big(\|u\|_2\|\theta\|_1 + \int_0^t \|u_t\|_2\|\theta\|_1 ds\Big). \end{aligned} \tag{4.28}$$

For J_4, we use the inverse inequality to obtain

$$J_4 \leq 2\int_0^t |\epsilon_a(\theta, \theta_t)| ds \leq Ch\int_0^t \|\theta\|_1\|\theta_t\|_1 ds \leq C\int_0^t \|\theta\|_1\|\theta_t\|_0 ds. \tag{4.29}$$

Finally for J_5, apply (4.4) to arrive at

$$|J_5| \leq 2\int_0^t \|\rho_{tt}\|_0\|\theta_t\|_0 ds \leq Ch^2\int_0^t \|u_{tt}\|_2\|\theta_t\|_0 ds. \tag{4.30}$$

Substituting the estimates (4.26)-(4.30) in (4.20), we use $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2,\ a,\ b \geq 0,\ \epsilon > 0$. Further, apply standard kick back arguments to obtain

$$\begin{aligned} \|\theta_t\|_0^2 + \|\theta\|_1^2 &\leq Ch^4\Big[\|w\|_2^2 + \|f\|_1^2 + \|u_{tt}\|_1^2 + \|u\|_2^2 \\ &\quad + \int_0^t \left(\|f_t\|_1^2 + \|u_t\|_2^2 + \|u_{tt}\|_2^2 + \|u_{ttt}\|_1^2\right) ds\Big] \\ &\quad + C\int_0^t (\|\theta_t\|_0^2 + \|\theta\|_1^2) ds. \end{aligned}$$

Using Grownwall's lemma, $t \leq T$ and the following estimates

$$\|u_{tt}(0)\|_1 \leq C(\|u(0)\|_3 + \|f(0)\|_1),$$

$$\|f\|_1 \leq C\left(\|f(0)\|_1 + \int_0^t \|f_t\|_1 ds\right),$$

$$\|u\|_2 \leq C\left(\|u(0)\|_2 + \int_0^t \|u_t\|_2 ds\right),$$

we arrive at

$$\begin{aligned} \|\theta_t\|_0^2 + \|\theta\|_1^2 &\leq Ch^4\Big[\|w\|_2^2 + \|g\|_3^2 + \|f(0)\|_1^2 \\ &\quad + \int_0^T \left(\|f_t\|_1^2 + \|u_t\|_2^2 + \|u_{tt}\|_2^2 + \|u_{ttt}\|_1^2\right) ds\Big]. \end{aligned}$$

This together with Lemma 2.1 yields

$$\|\theta_t\|_0^2 + \|\theta\|_1^2 \leq C(T)h^4\left(\|g\|_4^2 + \|w\|_3^2 + \int_0^T \left(\|f\|_2^2 + \|f_t\|_1^2 + \|f_{tt}\|_0^2 + \|f_{ttt}\|_0^2\right) ds\right),$$

and this completes the proof. ■

Remark 4.2.

The estimate for $\|\theta\|_\infty$ can be obtained by the following well known inequality

$$\|\chi\|_\infty \leq C \left(log\frac{1}{h}\right)^{1/2} \|\nabla\chi\| \;\forall \chi \in U_h. \tag{4.31}$$

Now from Lemma 4.3, we have

$$\begin{aligned} \|\theta\|_\infty^2 &\leq C(T)h^4 log\frac{1}{h}\Big[\|g\|_4^2 + \|w\|_3^2 + \\ &\quad + \int_0^T \left(\|f\|_2^2 + \|f_t\|_1^2 + \|f_{tt}\|_0^2 + \|f_{ttt}\|_0^2\right) ds\Big]. \end{aligned} \tag{4.32}$$

Theorem 4.3. *Let u and u_h be the solutions of (1.1) and (3.1) respectively. Further, let the assumptions of Lemma 4.3 hold. Then,*

$$\begin{aligned} \|u(t) - u_h(t)\|_\infty &\leq C(T)h^2 log\frac{1}{h}\Big(\|u\|_{2,\infty} + \|g\|_4 + \|w\|_3 + + \|\frac{\partial^3 f}{\partial t^3}\|_{L^2(L^2)} \\ &\quad + \sum_{j=0}^{2} \|\frac{\partial^j f}{\partial t^j}\|_{L^2(H^{2-j})}\Big), \end{aligned} \tag{4.33}$$

where $C(T)$ is a positive constant independent of h.

