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A UNIFORMLY OPTIMAL-ORDER ESTIMATE FOR BILINEAR FINITE ELEMENT METHOD FOR TRANSIENT ADVECTION-DIFFUSION EQUATIONS

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Abstract. We prove an optimal-order error estimate in a weighted energy norm for bilinear Galerkin finite element method for two-dimensional time-dependent advection-diffusion equations by the means of integral identities or expansions, in the sense that the generic constants in the estimates depend only on certain Sobolev norms of the true solution but not on the scaling parameter ε. These estimates, combined with a priori stability estimates of the governing partial differential equations, yield an ε-uniform estimate of the bilinear Galerkin finite element method, in which the generic constants depend only on the Sobolev norms of the initial and right data but not on the scaling parameter ε.

Key Words. convergence analysis, Galerkin methods, integral identity, integral expansion, uniform error estimates

1. Introduction

Time-dependent advection-diffusion equations, whih arise in mathematical models of petroleum reservoir simulation, environmental modeling, and other applications [3, 12], admit solutions with moving fronts and complex structures, and present serious mathematical and numerical difficulties [9, 13]. Many numerical methods have been developed to solve these problems and corresponding optimal-order convergence rates were proved [1, 5, 6, 9, 13, 14, 15, 18, 24]. However, these estimates have the major drawback that the generic constants in these estimates depend inversely on the scaling parameter ε, and so could blow up as ε tends to zero.

ε uniform estimates have been sought to address these issues and some progress has been made [13]. In the context of time-dependent advection-diffusion equations, suboptimal- and optimal-order ε uniform estimates were obtained primarily for Eulerian-Lagrangian methods [2, 19, 20, 21, 22, 23]. In essence, an ε uniform estimate is somewhat a restatement that the estimate is independent of the Peclet number. Eulerian-Lagrangian methods combine the advection and capacity terms to reformulate the governing equation as a parabolic equation in the Lagrangian coordinate to carry out the temporal discretization [6, 16, 17]. Thus, the corresponding Peclet number is formally zero. This explains why ε uniform estimates were proved only for Eulerian-Lagrangian methods, even if these methods are much more complex to analyze.

In this paper we prove an ε-uniform optimal-order error estimate for the bilinear Galerkin finite element method for time-dependent advection-diffuson equations,

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which, to the best knowledge of the authors, is the first work of this type. The primary advantage of Galerkin method resides in the simplicity of the implementation of the method. Due to the use of a standard temporal discretization, the advection term must be analyzed with care to ensure the impact of the Peclet number to be handled properly.

The rest of this paper is organized as follows. In §2 we recall preliminary results that are to be used in the paper. In §3 we revisit the problem formulation and approximation properties that are to be used in the analysis. In §4 we prove ε-uniform optimal-order error estimate for the problem. In §5 we prove auxiliary lemmas. §6 contains concluding remarks.

2. Problem formulation and Preliminaries

We consider a time-dependent advection-diffusion equation in two space dimensions

$$\begin{array}{rclcrcl} u_t + \nabla\cdot\big(\mathbf{v}(\mathbf{x},t)u - \varepsilon\mathbf{D}(\mathbf{x},t)\nabla u\big) &=& f(\mathbf{x},t), & (\mathbf{x},t) &\in& \Omega\times(0,T] \\ u(\mathbf{x},0) &=& u_o(\mathbf{x}), & \mathbf{x} &\in& \Omega. \end{array} \tag{2.1}$$

Here $\Omega=(a,b)\times(c,d)$ is a rectangular domain, $\mathbf{x}=(x,y)$, $\mathbf{v}(\mathbf{x},t)=(V_1(\mathbf{x},t),V_2(\mathbf{x},t))$ is a velocity field, $f(\mathbf{x},t)$ accounts for external sources and sinks, $u_o(\mathbf{x})$ is a prescribed initial data, $\mathbf{D}(\mathbf{x},t)=\big(D_{ij}(\mathbf{x},t)\big)_{i,j=1}^2$ is a diffusion-dispersion tensor that has uniform lower and upper bounds $0<D_{min}|\boldsymbol{\alpha}|^2\le\boldsymbol{\alpha}^T\mathbf{D}(\mathbf{x},t)\boldsymbol{\alpha}\le D_{max}|\boldsymbol{\alpha}|^2<+\infty$ for any $\boldsymbol{\alpha}\in\mathbf{R}^2$ and $(\mathbf{x},t)\in\Omega\times[0,T]$. Here $0<\varepsilon<<1$ is a parameter that scales the diffusion and characterizes the advection-dominance of Eq. (2.1), and $u(\mathbf{x},t)$ is the ε-dependent unknown function. Finally, problem (2.1) is closed by a boundary condition. Differential types of boundary conditions are considered in this paper, including a (homogeneous) Dirichlet boundary condition

$$u(\mathbf{x},t)=0, \qquad (\mathbf{x},t)\in\Gamma\times[0,T] \tag{2.2}$$

where $\Gamma:=\partial\Omega$ is the spatial boundary of Ω as well as a noflow boundary condition [3, 12] which describes an impermeable boundary and is characterized by $\mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})=0$. On the noflow boundary Γ a homogeneous diffusive flux boundary condition is imposed

$$-(\mathbf{D}\nabla u)(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})=0, \qquad (\mathbf{x},t)\in\Gamma\times[0,T]. \tag{2.3}$$

This type of boundary condition often arizes in applications such as petroleum reservoir simulation. Finally, a periodic boundary condition is also considered in this paper [12].

Let $W_p^k(\Omega)$ consist of functions whose weak derivatives up to order-k are p-th Lebesgue integrable in Ω, and $H^k(\Omega):=W_2^k(\Omega)$. Let $H_0^1(\Omega):=\Big\{v\in H^1(\Omega): v(\mathbf{x})=0, \quad \mathbf{x}\in\Gamma\Big\}$, and $H_E^m(\Omega)$ be the subspace of $H^m(\Omega)$, which consists of functions that are periodic with respect to the domain Ω. We also introduce the energy norm $\|f(\cdot,t)\|_{H_D^1(\Omega)}:=\big(\int_\Omega\nabla f(\mathbf{x},t)\cdot\mathbf{D}(\mathbf{x},t)\nabla f(\mathbf{x},t)d\mathbf{x}\big)^{\frac{1}{2}}$.

