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THE CHARACTERISTIC FINITE ELEMENT ALTERNATING-DIRECTION METHOD WITH MOVING MESHES FOR THE TRANSIENT BEHAVIOR OF A SEMICONDUCTOR DEVICE

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Abstract. For the transient behavior of a semiconductor device, the modified method of characteristics finite element alternating-direction procedures with moving meshes are put forward. Some techniques, such as calculus of variations, operator-splitting, characteristic method, generalized L^2 projection, energy method, negative norm estimate and prior estimates and techniques are employed. Optimal order estimates in L^2 norm are derived for the error in the approximation solution. Thus the well-known theoretical problem has been thoroughly and completely solved.

Key Words. semiconductor device, alternating-direction, moving meshes, characteristic finite element, L^2 error estimate

1. Introduction

With the rapid development of semiconductor devices, the traditional approximate method is no longer applicable. We must study the initial boundary value problems of quasilinear partial differential equations, namely, the so-called diffusion model. For high-dimensional problems, new numerical simulation techniques[1−3] are needed to obtain the solutions for semiconductor devices in complicated geometric shapes.

The mathematical model of the two-dimensional semiconductor device of heat conduction is described with the initial boundary value problem made up of four quasilinear partial differential equations[1−4]: one equation of the elliptic type for the electric potential, two of the convection-dominated diffusion type for the conservation of electron and hole concentrations, and the last one for heat conduction. The four equations, relevant initial condition and boundary condition make up a closed system. For two-dimensional problems, there are

$$-\Delta\psi = \alpha(p - e + N(X)), \quad X = (x_1, x_2)^T \in \Omega, \;\; t \in J = (0, \bar{T}], \tag{1}$$

$$\frac{\partial e}{\partial t} = \nabla \cdot \{D_e(X)\nabla e - \mu_e(X) e \nabla\psi\} - R_1(e, p, T), \quad (X, t) \in \Omega \times J, \tag{2}$$

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$$\frac{\partial p}{\partial t} = \nabla \cdot \{D_p(X)\nabla p + \mu_p(X)p\nabla\psi\} - R_2(e,p,T), \quad (X,t) \in \Omega \times J, \tag{3}$$

$$\begin{aligned}\rho(X)\tfrac{\partial T}{\partial t} - \Delta T = \{(D_p(X)\nabla p + \mu_p p\nabla\psi) - (D_e(X)\nabla e - \mu_e(X)e\nabla\psi)\}\\ \cdot\nabla\psi, \quad (X,t) \in \Omega \times J.\end{aligned} \tag{4}$$

The unknown functions are electrostatic potential ψ and electron and hole concentrations e, p and temperature T. All coefficients appearing in (1) $\sim$ (4) are positive. $\alpha = \frac{q}{\varepsilon}$, q and ε are constants (q is the electron charge, ε is the dielectric permittivity). The diffusion coefficients $D_s(X)$ $(s = e, p)$ are related to the mobilities $\mu_s(X)$ by the relation $D_s(X) = U_T\mu_s(X)$, where U_{T} is the thermal voltage. $N(X) = N_D(X) - N_A(X)$ is a given function, $N_D(X)$ and $N_A(X)$ being the donor and acceptor impurity concentrations. $R_i(e,p,T)(i = 1,2)$ is the recombination term. $\nabla = (\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2})^T$ and $\Delta = \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}$.

The initial conditions is

$$e(X,0) = e_0(X),\ p(X,0) = p_0(X),\ T(X,0) = T_0(X), \quad X \in \Omega. \tag{5}$$

Boundary condition:

$$\psi|_{\partial\Omega} = 0,\ \ \frac{\partial e}{\partial\gamma}\bigg|_{\partial\Omega} = \frac{\partial p}{\partial\gamma}\bigg|_{\partial\Omega} = \frac{\partial T}{\partial\gamma}\bigg|_{\partial\Omega} = 0, \quad (X,t) \in \partial\Omega \times J, \tag{6}$$

where γ is the outer normal vector of Ω.