Proof. Combine the estimates obtained in (4.5), (4.32) and use the triangular inequality to obtain the required result. ■

Remark 4.3. *We note that under the extra regularity assumptions on the initial functions f, g, w and choosing Ritz projection as an approximation of $u(0)$, we obtain the superconvergence result for θ in H^1- norm.*

5. Effect of Numerical Integration

To study the effect of numerical integration, the L^2 inner product $(\cdot,\cdot)$ and the bilinear form $a_h(\cdot,\cdot)$ appearing in (3.1) need to be evaluated using numerical quadrature formulae.

For a continuous function ϕ on a triangle K, consider the following quadrature formula defined by

$$\mathcal{Q}_{K,h}(\phi) = \frac{1}{3}|K|\sum_{l=1}^{3}\phi(P_l) \approx \int_K \phi(x)dx \;\; \forall K \in T_h, \tag{5.1}$$

where $P_l,\ 1 \leq l \leq 3$ denote the vertices of the triangle K and $|K|$ denotes the area of the triangle K. Note that the quadrature formula defined in (5.1) is exact for $\phi \in P_1(K)\ \forall K \in T_h$.

Using (5.1), we define the quadrature formula for discrete L^2 inner product as

$$\begin{aligned} (\chi, \Pi_h^*\psi)_h &= \sum_{K\in T_h} \mathcal{Q}_{K,h}(\chi\Pi_h^*\psi) \\ &= \sum_{P_i \in N_h^0} \chi(P_i)\psi(P_i)|S^*_{KP_i}| \;\forall \chi,\ \psi \in U_h. \end{aligned} \tag{5.2}$$

We note that $\|\chi\|_h^2 = (\chi,\chi)_h\ \forall \chi \in U_h$ is a norm on U_h which is equivalent to the L^2 norm, i.e., there exist positive constants C_9 and C_{10}, independent of h, such that

$$C_9\|\chi\|_0 \leq \|\chi\|_h \leq C_{10}\|\chi\|_0. \tag{5.3}$$

Setting the quadrature error $\bar{\epsilon}(\chi,\psi) = (\chi, \Pi_h^*\psi) - (\chi, \Pi_h^*\psi)_h$, we discuss below an estimate for $\bar{\epsilon}_h(\cdot,\cdot)$.

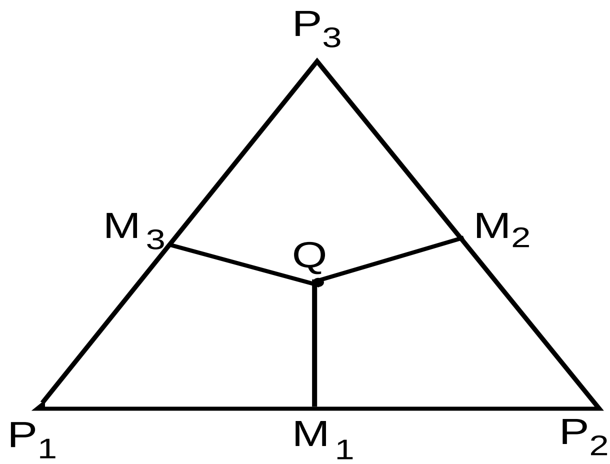

FIGURE 2

Lemma 5.1. *For $\chi,\ \psi \in U_h$, there exists a positive constant C, independent of h, such that*

$$|\bar{\epsilon}_h(\chi,\psi)| \le Ch^2\|\chi\|_1\|\psi\|_1. \tag{5.4}$$

Further, for $\chi \in H^2$ and $\psi \in U_h$, we have

$$|\bar{\epsilon}_h(\chi,\psi)| \le Ch^2\|\chi\|_2\|\psi\|_1. \tag{5.5}$$

Proof. Since the quadrature formula in (5.1) involves only the values of the functions at the interior nodes and $\Pi_h^* u_h(P_i) = u_h(P_i)\ \forall P_i \in N_h^0$ and $u_h \in U_h$, we obtain

$$(\chi,\psi)_h = (\chi, \Pi_h^*\psi)_h \quad \forall \chi,\ \psi \in U_h. \tag{5.6}$$

Therefore,

$$\begin{aligned} |\bar{\epsilon}_h(\chi,\psi)| &\le |(\chi,\psi) - (\chi,\Pi_h^*\psi)| + |(\chi,\psi) - (\chi,\psi)_h| \\ &\le |\epsilon_h(\chi,\psi)| + |(\chi,\psi) - (\chi,\psi)_h|. \end{aligned} \tag{5.7}$$

From [19],

$$|(\chi,\psi) - (\chi,\psi)_h| \le Ch^2\|\chi\|_1\|\psi\|_1 \ \ \forall \chi,\ \psi \in U_h, \tag{5.8}$$

and

$$|(\chi,\psi) - (\chi,\psi)_h| \le Ch^2\|\chi\|_2\|\psi\|_1 \quad \forall \chi \in H^2, \psi \in U_h. \tag{5.9}$$