For any Banach space X we introduce Sobolev spaces involving time [7]

$$
\begin{aligned}
W_p^k(t_1,t_2;X) \quad &:= \Big\{ f : \Big\| \frac{\partial^l f}{\partial t^l}(\cdot,t) \Big\|_X \in L^p(t_1,t_2),\ 0 \le l \le k,\ 1 \le p \le \infty \Big\}, \\
\|f\|_{W_p^k(t_1,t_2;X)} \quad &:= \begin{cases} \Big(\sum_{l=0}^{k} \int_{t_1}^{t_2} \Big\| \frac{\partial^l f}{\partial t^l}(\cdot,t) \Big\|_X^p dt \Big)^{1/p}, & 1 \le p < \infty, \\ \max_{0 \le l \le k} \operatorname{ess} \sup_{t \in (t_1,t_2)} \Big\| \frac{\partial^l f}{\partial t^l}(\cdot,t) \Big\|_X, & p = \infty. \end{cases}
\end{aligned}
$$

To define discrete norms, we define a uniform space-time partition on $\overline{\Omega} \times [0,T]$ by $x_i := a + i\Delta x$ for $i = 0,1,\dots,I$ with $\Delta x := (b-a)/I$; $y_j := c + j\Delta y$ for $j = 0,1,\dots,J$ with $\Delta y := (d-c)/J$; and $t_n := n\Delta t$ for $0 \le n \le N$ with $\Delta t := T/N$. We assume that $\Delta y/\Delta x$ has positive lower and upper bounds that are independent of the partition. We also define $h := \max\{\Delta x, \Delta y\}$.

$$
\begin{aligned}
\|f(\cdot,t)\|_{\hat{H}_D^1(\Omega)} \quad &:= \Big(\sum_{j=1}^{J} \sum_{i=1}^{I} (\mathbf{D}\nabla f)(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}, t) \cdot \nabla f(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}, t) \Delta x \Delta y \Big)^{\frac{1}{2}}, \\
|||f|||_{L_\varepsilon(0,T;\hat{H}_D^1(\Omega))} \quad &:= \|f\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\varepsilon}\|f\|_{L^2(0,T;\hat{H}_D^1(\Omega))}
\end{aligned}
$$

where $x_{i-\frac{1}{2}} := (x_{i-1} + x_i)/2$ and $y_{j-\frac{1}{2}} := (y_{j-1} + y_j)/2$.

3. Galerkin method and approximation properties

In this section we present the Galerkin formulation and study its approximation estimates that will be used in the proof of the main theorem.

3.1. Weak formulation and Galerkin method. Let $H = H_0^1(\Omega)$, $H^1(\Omega)$, or $H_E^1(\Omega)$ in the context of Dirichlet boundary condition, noflow boundary condition, or periodic boundary condition, respectively. We multiply the advection-diffusion term in (2.1) by any test function $w(\mathbf{x}) \in H$, and integrate it on Ω by parts to get

$$
\begin{aligned}
&\int_\Omega \nabla \cdot \big(\mathbf{v}(\mathbf{x},t)u - \varepsilon \mathbf{D}(\mathbf{x},t)\nabla u \big) w(\mathbf{x}) d\mathbf{x} \\
&\quad = - \int_\Omega \big(\mathbf{v}(\mathbf{x},t)u - \varepsilon \mathbf{D}(\mathbf{x},t)\nabla u \big) \cdot \nabla w(\mathbf{x}) d\mathbf{x} \\
&\qquad + \int_\Gamma \big(\mathbf{v}(\mathbf{x},t)u - \varepsilon \mathbf{D}(\mathbf{x},t)\nabla u \big) \cdot \mathbf{n}(\mathbf{x}) w(\mathbf{x}) d\mathbf{x}.
\end{aligned} \tag{3.4}
$$

The test function $w(\mathbf{x})$ vanishes on Γ for the Dirichlet boundary condition (2.2), while the noflow boundary condition (2.3) implies that the integrand vanishes. Finally, the boundary term naturally cancels for the periodic boundary condition. In short, in all the cases, the second term on the right side vanishes.

We approximate the time derivative by backward Euler difference quotient, multiply the governing equation (2.1) by any test function $w(\mathbf{x}) \in H$ and integrate the resulting equation on Ω. Then we combine it with (3.4), leading to a weak formulation for problem (2.1): For $n = 1,2,\cdots,N$, seek $u(\mathbf{x},t_n) \in H$ such that for any $w \in H$,

$$
\begin{aligned}
&\int_\Omega \frac{u(\mathbf{x},t_n) - u(\mathbf{x},t_{n-1})}{\Delta t} w(\mathbf{x}) d\mathbf{x} - \int_\Omega \mathbf{v}(\mathbf{x},t_n) u(\mathbf{x},t_n) \cdot \nabla w(\mathbf{x}) d\mathbf{x} \\
&\quad + \varepsilon \int_\Omega \mathbf{D}(\mathbf{x},t_n) \nabla u(\mathbf{x},t_n) \cdot \nabla w(\mathbf{x}) d\mathbf{x} \\
&= \int_\Omega f(\mathbf{x},t_n) w(\mathbf{x}) d\mathbf{x} - \frac{1}{\Delta t} E(u,w).
\end{aligned} \tag{3.5}
$$

Here $E(u,w)$ is the local truncation error of the weak formulation

$$E(u,w) = \int_\Omega w(\mathbf{x}) \int_{t_{n-1}}^{t_n} (t - t_{n-1}) \frac{\partial^2 u}{\partial t^2}(\mathbf{x},t) dt d\mathbf{x}.$$

Let $S_h(\Omega) \in H$ be the space of continuous and piecewise-bilinear functions with respect to the spatial partition which is subject to the appropriate boundary condition. The Galerkin method reads: Find $u_h(\cdot, t_n) \in S_h(\Omega)$ for $n = 1, \ldots, N$, such that for any $w_h \in S_h(\Omega)$

$$\begin{aligned}
&\int_\Omega \frac{u_h(\mathbf{x},t_n) - u_h(\mathbf{x},t_{n-1})}{\Delta t} w_h(\mathbf{x}) d\mathbf{x} - \int_\Omega \mathbf{v}(\mathbf{x},t_n) u_h(\mathbf{x},t_n) \cdot \nabla w_h(\mathbf{x}) d\mathbf{x} \\
&\quad + \varepsilon \int_\Omega \mathbf{D}(\mathbf{x},t_n) \nabla u_h(\mathbf{x},t_n) \cdot \nabla w_h(\mathbf{x}) d\mathbf{x} \\
&\quad = \int_\Omega f(\mathbf{x},t_n) w_h(\mathbf{x}) d\mathbf{x}.
\end{aligned} \tag{3.6}$$