In 1964 Gummel first proposed sequence iterative computation methods to treat this kind of problem[5], thus opening up a new field. Douglas et al. put forward the finite difference method for one-dimensional and two-dimensional simple models (without considering the temperature's effect or constant coefficients). They solved some practical problems and first obtained the theoretical analysis result[6,7]. However, optimal order error estimates in l^2 norm were not obtained yet. Based on what has been achieved, the author considers the finite difference method for a semiconductor device of heat conduction, and optimal order estimates in l^2 norm are obtained [8]. And the author first considers characteristic finite element method and theoretical analysis for numerical simulation of a semiconductor device [9,10]. In this paper, for modern numerical simulation the problems met are often large-scale and large-scope, and the mode number is as large as tens of thousands or even hundreds of millions. Thus we need the operator-splitting method to solve the problem. We shall apply the finite element method with moving meshes to concentration equations and heat-conduction equation. Moreover, in the process of solution, the electron, hole concentrations and heat conduction distribution front will push forward with increasing time, so the finite element mesh near the front will be locally densified. In such a way, we can ensure the accuracy of the numerical results without increasing the computation time as a whole. The densified meshes must move forward as time goes on, to keep themselves always near the concentration and temperature distribution front. At this point we need the new technique of alternating-direction and moving meshes mutual association scheme to solve the problem [11−15].

This thesis puts forward a kind of modified method of characteristics finite element alternating-direction procedures with moving meshes. Some techniques, such as calculus of variations, operator-splitting, characteristic methods, generalized L^2 projection, energy method, negative norm estimate and prior estimates are employed. Optimal order estimates in L^2 norm are derived for the error in approximate solution. The research is important both theoretically and practically for the model analysis, model numerical method and software development in the semiconductor device field.

To avoid technical boundary difficulties associated with the modified method of characteristics for (2) and (3), we assume that Ω is a rectangle and that (1)∼(4) are Ω-periodic. This is physically reasonable, because condition (6) can be treated as a reflection boundary. Throughout the rest of this paper, all functions will be assumed to be spatially Ω-periodic. The boundary conditions (6) can be dropped[16−18].

Generally, this is a positive definite problem:

$$\begin{aligned}&0 < D_* \le D_s(X) \le D^*, s = e, p; \quad 0 < \mu_* \le \mu_s(X) \le \mu^*, s = e, p;\\ &0 < p_* \le \rho(X) \le \rho^*,\end{aligned} \tag{7}$$

where $D_*, D^*, \mu_*, \mu^*, \rho_*$ and ρ^* are constants.

Our assumptions on the regularity of the solutions of (1)∼(6) are denoted collectively by

$$\begin{aligned}&\psi \in L^\infty(J; W^{r+1}(\Omega)), \quad e, p \in L^\infty(J; W^{k+1}(\Omega)),\\ &T \in L^\infty(J; W^{l+1}(\Omega)), \frac{\partial^2 e}{\partial \tau_e^2}, \frac{\partial^2 p}{\partial \tau_p^2}, \frac{\partial^2 T}{\partial t^2} \in L^\infty(J; L^\infty(\Omega)).\end{aligned} \tag{8}$$

The outline of this paper is as follows. In §2, some preparatory work is considered. In §3, the modified method of characteristics finite element alternating-direction procedures with moving meshes are formulated. In §4, some auxiliary elliptic projections are discussed. In §5, the convergence analysis is presented. In this paper M and ε stand for general positive constant and general positive small constant respectively, and have different meanings in different areas.

2. Some preparatory work

The weak form of problems (1)∼(6):

$$(\nabla\psi, \nabla v) = \alpha(p - e + N, v), \quad \forall v \in H^1(\Omega), \ \ t \in J, \tag{9a}$$

$$\begin{aligned}&(\frac{\partial e}{\partial t}, z) - (\mu_e \underline{u} \cdot \nabla e, z) + (D_e \nabla e, \nabla z) - (e\underline{u} \cdot \nabla \mu_e, z)\\ &\quad = \alpha(\mu_e e(p - e + N), z) - (R_1(e, p, T), z), \quad z \in H^1(\Omega), \ \ t \in J,\end{aligned} \tag{9b}$$

$$\begin{aligned}&(\frac{\partial p}{\partial t}, z) + (\mu_p u \cdot \nabla p, z) + (D_p \nabla p, \nabla z) + (p\underline{u} \cdot \nabla \mu_p, z)\\ &\quad = -\alpha(\mu_p p(p - e + N), z) - (R_2(e, p, T), z), \quad z \in H^1(\Omega), \ \ t \in J,\end{aligned} \tag{9c}$$

$$\begin{aligned}&(\frac{\partial T}{\partial t}, z) + (\nabla T, \nabla z) = ([(D_p \nabla p + \mu_p p \nabla\psi) - (D_e \nabla e - \mu_e e \nabla\psi)] \cdot \nabla\psi, z),\\ &\qquad\qquad\qquad\qquad\qquad\qquad z \in H^1(\Omega), \ \ t \in J,\end{aligned} \tag{9d}$$

where $\underline{u} = -\nabla\psi$.

Let Ω be the given plane region with coordinates $X = (x_1, x_2)^T \in R^2$.