Now using (4.6), we obtain

$$|\epsilon_h(\chi,\psi)| \le Ch^2\|\chi\|_1\|\psi\|_1 \quad \forall \chi,\ \psi \in U_h, \tag{5.10}$$

and

$$|\epsilon_h(\chi,\psi)| \le Ch^2\|\chi\|_2\|\psi\|_1 \quad \forall \chi \in H^2,\ \psi \in U_h. \tag{5.11}$$

Substitute (5.8)-(5.10) (respectively, (5.9) and (5.11)) in (5.7) to obtain (5.4) (respectively (5.5)). This completes the proof. ■

Let us introduce the following quadrature approximation over each element K by

$$\int_{\overline{M_l Q}\cap K} v(z)\ ds \approx \frac{\overline{M_l Q}}{2}\,(v(M_l) + v(Q)) = \tilde{\mathcal{Q}}_{h,l}(v), \tag{5.12}$$

where M_l is the mid point of $P_l P_{l+1}$ and Q is the barycenter of the triangle

$\triangle P_l P_{l+1} P_{l+2}$, (see FIGURE 2 for $l = 1$).
Now we introduce the following quadrature error:

$$\mathcal{E}_{\overline{M_l Q}\cap K}(v) = \int_{\overline{M_l Q}\cap K} v(s)ds - \tilde{\mathcal{Q}}_{h,l}(v).$$

Lemma 5.2. *For $v \in W^2_\infty(\overline{M_l Q}\cap K)$, there exists a positive constant C independent of h_K, such that*

$$|\mathcal{E}_{\overline{M_l Q}\cap K}(v)| \le C h_K^3 \|v\|_{2,\infty,\overline{M_l Q}\cap K}, \tag{5.13}$$

where h_K is the diam(K).

For a proof, we refer to Cai [4, pp. 732].
In order to replace the integral in the definition of $a_h(\cdot,\cdot)$, we note that

$$\begin{aligned} a_h(u_h, \Pi_h^* v_h) &= -\sum_{P_l\in N_h} v_i \int_{\partial K^*_{P_l}} A\nabla u_h.\mathbf{n}\ ds \quad \Big(v_i = v_h(P_i)\Big) \\ &= \sum_K I_K(u_h, \Pi_h^* v_h), \end{aligned}$$

where

$$\begin{aligned} I_K(u_h, \Pi_h^* v_h) &= -\sum_{P_l(1\le l\le 3)} v_l \int_{\partial K^*_{P_l}\cap K} A\nabla u_h.\mathbf{n_l} ds \\ &= \sum_{P_l(1\le l\le 3)} (v_{l+1} - v_l) \int_{\overline{M_l Q}\cap K} A\nabla u_h.\mathbf{n_l} ds, \end{aligned}$$

$v_4 = v_1$ and $\mathbf{n_l}$ is the outward unit normal vector to $\overline{M_l Q}$. Since $\nabla u_h.\mathbf{n_l}$ is constant on each element K, we define the quadrature rule

$$\tilde{I}_K(u_h, \Pi_h^* v_h) = \sum_{P_l(1\le l\le 3)} \mathcal{E}_{\overline{M_l Q}\cap K}(A)\nabla u_h.\mathbf{n_l}(v_{l+2} - v_{l+1}). \tag{5.14}$$

Thus, the bilinear form $a_h(\cdot,\cdot)$ in (3.1) is approximated by $\tilde{a}_h(\cdot,\cdot)$ using the quadrature formula as follows:

$$\tilde{a}_h(\chi, \Pi_h^*\psi) = \sum_{K\in T_h} \tilde{I}_K(\chi, \Pi_h^*\psi).$$

Now let us introduce the following error functional for the bilinear form $a_h(\cdot,\cdot)$:

$$\bar{\epsilon}_a(\chi,\psi) = a_h(\chi, \Pi_h^*\psi) - \tilde{a}_h(\chi, \Pi_h^*\psi) \quad \forall \chi,\ \psi \in U_h. \tag{5.15}$$

Lemma 5.3. *If $A(x) = (a_{ij}(x))^2_{i,j=1}$ with each $a_{ij} \in W^2_\infty$, then*

$$\bar{\epsilon}_a(\chi,\psi) \le Ch^2 \|\chi\|_1 \|\psi\|_1 \quad \forall \chi,\ \psi \in U_h,$$

where C is a positive constant independent of h.