3.2. Approximation properties. Let $\Pi_h v \in S_h(\Omega)$ be the piecewise-bilinear interpolation of v for any $v \in C(\overline{\Omega})$. The following estimates are well known [4]

$$\begin{aligned}
\|\Pi_h v - v\|_{H^k(\Omega)} &\le C_1 h^{2-k} \ \|v\|_{H^2(\Omega)} \quad \forall v \in H^2(\Omega), \ k = 0, 1, \\
\|v_h\|_{H^1(\Omega)} + \|v_h\|_{L^\infty(\Omega)} &\le C_2 h^{-1} \|v_h\|_{L^2(\Omega)} \quad \forall v_h \in S_h(\Omega).
\end{aligned} \tag{3.7}$$

However, as we shall see in subsequent sections, these approximation properties are not enough for the derivation of an ε-uniform estimate. Fortunately we have the following superconvergence estimate [11].

Lemma 3.1. *Let $\Omega_{i,j} := (x_{i-1}, x_i) \times (y_{j-1}, y_j)$. The following estimates holds for any bilinear function z_h on $\Omega_{i,j}$ and any $v \in H^3(\Omega_{i,j})$*

$$\begin{aligned}
&\int_{\Omega_{i,j}} (v - \Pi_h v) z_{hx} dxdy \\
&\quad = -\frac{(\Delta x)^2}{12} \int_{\Omega_{i,j}} v_{xx} z_{hx} dxdy - \frac{(\Delta y)^2}{12} \int_{\Omega_{i,j}} v_{yy} z_{hx} dxdy \\
&\quad + O(h^2) |v|_{H^3(\Omega_{i,j})} \|z_h\|_{L^2(\Omega_{i,j})}, \\
&\int_{\Omega_{i,j}} (v - \Pi_h v) z_{hy} dxdy \\
&\quad = -\frac{(\Delta x)^2}{12} \int_{\Omega_{i,j}} v_{xx} z_{hy} dxdy - \frac{(\Delta y)^2}{12} \int_{\Omega_{i,j}} v_{yy} z_{hy} dxdy \\
&\quad + O(h^2) |v|_{H^3(\Omega_{i,j})} \|z_h\|_{L^2(\Omega_{i,j})},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\int_{\Omega_{i,j}} (v - \Pi_h v)_x z_{hx} dxdy &= O(h^2) |v|_{H^3(\Omega_{i,j})} \|z_{hx}\|_{L^2(\Omega_{i,j})}, \\
\int_{\Omega_{i,j}} (v - \Pi_h v)_y z_{hy} dxdy &= O(h^2) |v|_{H^3(\Omega_{i,j})} \|z_{hy}\|_{L^2(\Omega_{i,j})},
\end{aligned} \tag{3.9}$$

and

$$
\begin{aligned}
&\int_{\Omega_{i,j}} (v - \Pi_h v)_x z_{hy} dxdy \\
&\qquad = \frac{(\Delta x)^2}{12} \int_{\Omega_{i,j}} v_{xx} z_{hxy} dxdy + O(h^2)|v|_{H^3(\Omega_{i,j})}|z_h|_{H^1(\Omega_{i,j})}, \\
&\int_{\Omega_{i,j}} (v - \Pi_h v)_y z_{hx} dxdy \\
&\qquad = \frac{(\Delta y)^2}{12} \int_{\Omega_{i,j}} v_{yy} z_{hxy} dxdy + O(h^2)|v|_{H^3(\Omega_{i,j})}|z_h|_{H^1(\Omega_{i,j})}.
\end{aligned} \tag{3.10}
$$

In this paper we use C to denote a general positive constant which could assume different values at different occurrences.

4. An Optimal-Order Error Estimate for the Galerkin methods

Theorem. Assume $u \in H^2(0,T;L^2) \cap H^1(0,T;H^2) \cap L^\infty(0,T;H^3)$ and the coefficients are smooth. Then the following superconvergence estimate holds uniformly with respect to ε for problem (2.1) with a homogeneous Dirichlet boundary condition (2.2) or a periodic boundary condition

$$
\begin{aligned}
&|||u_h - u|||_{L_\varepsilon(0,T;\hat{H}^1_D)} \\
&\qquad \le C\Delta t \|u\|_{H^2(0,T;L^2)} + Ch^2\big(\|u\|_{H^1(0,T;H^2)} + \|u\|_{L^\infty(0,T;H^3)}\big).
\end{aligned} \tag{4.11}
$$

Here the constant C is independent of u, ε, h, or Δt. Further, if the diffusion tensor $\mathbf{D}$ is diagonal, then (4.11) also holds for the noflow boundary condition (2.3).

Remark. The estimate (4.11), combined with a priori stability estimate of the governing equation (2.1) [7, 19], yields an ε-uniform estimate of the bilinear Galerkin finite element method, in which the generic constants depend only on certain Sobolev norms of the initial and right data but not on the scaling parameter ε.