Let $V_h = V_{h_\psi}$ be a family of finite-dimensional subspace of $H^1(\Omega)$, parameterized by h_ψ, with the following properties: for $v \in W^{r+1,q}(\Omega)$, $1 \le q \le \infty$,

$$\inf_{v_h \in V_h} \|v - v_h\|_{1,q} \le M\|v\|_{r+1,q,\Omega} h_\psi^r. \tag{10}$$

Let $N_{h_c}(\Omega), M_{h_T}(\Omega)$ be two families of finite-dimensional subspace $H^1(\Omega)$, parameterized by h_c and h_T, with the following properties:

(a) For $v \in W^{k+1,q}(\Omega)$, $1 \le q \le \infty$,

$$\inf_{v_h \in N_{h_c}(\Omega)} \|v - v_h\|_{1,q} \le M\|v\|_{k+1,q,\Omega} h_c^k.$$

For $z \in W^{l+1,q}(\Omega),\ 1 \leq q \leq \infty$,

$$\inf_{z_h \in M_{h_T}(\Omega)} \|z - z_h\|_{1,q} \leq M\|z\|_{l+1,q,\Omega} h_T^l.$$

(b) For $v_h \in N_{h_c}(\Omega),\ z_h \in M_{h_T}(\Omega)$,

$$\|v_h\|_{H^1(\Omega)} \leq M h_c^{-1} \|v_h\|_{L^2(\Omega)}, \|z_h\|_{H^1(\Omega)} \leq M h_T^{-1} \|z_h\|_{L^2(\Omega)},$$

$$\|v_h\|_{W^j_\infty(\Omega)} \leq M h_c^{-1} \|v_h\|_{H^j(\Omega)},\ \ j = 0, 1,$$

$$\|z_h\|_{W^j_\infty(\Omega)} \leq M h_T^{-1} \|z_h\|_{H^j(\Omega)},\ \ j = 0, 1.$$

For the alternating-direction methods, assume that $N_{h_c}(\Omega) \subset G, M_{h_T}(\Omega) \subset G$, where $G = \{\, w |\, w, \dfrac{\partial w}{\partial x_1}, \dfrac{\partial w}{\partial x_2}, \dfrac{\partial^2 w}{\partial x_1 \partial x_2} \in L^2(\Omega) \,\}$. Assume that

$$\inf_{\varphi \in N_h} \{ \sum_{m=0}^{2} h_c^m \sum_{\substack{i+j=m \\ i,j=0,1}} \left\| \frac{\partial^m (u - \varphi)}{\partial x_1^i \partial x_2^j} \right\|_0 \} \leq M h_c^{k+1} \|u\|_{k+1},$$

$$\inf_{\psi \in M_h} \{ \sum_{m=0}^{2} h_T^m \sum_{\substack{i+j=m \\ i,j=0,1}} \left\| \frac{\partial^m (v - \psi)}{\partial x_1^i \partial x_2^j} \right\|_0 \} \leq M h_T^{l+1} \|v\|_{l+1}.$$

We shall also assume that $N_{h_c}(\Omega)$ and $M_{h_T}(\Omega)$ are spanned by a tensor product basis.

We discuss characteristic finite element operator-splitting methods with moving meshes approximation of concentration equations (2) and (3).We subdivide region $\Omega_1 = [a_1, b_1] \times [c_1, d_1]$. The coding of nodes: $\{\, x_{1,\alpha} \mid 0 \leq \alpha \leq N_{x_1} \,\}, \{\, x_{2,\beta} \mid 0 \leq \beta \leq N_{x_2} \,\}$. The global coding of two-dimensional mesh region $i,\ i = 1, 2, \cdots, N$; $N = (N_{x_1} + 1)(N_{x_2} + 1)$. The tensor product index of node i is $(\alpha(i), \beta(i))$, where $\alpha(i)$ is the number of the x_1-axis, and $\beta(i)$ is the number of the x_2-axis. The tensor product basis can be rewritten as products of one-dimensional basis functions in the following manner:

$$N_i(x) = \varphi_{\alpha(i)}(x_1)\psi_{\beta(i)}(x_2) = \varphi_\alpha(x_1)\psi_\beta(x_2), \quad 1 \leq i \leq N. \tag{11}$$

This tensor product basis is very easy to construct in R^2 [19,20]. Now we construct the finite element subspace. Its index is k, simplified as N_h. We note that here the subdivision and the structure of basis functions are changed with time t^n, so N_h^n is used.