Proof. Note that

$$\begin{aligned} |I_K(u_h, \Pi_h^* v_h) - \tilde{I}_K(u_h, \Pi_h^* v_h)| &= |\sum_{P_l(1\le l\le 3)} \mathcal{E}_{\overline{M_l Q}\cap K}(A)\nabla u_h.\mathbf{n_l}(v_{l+1} - v_l)| \\ &\le \sum_{P_l(1\le l\le 3)} \sum_{m=1}^{2} \Big[|\mathcal{E}_{\overline{M_l Q}\cap K}(a_{m1})||\frac{\partial u_h}{\partial x}| \\ &\quad + |\mathcal{E}_{\overline{M_l Q}\cap K}(a_{2m})||\frac{\partial u_h}{\partial y}|\Big]|v_{l+1} - v_l|. \end{aligned}$$

Using Lemma 5.2, we obtain

$$\begin{aligned} |I_K(u_h,\Pi_h^*v_h)-\tilde{I}_K(u_h,\Pi_h^*v_h)| \quad &\le \quad Ch^3\sum_{l=1}^{3}\sum_{m=1}^{2}\Big[\|a_{m1}\|_{2,\infty,\overline{M_lQ}\cap K}|\frac{\partial u_h}{\partial x}| \\ &\quad +\|a_{2m}\|_{2,\infty,\overline{M_lQ}\cap K}|\frac{\partial u_h}{\partial y}|\Big]|v_{l+1}-v_l|. \end{aligned} \tag{5.16}$$

Now on each K, we use Taylor expansion and (2.9) to find that

$$\begin{aligned} |v_{l+1}-v_l| \quad &\le \quad h\left(|\frac{\partial v_h}{\partial x}|+|\frac{\partial v_h}{\partial y}|\right) \\ &\le \quad 2^{1/2}\left[\left(|\frac{\partial v_h}{\partial x}|^2+|\frac{\partial v_h}{\partial y}|^2\right).S_{K_Q}\right]^{1/2} \\ &= \quad C|v_h|_{1,h,K}. \end{aligned} \tag{5.17}$$

We also note that

$$h|\frac{\partial u_h}{\partial x}|\le C|u_h|_{1,h,K},\ h|\frac{\partial u_h}{\partial y}|\le C|u_h|_{1,h,K}. \tag{5.18}$$

Substitute (5.17), (5.18) in (5.16) to obtain

$$|I_K(u_h,\Pi_h^*v_h)-\tilde{I}_K(u_h,\Pi_h^*v_h)|\le Ch^2\|u_h\|_{1,h,K}\|v_h\|_{1,h,K}.$$

Taking summation on $K\in T_h$, we arrive at

$$\sum_{K\in T_h}|I_K(u_h,\Pi_h^*v_h)-\tilde{I}_K(u_h,\Pi_h^*v_h)|\le Ch^2\|u_h\|_{1,h}\|v_h\|_{1,h}.$$

Since the norms $\|\cdot\|_{1,h}$ and $\|\cdot\|_1$ are equivalent, from (3.3), we obtain

$$|a_h(u_h,\Pi_h^*v_h)-\tilde{a}_h(u_h,\Pi_h^*v_h)|\le Ch^2\|u_h\|_1\|v_h\|_1,$$

and this completes the proof. ∎

Now the semidiscrete finite volume element method with quadrature is to seek $u_h:(0,T]\longrightarrow U_h$ satisfying

$$(u_{h,tt},v_h)_h+\tilde{a}_h(u_h,v_h)=(f,v_h)_h\ \forall v_h\in V_h \tag{5.19}$$

with $u_h(0)$ and $u_{h,t}(0)$ in U_h to be defined later.

5.1. Optimal L^2 error estimate.

Theorem 5.1. *Let u and u_h be the solutions of (1.1) and (5.19) respectively, and assume that $f\in L^2(H^2)$, f_t, $f_{tt}\in L^2(L^2)$, $g\in H^3\cap H_0^1$ and $w\in H^2\cap H_0^1$. Further, let $u_h(0)=\Pi_h g$ and $u_{h,t}(0)=R_h w$. Then, there exists a positive constant $C(T)$ independent of h, such that*

$$\|u(t)-u_h(t)\|_0\le C(T)h^2\left(\|g\|_3+\|w\|_2+\|f\|_{L^2(H^2)}+\sum_{j=1}^{2}\|\frac{\partial^j f}{\partial t^j}\|_{L^2(L^2)}\right).$$

Proof. The equation (5.19) can be written equivalently as

$$(u_{h,tt},\Pi_h^*\chi)_h+\tilde{a}_h(u_h,\Pi_h^*\chi)=(f,\Pi_h^*\chi)_h\quad\forall\chi\in U_h. \tag{5.20}$$