Proof. We let $e = u_h - u$, and choose $w(\mathbf{x})$ in the reference equation (3.5) to be $w_h(\mathbf{x}) \in S_h(\Omega)$. We then subtract (3.5) from the Galerkin formulation (3.6) to get an error equation for any $w_h(\mathbf{x}) \in S_h(\Omega)$

$$
\begin{aligned}
&\int_\Omega e(\mathbf{x}, t_n) w_h(\mathbf{x}) d\mathbf{x} + \varepsilon \Delta t \int_\Omega \mathbf{D}(\mathbf{x}, t_n) \nabla e(\mathbf{x}, t_n) \cdot \nabla w_h(\mathbf{x}) d\mathbf{x} \\
&\qquad -\Delta t \int_\Omega \mathbf{v}(\mathbf{x}, t_n) e(\mathbf{x}, t_n) \cdot \nabla w_h(\mathbf{x}) d\mathbf{x} \\
&\qquad = \int_\Omega e(\mathbf{x}, t_{n-1}) w_h(\mathbf{x}) d\mathbf{x} + E(u, w_h).
\end{aligned} \tag{4.12}
$$

The standard technique for deriving optimal-order error estimates for Galerkin finite element methods for parabolic equations is using a Ritz projection [14, **?**] $R_h u$ of the true solution u of problem (2.1). This would split the global truncation error $e = u_h - u$ as $e = (u_h - R_h u) + (R_h u - u)$. However, the upper bound of $R_h u - u$ would depend inversely on ε, which could blow up as ε tends to zero. To derive an ε-uniform estimate we decompose the global truncation error e as $e = \xi_h + \eta$, where $\xi_h = u_h - \Pi u$ and $\eta = \Pi u - u$ with $\Pi u \in S_h(\Omega)$ being the piecewise bilinear interpolation of u.

The estimate for η is given in (3.7), so we need only to estimate ξ_h. We choose $w_h(\mathbf{x}) = \xi_h(\mathbf{x}, t_n)$ and rewrite the error equation in terms of ξ_h and η as follows:

$$\begin{aligned}
&\int_\Omega \xi_h^2(\mathbf{x}, t_n) d\mathbf{x} + \varepsilon \Delta t \int_\Omega \mathbf{D}(\mathbf{x}, t_n) \nabla \xi_h(\mathbf{x}, t_n) \cdot \nabla \xi_h(\mathbf{x}, t_n) d\mathbf{x} \\
&\quad = \int_\Omega \xi_h(\mathbf{x}, t_{n-1}) \xi_h(\mathbf{x}, t_n) d\mathbf{x} + \int_\Omega \eta(\mathbf{x}, t_{n-1}) \xi_h(\mathbf{x}, t_n) d\mathbf{x} \\
&\quad\quad - \int_\Omega \eta(\mathbf{x}, t_n) \xi_h(\mathbf{x}, t_n) d\mathbf{x} + \Delta t \int_\Omega \mathbf{v}(\mathbf{x}, t_n) \xi_h(\mathbf{x}, t_n) \cdot \nabla \xi_h(\mathbf{x}, t_n) d\mathbf{x} \\
&\quad\quad + \Delta t \int_\Omega \mathbf{v}(\mathbf{x}, t_n) \eta(\mathbf{x}, t_n) \cdot \nabla \xi_h(\mathbf{x}, t_n) d\mathbf{x} \\
&\quad\quad - \varepsilon \Delta t \int_\Omega \mathbf{D}(\mathbf{x}, t_n) \nabla \eta(\mathbf{x}, t_n) \cdot \nabla \xi_h(\mathbf{x}, t_n) d\mathbf{x} + E(u, \xi_h).
\end{aligned} \tag{4.13}$$

We need only to estimate the right side of (4.13) term by term. The first term on the right side of (4.13) can be bounded by Cauchy inequality.

$$\Big| \int_\Omega \xi_h(\mathbf{x}, t_{n-1}) \xi_h(\mathbf{x}, t_n) d\mathbf{x} \Big| \le \frac{1}{2} \|\xi_h(\cdot, t_n)\|_{L^2}^2 + \frac{1}{2} \|\xi_h(\cdot, t_{n-1})\|_{L^2}^2.$$

The second and third terms on the right-hand side of Eq. (4.13) are bounded by

$$\begin{aligned}
&\Big| \int_\Omega (\eta(\mathbf{x}, t_n) - \eta(\mathbf{x}, t_{n-1})) \xi_h(\mathbf{x}, t_n) d\mathbf{x} \Big| \\
&\quad = \Big| \int_\Omega \int_{t_{n-1}}^{t_n} \eta_t(\mathbf{x}, t) dt \xi_h(\mathbf{x}, t_n) d\mathbf{x} \Big| \\
&\quad \le C \Delta t \|\xi_h(\cdot, t_n)\|^2 + C h^4 \|u\|_{H^1(t_{n-1}, t_n; H^2)}^2.
\end{aligned}$$

We rewrite the fourth term on the right side of (4.13), integrate it by parts, and incorporate one of the Dirichlet, noflow, or periodic boundary conditions to obtain

$$\begin{aligned}
&\Big| \Delta t \int_\Omega \mathbf{v}(\mathbf{x}, t_n) \xi_h(\mathbf{x}, t_n) \cdot \nabla \xi_h(\mathbf{x}, t_n) d\mathbf{x} \Big| \\
&= \frac{\Delta t}{2} \Big| \int_\Omega \mathbf{v}(\mathbf{x}, t_n) \cdot \nabla \xi_h^2(\mathbf{x}, t_n) d\mathbf{x} \Big| \\
&= \frac{\Delta t}{2} \Big| \int_\Gamma \mathbf{v}(\mathbf{x}, t_n) \cdot \mathbf{n}(\mathbf{x}) \xi_h^2(\mathbf{x}, t_n) d\mathbf{x} - \int_\Omega \nabla \cdot \mathbf{v}(\mathbf{x}, t_n) \xi_h^2(\mathbf{x}, t_n) d\mathbf{x} \Big| \\
&= \frac{\Delta t}{2} \Big| \int_\Omega \nabla \cdot \mathbf{v}(\mathbf{x}, t_n) \xi_h^2(\mathbf{x}, t_n) d\mathbf{x} \Big| \\
&\le C \Delta t \|\xi_h(\cdot, t_n)\|^2.
\end{aligned}$$

A standard estimate of the fifth term on the right-hand side of (4.13) yields

$$\begin{aligned}
&\Big| \Delta t \int_\Omega \mathbf{v}(\mathbf{x}, t_n) \eta(\mathbf{x}, t_n) \cdot \nabla \xi_h(\mathbf{x}, t_n) d\mathbf{x} \Big| \\
&\quad \le C \Delta t \|\xi_h(\cdot, t_n)\|_{H^1} \|\eta(\cdot, t_{n-1})\|_{L^2} \\
&\quad \le C \Delta t \ h^2 \|\xi_h(\cdot, t_n)\|_{H^1} \|u(\cdot, t_{n-1})\|_{H^2} \\
&\quad \le C \Delta t \ h \|\xi_h(\cdot, t_n)\|_{L^2} \|u(\cdot, t_{n-1})\|_{H^2} \\
&\quad \le C \Delta t \|\xi_h(\cdot, t_n)\|_{L^2}^2 + C \Delta t \ h^2 \|u(\cdot, t_{n-1})\|_{H^2}^2.
\end{aligned}$$