Next, we discuss a finite element method with moving meshes approximation of heat conduction equation (4). Similarly, we subdivide region. $\Omega = [a_2, b_2] \times [c_2, d_2]$, the coding of nodes: $\{\, x_{1,\alpha} \mid 0 \leq \alpha \leq M_{x_1} \,\}, \{\, x_{2,\beta} \mid 0 \leq \beta \leq M_{x_2} \,\}$. The global coding $j,\ j = 1, 2, \cdots, M;\ M = (M_{x_1} + 1)(M_{x_2} + 1)$. The tensor product index of node j is $(\lambda(j), \mu(j))$. The tensor product basis can be rewritten as products of one-dimensional basis function in the following manner:

$$M_j(x) = \Phi_{\lambda(j)}(x_1)\Psi_{\mu(j)}(x_2) = \Phi_\lambda(x_1)\Psi_\mu(x_2), \quad 1 \leq j \leq M. \tag{12}$$

It is easy to construct this tensor product basis which is the finite element subspace mentioned above; whose index is l and can be simplified as M_h. Similarly, this finite element space is written as M_h^n.

3. The modified method of characteristics finite element alternating-direction procedures with moving meshes

Noting that the electric potential of time t changes very slowly, we adopt big step calculation. However, for the concentration, we adopt small step calculation. We shall use the following notations: Δt_c—the time step for the concentration equation; Δt_ψ—the time step for the potential equation; $j=\Delta t_\psi/\Delta t_c$, $t^n=n\Delta t_c$, $t_m=m\Delta t_\psi$, $\psi^n=\psi(t^n), \psi_m=\psi(t_m)$.

$$E\psi^n=\begin{cases}\psi_0, \quad t^n\le t_1,\\ (1+\dfrac{v}{j})\psi_m-\dfrac{v}{j}\psi_{m-1}, \quad t_m<t^n\le t_{m+1},\ \ t^n=t_m+v\Delta t_c,\end{cases}$$

where subscripts correspond to potential time levels, and superscripts to concentration levels.

The finite element scheme of electric potential equation (1):

$$(\nabla\psi_{h,m},\nabla v_h)=\alpha(p_{h,m}-e_{h,m}+N_m,v_h), \quad \forall v_h\in V_h, \tag{13}$$

where the approximate electric intensity $\underline{U}_{h,m}=-\nabla\psi_{h,m}$.

As the flow is essentially in the characteristic direction, we apply the modified method for characteristic procedure to the first-order parts of (2) and (3), thus ensuring a high accuracy of the numerical results[16–18,21–24]. We write eqs. (2) and (3) in the form

$$\frac{\partial e}{\partial t}=\nabla\cdot(D_e\nabla e)+\mu_e\underline{u}\cdot\nabla e+e\underline{u}\cdot\nabla\mu_e+\alpha\mu_e(x)e(p-e+N(x))-R_1(e,p,T), \tag{14a}$$

$$\frac{\partial p}{\partial t}=\nabla\cdot(D_p\nabla p)-\mu_p\underline{u}\cdot\nabla p-p\underline{u}\cdot\nabla\mu_p-\alpha\mu_p(x)p(p-e+N(x))-R_2(e,p,T), \tag{14b}$$

where $\underline{u}=-\nabla\psi$. Let $\tau_e=\tau_e(X,t)$ be the unit vector in the direction $(-\mu_e u_1,-\mu_e u_2,1)$ and $\tau_p=\tau_p(X,t)$ be unit vector in the direction $(\mu_p u_1,\mu_p u_2,1)$. Setting $\Phi_s=[1+\mu_s^2|\underline{u}|^2]^{1/2}, s=e,p$, we have

$$\Phi_e\frac{\partial}{\partial\tau_e}=\frac{\partial}{\partial t}-\mu_e\underline{u}\cdot\nabla, \ \ \Phi_p\frac{\partial}{\partial\tau_p}=\frac{\partial}{\partial t}+\mu_p\underline{u}\cdot\nabla.$$

We write eqs. (14a) and (14b) in the form

$$\Phi_e\frac{\partial e}{\partial\tau_e}-\nabla\cdot(D_e\nabla e)-\alpha\mu_e(X)e(p-e+N(X))-e\underline{u}\cdot\nabla\mu_e=-R_1(e,p,T), \tag{15a}$$

$$\Phi_p\frac{\partial p}{\partial\tau_p}-\nabla\cdot(D_p\nabla p)+\alpha\mu_p(X)p(p-e+N(X))+p\underline{u}\cdot\nabla\mu_p=-R_2(e,p,T). \tag{15b}$$

Approximate $\dfrac{\partial e^{n+1}}{\partial\tau_e}=\dfrac{\partial e}{\partial\tau_e}(X,t^{n+1})$ by a backward difference quotient in the τ_e-direction.