Subtract (5.20) from (4.9) to obtain

$$\begin{aligned} (u_{tt},\Pi_h^*\chi)-(u_{h,tt},\Pi_h^*\chi)_h \quad &+ \quad a_h(u,\Pi_h^*\chi)-\tilde{a}_h(u_h,\Pi_h^*\chi) \\ &= (f,\Pi_h^*\chi)-(f,\Pi_h^*\chi)_h\quad\forall\chi\in U_h. \end{aligned} \tag{5.21}$$

Use the definition of Ritz projection and ϵ_a, integrate from 0 to t to obtain

$$\begin{aligned}(\theta_t, \Pi_h^*\chi)_h + a(\hat{\theta}, \chi) &= -(\rho_t, \Pi_h^*\chi) + \epsilon_a(\hat{\rho}, \chi) + \epsilon_a(\hat{\theta}, \chi) \\ &\quad + \bar{\epsilon}_a(\hat{\theta}, \chi) - \bar{\epsilon}_a(R_h\hat{u}, \chi) + \bar{\epsilon}_h(\hat{f}, \chi) - \bar{\epsilon}_h(R_h u_t, \chi) \\ &\quad + (u_t(0), \Pi_h^*\chi) - (u_{h,t}(0), \Pi_h^*\chi)_h. \qquad (5.22)\end{aligned}$$

Put $\chi = \theta = \hat{\theta}_t$. Now use (5.6) and symmetry of the bilinear form $a(\cdot,\cdot)$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big[(\theta,\theta)_h + a(\hat{\theta},\hat{\theta})\Big] &= -(\rho_t, \Pi_h^*\theta) + \epsilon_a(\hat{\rho}, \theta) + \epsilon_a(\hat{\theta}, \theta) \\ &\quad + \bar{\epsilon}_a(\hat{\theta}, \theta) - \bar{\epsilon}_a(R_h\hat{u}, \theta) + \bar{\epsilon}_h(\hat{f}, \theta) - \bar{\epsilon}_h(R_h u_t, \theta) \\ &\quad + \left[(u_t(0), \Pi_h^*\theta) - (u_{h,t}(0), \Pi_h^*\theta)_h\right]. \qquad (5.23)\end{aligned}$$

Integrate (5.23) from 0 to t to obtain

$$\begin{aligned}\|\theta(t)\|_h^2 + a(\hat{\theta},\hat{\theta}) &= \|\theta(0)\|_h^2 + 2\int_0^t \Big[(\rho_t, \Pi_h^*\theta) + \epsilon_a(\hat{\rho}, \theta) + \epsilon_a(\hat{\theta}, \theta)\Big]\, ds \\ &\quad + 2\int_0^t \bar{\epsilon}_a(\hat{\theta}, \theta) ds - 2\int_0^t \bar{\epsilon}_a(R_h\hat{u}, \theta) ds - 2\int_0^t \bar{\epsilon}_h(R_h u_t, \theta) ds \\ &\quad + 2\int_0^t \bar{\epsilon}_h(\hat{f}, \theta) ds + \Big[(u_t(0), \Pi_h^*\hat{\theta}) - (u_{h,t}(0), \Pi_h^*\hat{\theta})_h\Big] \\ &= \|\theta(0)\|_h^2 + 2\int_0^t \Big[(\rho_t, \Pi_h^*\theta) + \epsilon_a(\hat{\rho}, \theta) + \epsilon_a(\hat{\theta}, \theta)\Big]\, ds \\ &\quad + J_1 + J_2 + J_3 + J_4 + J_5. \qquad (5.24)\end{aligned}$$

We have already derived other estimates in Theorem 4.1 except J_1 to J_5.
For J_1, use Lemma 5.3 and inverse inequality to obtain

$$\begin{aligned}|J_1| \le 2\int_0^t |\bar{\epsilon}_a(\hat{\theta}, \theta)| ds &\le Ch^2 \int_0^t \|\theta\|_1 \|\hat{\theta}\|_1 ds \\ &\le C\int_0^t \|\theta\|_0 \|\hat{\theta}\|_1 ds. \qquad (5.25)\end{aligned}$$

Now for J_2, use Lemma 5.3 and the stability result $\|R_h u\|_1 \le C\|u\|_1$ to obtain

$$\begin{aligned}|J_2| &\le 2|\bar{\epsilon}_a(R_h\hat{u}, \hat{\theta})| + 2\int_0^t |\bar{\epsilon}_a(R_h u, \hat{\theta})| ds \\ &\le Ch^2\Big(\|\hat{u}\|_1 \|\hat{\theta}\|_1 + \int_0^t \|u\|_1 \|\hat{\theta}\|_1 ds\Big). \qquad (5.26)\end{aligned}$$