This will result in a suboptimal-order estimate of order $O(h + \Delta t)$ for the Galerkin method. To derive optimal order error estimate, we need to estimate this term more carefully. We utilize the superconvergence results in Lemma 3.1 to prove the following estimate in Lemma 5.1

$$
\begin{aligned}
&\Big|\Delta t \int_\Omega \mathbf{v}(\mathbf{x},t_n)\eta(\mathbf{x},t_n)\cdot\nabla\xi_h(\mathbf{x},t_n)d\mathbf{x}\Big| \\
&\quad\le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2} + C\Delta t\, h^4\|u\|^2_{L^\infty(0,T;H^3)}
\end{aligned}
\tag{4.14}
$$

We now turn to the sixth term on the right side of Eq. (4.13). If η were the error of Ritz projection, this term would have vanished naturally. In the current context this term does not vanish. Fortunately, we can utilize Lemma 3.1 to prove the following result in Lemma 5.2

$$
\begin{aligned}
&\Big|\varepsilon\Delta t \int_\Omega \mathbf{D}(\mathbf{x},t_n)\nabla\eta(\mathbf{x},t_n)\cdot\nabla\xi_h(\mathbf{x},t_n)d\mathbf{x}\Big| \\
&\quad\le \delta\varepsilon\Delta t\|\xi_h(\cdot,t_n)\|^2_{H^1_D} + C\varepsilon\Delta t\, h^4\|u\|^2_{L^\infty(0,T;H^3)}.
\end{aligned}
\tag{4.15}
$$

We use the expression of $E(u,\xi_h)$ (below (3.5)) to bound this term by

$$
\begin{aligned}
\big|E(u,\xi_h)\big| &\le C(\Delta t)^{3/2}\|\xi_h(\cdot,t_n)\|_{L^2}\|u\|_{H^2(t_{n-1},t_n;L^2)} \\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2} + C(\Delta t)^2\|u\|^2_{H^2(t_{n-1},t_n;L^2)}.
\end{aligned}
$$

We incorporate the preceding estimates into (4.13) to obtain

$$
\begin{aligned}
&\|\xi_h(\cdot,t_n)\|^2_{L^2} + \varepsilon\Delta t\|\xi_h(\cdot,t_n)\|^2_{H^1_D} \\
&\le \frac{1+C\Delta t}{2}(\|\xi_h(\cdot,t_n)\|^2_{L^2} + \|\xi_h(\cdot,t_{n-1})\|^2_{L^2}) \\
&\quad+\frac{1}{2}\varepsilon\Delta t\|\xi_h(\cdot,t_n)\|^2_{H^1_D} + C(\Delta t)^2\|u\|_{H^2(t_{n-1},t_n;L^2)} \\
&\quad+Ch^4(\|u\|^2_{H^1(t_{n-1},t_n;H^2)} + \Delta t\|u\|^2_{L^\infty(0,T;H^3)}).
\end{aligned}
$$

We sum the estimate for $n=1,\dots,N_1(\le N)$ and cancel like terms to obtain

$$
\begin{aligned}
&\|\xi_h(\cdot,t_{N_1})\|^2_{L^2} + \varepsilon\Delta t\sum_{n=1}^{N_1}\|\xi_h(\cdot,t_n)\|^2_{H^1_D} \\
&\le C\Delta t\sum_{n=0}^{N_1-1}\|\xi_h(\cdot,t_n)\|^2_{L^2} + C(\Delta t)^2\|u\|_{H^2(0,T;L^2)} \\
&\quad+Ch^4\big(\|u\|^2_{H^1(0,T;H^2)} + \|u\|^2_{L^\infty(0,T;H^3)}\big).
\end{aligned}
$$

We apply Gronwall inequality to conclude

$$
\begin{aligned}
&|||\xi_h|||_{L_\varepsilon(0,T;\hat{H}^1_D)} \\
&\quad\le C\Delta t\|u\|_{H^2(0,T;L^2)} + Ch^2\big(\|u\|_{H^1(0,T;H^2)} + \|u\|_{L^\infty(0,T;H^3)}\big).
\end{aligned}
\tag{4.16}
$$

We combine (4.16) with (3.7) to finish the proof.

5. Auxiliary Estimate on η

In this section we utilize Lemma 3.1 to prove the two superconvergence estimates on η that were used in the proof of the main theorem.

Lemma 5.1. *Assume $\mathbf{v}\in L^\infty(0,T;W^1_\infty)$ and $u\in L^\infty(0,T;H^3)$. Then the superconvergence estimate (4.14) holds.*

Proof. We decompose the fifth term on the right side of (4.13) as follows:

$$\begin{aligned}&\Delta t\int_\Omega \mathbf{v}(\mathbf{x},t_n)\eta(\mathbf{x},t_n)\cdot\nabla\xi_h(\mathbf{x},t_n)d\mathbf{x}\\&\quad=\Delta t\int_\Omega V_1(x,y,t_n)\xi_{hx}(x,y,t_n)\eta(x,y,t_n)dxdy\\&\qquad+\Delta t\int_\Omega V_2(x,y,t_n)\xi_{hy}(x,y,t_n)\eta(x,y,t_n)dxdy.\end{aligned}$$

By symmetry we need only to estimate the first term on the right side. Let $\bar{V}_1(x,y,t_n) := \frac{1}{\Delta x\Delta y}\int_{\Omega_{i,j}} V_1(x,y,t_n)dxdy$ be the cell average of V. We rewrite the first term on the right side by

$$\begin{aligned}&\Delta t\int_\Omega V_1(x,y,t_n)\xi_{hx}(x,y,t_n)\eta(x,y,t_n)dxdy\\&\quad=\Delta t\int_\Omega \bar{V}_1(x,y,t_n)\xi_{hx}(x,y,t_n)\eta(x,y,t_n)dxdy\\&\qquad+\Delta t\int_\Omega (V_1(x,y,t_n)-\bar{V}_1(x,y,t_n))\xi_{hx}(x,y,t_n)\eta(x,y,t_n)dxdy.\end{aligned}\tag{5.17}$$

We use the inverse estimate (3.7) to bound the second term by

$$\begin{aligned}&\left|\Delta t\int_\Omega (V_1(x,y,t_n)-\bar{V}_1(x,y,t_n))\xi_{hx}(x,y,t_n)\eta(x,y,t_n)dxdy\right|\\&\quad\le C\Delta t h^3\|\xi_{hx}(\cdot,t_n)\|_{L^2}\|u\|_{L^\infty(0,T;H^2)}\\&\quad\le C\Delta t h^2\|\xi_{h}(\cdot,t_n)\|_{L^2}\|u\|_{L^\infty(0,T;H^2)}\\&\quad\le C\Delta t\|\xi_{h}(\cdot,t_n)\|^2_{L^2}+C\Delta t h^4\|u\|^2_{L^\infty(0,T;H^2)}.\end{aligned}\tag{5.18}$$

Because $\bar{V}_1$ is constant on each cell $\Omega_{i,j}$, we can use (3.8) to estimate the first term on the right side of (5.17).

$$\begin{aligned}&\Delta t\int_\Omega \bar{V}_1(x,y,t_n)\xi_{hx}(x,y,t_n)\eta(x,y,t_n)dxdy\\&\quad=\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\int_{\Omega_{i,j}}\bar{V}_1(x,y,t_n)\xi_{hx}(x,y,t_n)\eta(x,y,t_n)dxdy\\&\quad=\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{\Omega_{i,j}}\bar{V}_1(x,y,t_n)u_{xx}(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\&\qquad+\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{\Omega_{i,j}}\bar{V}_1(x,y,t_n)u_{yy}(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\&\qquad+O(h^2)|u(\cdot,t_n)|_{H^3}\|\xi_h(\cdot,t_n)\|_{L^2}\\&\quad=\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{\Omega_{i,j}}V_1(x,y,t_n)u_{xx}(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\&\qquad+\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{\Omega_{i,j}}V_1(x,y,t_n)u_{yy}(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\end{aligned}$$

$$
\begin{aligned}
&+\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{\Omega_{i,j}}(\bar{V}_1(x,y,t_n)-V_1(x,y,t_n))u_{xx}(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\
&+\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{\Omega_{i,j}}(\bar{V}_1(x,y,t_n)-V_1(x,y,t_n))u_{yy}(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\
&+O(h^2)|u(\cdot,t_n)|_{H^3}\|\xi_h(\cdot,t_n)\|_{L^2}.
\end{aligned}
$$

The last three terms can be bounded in the same manner as (5.18). We need only to estimate the first term on the right side. The second term can be estimated similarly. We integrate the first term by parts to get

(5.19)

$$
\begin{aligned}
&\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{\Omega_{i,j}}V_1(x,y,t_n)u_{xx}(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\
&\quad=\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{y_{j-1}}^{y_j}V_1(x_i,y,t_n)u_{xx}(x_i,y,t_n)\xi_h(x_i,y,t_n)dy\\
&\qquad-\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{y_{j-1}}^{y_j}V_1(x_{i-1},y,t_n)u_{xx}(x_{i-1},y,t_n)\xi_h(x_{i-1},y,t_n)dy\\
&\qquad-\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{\Omega_{i,j}}(V_1u_{xx})_x(x,y,t_n)\xi_h(x,y,t_n)dxdy\\
&\quad=\Delta t\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{y_{j-1}}^{y_j}V_1(b,y,t_n)u_{xx}(b,y,t_n)\xi_h(b,y,t_n)dy\\
&\qquad-\Delta t\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{y_{j-1}}^{y_j}V_1(a,y,t_n)u_{xx}(a,y,t_n)\xi_h(a,y,t_n)dy\\
&\qquad-\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta x)^2}{12}\int_{\Omega_{i,j}}(V_1u_{xx})_x(x,y,t_n)\xi_h(x,y,t_n)dxdy.
\end{aligned}
$$

Here we have used the continuity of V_1, u_{xx}, and ξ_h to cancel the integrals on the interior element edges. The third term on the right side of (5.19) can be bounded by $C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2+C\Delta t h^4\|u\|^2_{L^\infty(0,T;H^3)}$. We now bound the first and the second terms on the right-hand side of (5.19). In the case of Dirichlet boundary condition, these two terms vanish since $\xi_h(a,y,t_n)=\xi_h(b,y,t_n)=0$ for $y\in[c,d]$. In the case of noflow boundary condition, $V_1(a,y,t)=V_1(b,y,t)=0$ for $y\in[c,d]$. Finally in the case of periodic boundary condition the two terms cancel with each other. In short, the first two terms on the right side of (5.19) vanish in all the cases. We combine the preceding estimates to finish the proof.

Lemma 5.2. *If $u\in L^\infty(0,T;H^3)$, then the superconvergence estimate (4.15) holds for the Dirichlet boundary condition (2.2) and a periodic boundary condition. In addition, if $\mathbf{D}$ is diagonal, (4.15) also holds for the noflow boundary condition (2.3).*

Proof. We rewrite the sixth term on the right side of (4.13) as follows:

$$
\begin{aligned}
&\varepsilon\Delta t\int_\Omega(\mathbf{D}\nabla\eta)(\mathbf{x},t_n)\cdot\nabla\xi_h(\mathbf{x},t_n)d\mathbf{x}\\
&\quad=\varepsilon\Delta t\int_\Omega D_{11}\eta_x(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
&+\varepsilon\Delta t\int_\Omega D_{22}\eta_y(x,y,t_n)\xi_{hy}(x,y,t_n)dxdy\\
&+\varepsilon\Delta t\int_\Omega D_{12}\eta_y(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\
&+\varepsilon\Delta t\int_\Omega D_{21}\eta_x(x,y,t_n)\xi_{hy}(x,y,t_n)dxdy.
\end{aligned}
$$