To bound J_3 and J_4, an application of Lemma 5.1 yields

$$\begin{aligned}|J_3| &\le 2|\bar{\epsilon}_h(R_h u_t, \hat{\theta})| + 2\int_0^t |\bar{\epsilon}_h(R_h u_{tt}, \hat{\theta})| ds \\ &\le Ch^2\Big(\|u_t\|_1 \|\hat{\theta}\|_1 + \int_0^t \|u_{tt}\|_1 \|\hat{\theta}\|_1 ds\Big) \qquad (5.27)\end{aligned}$$

and

$$\begin{aligned}|J_4| &\le 2|\bar{\epsilon}_h(\hat{f}, \hat{\theta})| + 2\int_0^t |\bar{\epsilon_h}(f, \hat{\theta})| ds \\ &\le Ch^2\Big(\|\hat{f}\|_2 \|\hat{\theta}\|_1 + \int_0^t \|f\|_2 \|\hat{\theta}\|_1 ds\Big). \qquad (5.28)\end{aligned}$$

Since $u_{h,t}(0) = R_h u_t(0)$,

$$\begin{aligned} |J_5| \leq |(u_t(0), \Pi_h^* \hat{\theta}) - (u_{h,t}(0), \Pi_h^* \hat{\theta})_h| &\leq |(u_t(0) - R_h u_t(0), \Pi_h^* \hat{\theta})| + |\bar{\epsilon_h}(R_h u_t(0), \hat{\theta})| \\ &\leq Ch^2 \left(\|u_t(0)\|_2 + \|R_h u_t(0)\|_1\right) \|\hat{\theta}\|_1 \\ &\leq Ch^2 \|u_t(0)\|_2 \|\hat{\theta}\|_1 \leq Ch^2 \|u_t(0)\|_2 \|\hat{\theta}\|_1. \end{aligned} \tag{5.29}$$

Substitute (5.25)-(5.29) in (5.24). Use the coercivity property of the bilinear form $a(\cdot,\cdot)$ and the equivalence of the norms $\|\cdot\|_h$ and $\|\cdot\|_0$. Then as in Theorem 4.1, apply standard kick back arguments with Gronwall's Lemma to complete the rest of the proof. ■

5.2. H^1 error estimate.

Theorem 5.2. *Under the assumptions of Theorem 4.1, there exists a positive constant $C(T)$, independent of h, such that*

$$\|u(t) - u_h(t)\|_1 \leq C(T) h \left(\|g\|_3 + \|w\|_2 + \|f\|_{L^2(H^2)} + \sum_{j=1}^{2} \|\frac{\partial^j f}{\partial t^j}\|_{L^2(L^2)} \right).$$

Proof. Put $\chi = \theta_t$ in (5.21) and use definition of Ritz projection with (4.10) to find that

$$\begin{aligned} (\theta_{tt}, \Pi_h^* \theta_t)_h + a(\theta, \theta_t) &= \epsilon_h(f - u_{tt}, \theta_t) - \epsilon_a(R_h u, \theta_t) \\ &\quad + \epsilon_a(\theta, \theta_t) - (\rho_{tt}, \Pi_h^* \theta_t) + \bar{\epsilon}_a(\theta, \theta_t) - \bar{\epsilon}_a(R_h u, \theta_t) \\ &\quad + \bar{\epsilon}_h(f, \theta_t) - \bar{\epsilon}_h(R_h u_{tt}, \theta_t) \end{aligned} \tag{5.30}$$

Using (5.6) and the symmetric property of $a(\cdot,\cdot)$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[(\theta_t, \theta_t)_h + a(\theta, \theta)\right] &= \epsilon_h(f - u_{tt}, \theta_t) - \epsilon_a(R_h u, \theta_t) \\ &\quad + \epsilon_a(\theta, \theta_t) - (\rho_{tt}, \Pi_h^* \theta_t) + \bar{\epsilon}_a(\theta, \theta_t) - \bar{\epsilon}_a(R_h u, \theta_t) \\ &\quad + \bar{\epsilon}_h(f, \theta_t) - \bar{\epsilon}_h(R_h u_{tt}, \theta_t). \end{aligned}$$

Integrating from 0 to t and using the equivalence of the norms (5.3), we obtain

$$\begin{aligned} \|\theta_t\|_h^2 + a(\theta, \theta) &= \Big\{ \|\theta_t(0)\|_h^2 + a(\theta(0), \theta(0)) + 2\int_0^t \Big[\epsilon_h(f - u_{tt}, \theta_t) - \epsilon_a(R_h u, \theta_t) \\ &\quad + \epsilon_a(\theta, \theta_t) - (\rho_{tt}, \Pi_h^* \theta_t)\Big] ds \Big\} + 2\int_0^t \bar{\epsilon}_a(\theta, \theta_t) ds \\ &\quad - 2\int_0^t \bar{\epsilon}_a(R_h u, \theta_t) ds + 2\int_0^t \bar{\epsilon}_h(f, \theta_t) ds - 2\int_0^t \bar{\epsilon}_h(R_h u_{tt}, \theta_t) ds \\ &= I + J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{5.31}$$

We have already derived estimates for I in Theorem 4.2.
For J_1, use Lemma 5.3 and inverse inequality to obtain

$$|J_1| \leq 2\int_0^t |\bar{\epsilon}_a(\theta,\ \theta_t)| ds \leq Ch^2 \int_0^t \|\theta\|_1 \|\theta_t\|_1 ds \leq C\int_0^t \|\theta\|_1 \|\theta_t\|_0 ds. \tag{5.32}$$

Again from Lemma 5.3 and stability of the Ritz projection, it follows that

$$\begin{aligned} |J_2| &\leq 2\int_0^t |\bar{\epsilon}_a(R_h u, \theta_t)| ds \leq Ch^2 \int_0^t \|u\|_1 \|\theta_t\|_1 ds \\ &\leq Ch \int_0^t \|u\|_1 \|\theta_t\|_0 ds. \end{aligned} \tag{5.33}$$

To bound J_3 and J_4, we use Lemma 5.1 to obtain

$$\begin{aligned} |J_3| + |J_4| &\leq 2\int_0^t |\bar{\epsilon}_h(f,\theta_t)|ds + 2\int_0^t |\bar{\epsilon}_h(R_h u_{tt},\theta_t)|ds \\ &\leq Ch^2\int_0^t (\|f\|_2 + \|u_{tt}\|_1)\|\theta_t\|_1 ds \\ &\leq Ch\int_0^t (\|f\|_2 + \|u_{tt}\|_1)\|\theta_t\|_0 ds. \end{aligned} \tag{5.34}$$

Substitute (5.32)-(5.34) in (5.31). We use the coercivity property of the bilinear form $a(\cdot,\cdot)$ and equivalence of norms. Then proceed as in Theorem 4.2 to complete the rest of the proof. ■

5.3. Maximum norm estimate.

Theorem 5.3. *Let u and u_h be the solutions of (1.1) and (3.1) respectively. Further, let the assumptions of the Lemma 4.3 hold. Then,*

$$\begin{aligned} \|u(t) - u_h(t)\|_\infty &\leq C(T)h^2 log\frac{1}{h}\Big(\|u\|_{2,\infty} + \|g\|_4 + \|w\|_3 + \|\frac{\partial^3 f}{\partial t^3}\|_{L^2(L^2)} \\ &\quad + \sum_{j=0}^{2}\|\frac{\partial^j f}{\partial t^j}\|_{L^2(H^{2-j})}\Big), \end{aligned} \tag{5.35}$$

where $C(T)$ is a positive constant, independent of h.

Proof: Since $u_h(0) = R_h u(0)$, then $\theta(0) = 0$. Now we modify our estimates for J_2 to J_4 in (5.31) to obtain a superconvergence result for θ in H^1- norm.
For J_2, use Lemma 5.3 to find that

$$\begin{aligned} |J_2| &\leq 2|\bar{\epsilon}_a(R_h u,\theta)| + 2\int_0^t |\bar{\epsilon}_a(R_h u_t,\theta)|ds \\ &\leq Ch^2\Big(\|u\|_1\|\theta\|_1 + \int_0^t \|u_t\|_1\|\theta\|_1 ds\Big). \end{aligned} \tag{5.36}$$

For J_3 and J_4, using Lemma 5.1, we obtain

$$\begin{aligned} |J_3| &\leq 2|\bar{\epsilon}_h(f,\theta)| + 2\int_0^t |\bar{\epsilon}_h(f_t,\theta)|ds \\ &\leq Ch^2\left(\|f\|_2\|\theta\|_1 + \int_0^t \|f_t\|_2\|\theta\|_1 ds\right). \end{aligned} \tag{5.37}$$

and

$$\begin{aligned} |J_4| &\leq 2|\bar{\epsilon}_h(R_h u_{tt},\theta)| + 2\int_0^t |\bar{\epsilon}_h(R_h u_{ttt},\theta)|ds \\ &\leq Ch^2\left(\|u_{tt}\|_1\|\theta\|_1 + \int_0^t \|u_{ttt}\|_1\|\theta\|_1 ds\right) \end{aligned} \tag{5.38}$$