If $\mathbf{D}$ is a diagonal matrix, e.g. $D_{12} = D_{21} = 0$, the third and the fourth terms on the right side of (5.20) vanish and we need only to pay attention to the first term since the second term can be estimated by symmetry. Let $\bar{D}_{11}(x,y,t_n)|_{\Omega_{i,j}} := \frac{1}{\Delta x\Delta y}\int_{\Omega_{i,j}} D_{11}(x,y,t_n)dxdy$ be the cell average of D_{11}. Then we use (3.9) to bound the first term on the right side of (5.20) as follows:

$$
\begin{aligned}
&\left|\varepsilon\Delta t\int_\Omega D_{11}\eta_x(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\right|\\
&\quad=\left|\varepsilon\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\int_{\Omega_{i,j}}\bar{D}_{11}\eta_x(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\right.\\
&\qquad\left.+\varepsilon\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\int_{\Omega_{i,j}}(D_{11}-\bar{D}_{11})\eta_x(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\right|\\
&\quad\le C\varepsilon\Delta t h^2|\xi_h(\cdot,t_n)|_{H^1}(|u(\cdot,t_n)|_{H^3}+\|u(\cdot,t_n)\|_{H^2})\\
&\quad\le \delta\varepsilon\Delta t\|\xi_h(\cdot,t_n)\|^2_{H^1_D}+C\varepsilon\Delta t h^4\|u\|^2_{L^\infty(0,T;H^3)},
\end{aligned}
$$

Notice that this estimate is valid for all the three types of boundary conditions considered in this paper.

When $\mathbf{D}$ is a full tensor, we still need to consider the third term on the right side of (5.20) and the fourth term can be estimated similarly. Again, let $\bar{D}_{12}(x,y,t_n)|_{\Omega_{i,j}} := \frac{1}{\Delta x\Delta y}\int_{\Omega_{i,j}} D_{12}(x,y,t_n)dxdy$ be the cell average of D_{12}. We decompose the third term as

$$
\begin{aligned}
&\varepsilon\Delta t\int_\Omega D_{12}\eta_y(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\
&\quad=\varepsilon\Delta t\int_\Omega \bar{D}_{12}\eta_y(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\
&\qquad+\varepsilon\Delta t\int_\Omega (D_{12}-\bar{D}_{12})\eta_y(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy.
\end{aligned}\tag{5.21}
$$

The second term on the right side of (5.21) can be bounded by

$$
\begin{aligned}
&\left|\varepsilon\Delta t\int_\Omega (D_{12}-\bar{D}_{12})\eta_y(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\right|\\
&\quad\le C\varepsilon\Delta t h^2\|\xi_h(\cdot,t_n)\|_{H^1_D}\|u\|_{L^\infty(0,T;H^2)}\\
&\quad\le \delta\varepsilon\Delta t\|\xi_h(\cdot,t_n)\|^2_{H^1_D}+C\varepsilon\Delta t h^4\|u\|^2_{L^\infty(0,T;H^2)}.
\end{aligned}\tag{5.22}
$$

We use (3.10) to estimate the first term on the right side of (5.21).

$$
\begin{aligned}
&\int_\Omega \bar{D}_{12}\eta_y(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\\
&\quad=\sum_{i=1}^{I}\sum_{j=1}^{J}\int_{\Omega_{i,j}}\bar{D}_{12}\eta_y(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy
\end{aligned}\tag{5.23}
$$

$$
\begin{aligned}
&= -\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{\Omega_{i,j}}\bar{D}_{12}u_{yy}(x,y,t_n)\xi_{hxy}(x,y,t_n)dxdy \\
&\quad +O(h^2)|u(\cdot,t_n)|_{H^3}|\xi_h(\cdot,t_n)|_{H^1} \\
&= -\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{\Omega_{i,j}}(D_{12}u_{yy})(x,y,t_n)\xi_{hxy}(x,y,t_n)dxdy \\
&\quad +O(h^2)|u(\cdot,t_n)|_{H^3}|\xi_h(\cdot,t_n)|_{H^1}.
\end{aligned}
$$

We integrate the first term by parts to get

$$
\begin{aligned}
&\varepsilon\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{\Omega_{i,j}}(D_{12}u_{yy})(x,y,t_n)\xi_{hxy}(x,y,t_n)dxdy \\
&= \varepsilon\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{x_{i-1}}^{x_i}(D_{12}u_{yy})(x,y_j,t_n)\xi_{hx}(x,y_j,t_n)dx \\
&\quad -\varepsilon\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{x_{i-1}}^{x_i}(D_{12}u_{yy})(x,y_{j-1},t_n)\xi_{hx}(x,y_{j-1},t_n)dx \\
&\quad -\varepsilon\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{\Omega_{i,j}}(D_{12}u_{yy})_y(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy
\end{aligned}
$$

The third term can be bounded by

$$
\begin{aligned}
&\left|\varepsilon\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{\Omega_{i,j}}(D_{12}u_{yy})_y(x,y,t_n)\xi_{hx}(x,y,t_n)dxdy\right| \\
&\quad \leq \delta\varepsilon\Delta t\|\xi_h(\cdot,t_n)\|_{H_D^1}^2 + C\varepsilon\Delta t h^4\|u\|_{L^\infty(0,T;H^3)}^2.
\end{aligned}
$$

Note that $\xi_{hx}(x,y,t_n)$ is continuous across each $y = y_j$, so the integrands in the first two terms cancel with each other in the interior edges, leading to

$$
\begin{aligned}
&\varepsilon\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{x_{i-1}}^{x_i}(D_{12}u_{yy})(x,y_j,t_n)\xi_{hx}(x,y_j,t_n)dx \\
&\quad -\varepsilon\Delta t\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{(\Delta y)^2}{12}\int_{x_{i-1}}^{x_i}(D_{12}u_{yy})(x,y_{j-1},t_n)\xi_{hx}(x,y_{j-1},t_n)dx \\
&= \varepsilon\Delta t\sum_{i=1}^{I}\frac{(\Delta y)^2}{12}\int_{x_{i-1}}^{x_i}(D_{12}u_{yy})(x,d,t_n)\xi_{hx}(x,d,t_n)dx \\
&\quad -\varepsilon\Delta t\sum_{i=1}^{I}\frac{(\Delta y)^2}{12}\int_{x_{i-1}}^{x_i}(D_{12}u_{yy})(x,c,t_n)\xi_{hx}(x,c,t_n)dx.
\end{aligned}
$$

The last two terms cancel each other in the case of periodic boundary condition and vanish in the case of Dirichlet boundary condition (2.2). Thus, we finish the proof.