Substitute (5.36)-(5.38) in (5.31) and use the coercivity property of the bilinear form $a(\cdot,\cdot)$ with equivalence of norms (5.3). Then proceed as in Lemma 4.3 to arrive at

$$\begin{aligned} \|\theta_t\|_0^2 + \|\theta\|_1^2 &\leq C(T)h^4\Big(\|g\|_4^2 + \|w\|_3^2 + \|\frac{\partial^3 f}{\partial t^3}\|_{L^2(L^2)}^2 \\ &\quad + \sum_{j=0}^{2}\|\frac{\partial^j f}{\partial t^j}\|_{L^2(H^{2-j})}^2\Big). \end{aligned} \tag{5.39}$$

Use (4.31) and (5.39) to complete the rest of the proof. ■

6. Numerical Experiment

In this section, we discuss a numerical result to illustrate the performance of the finite volume element method applied to (1.1) by taking an example.
Choose $g(x,y) = xy(x-1)(y-1)$, $w(x,y) = xy(x-1)(y-1)$, $A = \begin{pmatrix} 1+x^2 & 0 \\ 0 & 1+x^2 \end{pmatrix}$ and $\Omega = (0,1)\times(0,1)$. The load function f is chosen so that the exact solution is $u = e^t xy(x-1)(y-1)$.
Let T_h be an admissible regular, uniform triangulation of $\overline{\Omega}$ into closed triangles and $0 = t_0 < t_1 < \cdots t_M = T$ be a given partition of the time interval $(0,T]$ with step length $\Delta t = \frac{T}{M}$ for some positive integer M. Let U^n denote the approximation of u_h at $t = t_n$. Set $\partial_t U^n = \dfrac{U^{n+1}-U^n}{\Delta t}$ and $\bar{\partial}_t U^n = \dfrac{U^n - U^{n-1}}{\Delta t}$. Then, the time discretization scheme is defined as:
Given U^0 and U^1, find $U^n \in U_h$, such that

$$(\partial_t \bar{\partial}_t U^n, \Pi_h^* \chi)_h + \tilde{a}_h(U^n, \Pi_h^*\chi) = (f^n, \Pi_h^*\chi)_h \quad \forall \chi \in U_h. \tag{6.1}$$

Let $\{\phi_j\}_{j=1,2,\cdots,N}$ be the standard nodal basis functions for the trial space U_h and $\{\chi_j\}_{j=1,2,\cdots,N}$ be the characteristic functions corresponding to the control volumes which form basis functions for the test space V_h. U^n can be expressed as

$$U^n = \sum_{j=1}^{N} \alpha_j^n \phi_j(x), \quad \text{where } \alpha_j^n = U^n(x_j).$$

Then, (6.1) can be written as the following system of linear equations which can be solved for $\bar{\alpha}^n$.

$$A\bar{\alpha}^{n+1} = F^n - B\bar{\alpha}^n - A\bar{\alpha}^{n-1},$$

Here $\bar{\alpha}^n = (\alpha_1^n,\ \alpha_2^n, \cdots, \alpha_N^n)^t$, $A = (\phi_j, \chi_j)_h$, $C = (\Delta t)^2 \tilde{a}_h(\phi_j, \chi_j)$, $B = -2A + C$ and $F^n = (\Delta t)^2 (f^n, \chi_j)_h$.
The order of convergence is computed in L^∞- norm. FIGURE 3 shows that the computed order of convergence for $\|u - u_h\|_\infty$ in the log-log scale matches with the theoretical order of convergence derived in Theorem 5.3.

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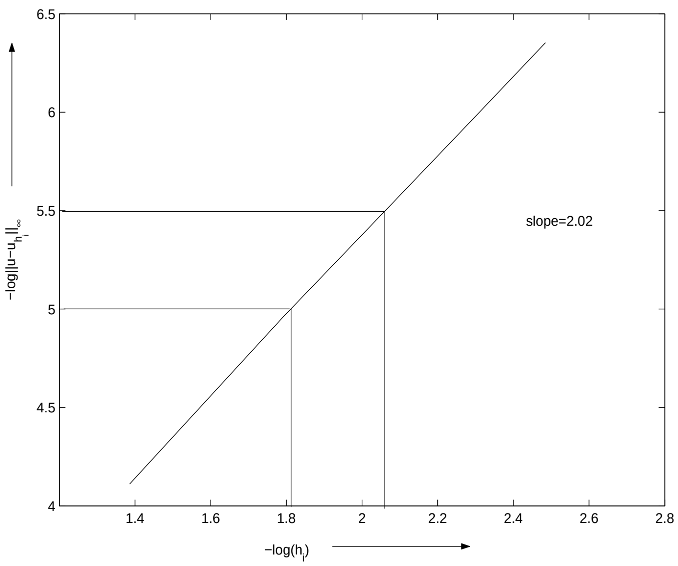

FIGURE 3. Convergence order estimate in L_∞ norm

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