6. Concluding Remarks

In the context of stationary advection-diffusion equations, the location of internal and boundary layers is known a priori. A piecewise-uniform ε-dependent mesh was proposed and analyzed by Shishkin to resolve the boundary and internal layers. Moreover, an ε-uniform L^∞ error estimate was proved for numerical methods with Shishkin mesh [8, 13]. However, in the context of time-dependent advection-diffusion equations, the fronts are dynamic and do not always coincide with the spatial mesh. Thus, although an ε-uniform error estimate in the L^∞-norm is ideal, it is generally impossible especially in the context of multiple space dimensions and in the limiting case of $\varepsilon = 0$. This is why L^∞ norm is not used in the numerical methods for hyperbolic conservation laws [10].

In this paper we derived ε-uniform error estimates in the ε-weighted energy norm, which is in fact related to the $\|\cdot\|_{L^\infty}$ in the context of advection-diffusion equations. As a matter of fact, in the context of an exponential layer [8, 13], both the ε-weighted energy norm and the L^∞ norm of the global truncation error are of order $O(1)$. Thus, both norms are comparable recognize the exponential layer. On the other hand, when problem (2.1) has a smooth solution, both the ε-weighted energy norm and the L^∞ norm of the global truncation error are of order $O(h^2)$ and are still comparable. Nevertheless, in the context of a parabolic layer, the L^∞ norm of the global truncation error is of order $O(1)$ but the ε-weighted energy norm of the global truncation error is of order $O(\varepsilon^{1/4})$. In other word, the L^∞ norm still recognizes the parabolic layer, but the ε-weighted norm does not.

In summary, the ε-weighted norm is comparable to the L^∞ norm in the context of a smooth solution or an exponential layer. An exponential layer exhibits the strongest layer behavior and is of the major concern from a numerical and analysis viewpoint. On the other hand, in the context of a parabolic layer, the ε-weighted energy norm of the global truncation error is somewhat weaker than the L^∞ norm of the error. In short, the ε-weighted norm is probably a natural measure for time-dependent advection-diffusion equations and is closely related to the L^∞ norm. The L^∞ norm is an ideal but impossible measure in this context.

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References

[1] T. Arbogast and M.F. Wheeler, A characteristic-mixed finite element method for advection-dominated transport problems, *SIAM Numer. Anal.*, **32**, (1995), 404–424.

[2] M. Bause and P. Knabner, Uniform error analysis for Lagrange-Galerkin approximations of convection-dominated problems, *SIAM J. Numer. Anal.*, 39:1954–1984, 2002.

[3] Bear, J. (1972). *Dynamics of Fluids in Porous Materials*. American Elsevier, New York.

[4] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[5] J. Douglas, Jr., C.-S. Huang, and F. Pereira, The modified method of characteristics with adjusted advection, *Numer. Math.*, 83 (1999), 353–369.

[6] J. Douglas, Jr. and T.F. Russell, Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures, *SIAM Numer. Anal.*, 19 (1982), 871–885.

[7] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, V 19, American Mathematical Society, Rhode Island, 1998.

[8] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, and G.I. Shishkin. Robust Computational Techniques for Boundary Layers. Applied Mathematics 16, Chapman & Hall/CRC, Boca Raton, Florida, 2000.

[9] C. Johnson: *Numerical Solutions of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, Cambridge, 1987.

[10] R.J. LeVeque, *Finite volume methods for hyperbolic problems*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.

[11] Q. Lin and J. Lin, Finite Element Methods: Accuracy And Improvement, Science press, Beijing, 2006.

[12] D.W. Peaceman, *Fundamentals of Numerical Reservoir Simulation*, Elsevier, Amsterdam, 1977.
[13] H.-G. Roos, M. Stynes, and L. Tobiska. Robust Numerical Methods for Singularly Perturbed Differential Equations, second edition, Springer-Verlag, Berlin, 2008
[14] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Lecture Notes in Mathematics 1054, Springer-Verlag, New York, 1984.
[15] H. Wang, An optimal-order error estimate for an ELLAM scheme for two-dimensional linear advection-diffusion equations, *SIAM J. Numer. Anal.*, 37 (2000) 1338–1368.
[16] H. Wang, H.K. Dahle, R.E. Ewing, M.S. Espedal, R.C. Sharpley, and S. Man, An ELLAM Scheme for advection-diffusion equations in two dimensions, *SIAM J. Sci. Comput.*, 20 (1999) 2160–2194.
[17] H. Wang, R.E. Ewing, G. Qin, S.L. Lyons, M. Al-Lawatia, and S. Man, A family of Eulerian-Lagrangian localized adjoint methods for multi-dimensional advection-reaction equations, *J. Comput. Phys.*, 152 (1999) 120–163.
[18] H. Wang, R.E. Ewing, and T.F. Russell, Eulerian-Lagrangian localized methods for convection-diffusion equations and their convergence analysis, *IMA J. Numer. Anal.*, 15 (1995), 405–459.
[19] H. Wang and K. Wang, Uniform estimates for Eulerian-Lagrangian methods for singularly perturbed time-dependent problems, *SIAM J. Numer. Anal.*, 45 (2007), 1305–1329.
[20] K. Wang, A uniformly optimal-order error estimate of an ELLAM scheme for unsteady-state advection-diffusion equations, *International Journal of Numerical Analysis and Modeling*, 5 (2008), 286–302
[21] K. Wang, A uniform optimal-order error estimate for an Eulerian-Lagrangian discontinuous Galerkin method for transient advection-diffusion equations, *Numerical Methods for PDEs*, 25 (2009), 87-109.
[22] K. Wang and H. Wang, A Uniform Estimate for the ELLAM Scheme for Transport Equations, *Numer. Methods for PDEs*, 24 (2008), 535–554.
[23] K. Wang and H. Wang, A uniform estimate for the MMOC for two-dimensional advection-diffusion equations, *Numer. Methods for PDEs*, 26 (2010), 1054–1069.
[24] K. Wang, H. Wang, and M. Al-Lawatia, An Eulerian-Lagrangian discontinuous Galerkin method for transient advection-diffusion equations, *Numerical Methods for PDEs*, 23 (2007), 1343–1367.

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