$$\frac{\partial e^{n+1}}{\partial\tau_e}(X)\approx\frac{e^{n+1}(X)-e^n(X+\mu_e\underline{u}^{n+1}\Delta t)}{\Delta t_c(1+\mu_e^2|\underline{u}|^2)^{1/2}}. \tag{16a}$$

Similarly, we have

$$\frac{\partial p^{n+1}}{\partial\tau_p}(X)\approx\frac{p^{n+1}(X)-p^n(X-\mu_p\underline{u}^{n+1}\Delta t)}{\Delta t_c(1+\mu_p^2|\underline{u}|^2)^{1/2}}. \tag{16b}$$

The equivalent weak form of problems (9b), (9c) and (9d) becomes:

$$\begin{aligned}(\Phi_e\frac{\partial e}{\partial\tau_e},z)+(D_e\nabla e,\nabla z)-(e\underline{u}\cdot\nabla\mu_e,z)-\alpha(\mu_e e(p-e+N),z)&\\ =-(R_1(e,p,T),z),&\end{aligned} \tag{17a}$$

$$(\Phi_p \frac{\partial p}{\partial \tau_p}, z) + (D_p \nabla p, \nabla z) + (p\underline{u} \cdot \nabla \mu_p, z) + \alpha(\mu_p p(p - e + N), z) \\ = -(R_2(e, p, T), z), \tag{17b}$$

$$(\rho \frac{\partial T}{\partial t}, v) + (\nabla T, \nabla v) = -([(D_p \nabla p - \mu_p p\underline{u}) - (D_e \nabla e + \mu_e e\underline{u})] \cdot \underline{u}, v). \tag{17c}$$

For the electron concentration equation, the modified method of characteristics with finite element procedure is defined as follows:

$$(\frac{e_h^{n+1} - \hat{e}_h^n}{\Delta t_c}, z_h) + (D_e \nabla e_h^{n+1}, \nabla z_h) - \alpha(\mu_e e_h^n(\hat{p}_h^n - \hat{e}_h^n + N), z_h) \\ -(\hat{e}_h^n E\underline{U}_h^{n+1} \cdot \nabla \mu_e, z_h) = -(R_1(\hat{e}_h^n, \hat{p}_h^n, T_h^n), z_h), \tag{18}$$

where $\underline{U}_{h,m} = -\nabla \Psi_{h,m}$, $\hat{e}_h^n = e_h^n(\hat{X}_e^n)$, $\hat{X}_e^n = X + \mu_e E\underline{U}_h^{n+1}\Delta t_c$.

Similarly, for hole concentration equation (17b) the characteristic finite element schemes:

$$(\frac{p_h^{n+1} - \hat{p}_h^n}{\Delta t_c}, z_h) + (D_p \nabla p_h^{n+1}, \nabla z_h) + \alpha(\mu_p p_n^n(\hat{p}_n^n - \hat{e}_n^n + N), z_n) \\ +(\hat{p}_n^n E\underline{U}_h^{n+1} \cdot \nabla \mu_p \cdot z_h) = -(R_2(\hat{e}_n^n, \hat{p}_n^n, T_n^n), z_n), \tag{19}$$

where $\hat{p}_h^n = p_h^n(\hat{X}_p^n), \hat{X}_p^n = X - \mu_p E\underline{U}_h^{n+1}\Delta t_c$.

The characteristic finite element alternating-direction schemes with moving meshes of problems (1)~(6): When $t = t_m$, and if $\{e_{h,m}, p_{h,m}, T_{h,m}\} \in N_{h,m} \times N_{h,m} \times M_{h,m}$ are known, from (13), we obtain approximation solution $\Psi_{h,m} \in V_h$. We find $t = t^n = t_m + \gamma \Delta t_c$, $v = 1, 2, \cdots, j$, the finite element solution $\{e_h^{n+1}, p_h^{n+1}, T_h^{n+1}\} \in N_h^{n+1} \times N_h^{n+1} \times M_h^{n+1}$.

First, we put forward generalized L^2 projection:

$$(\bar{e}_h^n - e_h^n, z_h) + \lambda_e \Delta t_c(\nabla(\bar{e}_h^n - e_h^n), \nabla z_h) + (\lambda_e \Delta t_c)^2(\frac{\partial^2(\bar{e}_h^n - e_h^n)}{\partial x_1 \partial x_2}, \frac{\partial^2 z_h}{\partial x_1 \partial x_2}) = 0, \\ \forall z_h \in N_h^{n+1}, \tag{20a}$$

$$(\bar{e}_h^0 - e_h^0, z_h) + \lambda_e \Delta t_c(\nabla(\bar{e}_h^0 - e_h^0), \nabla z_h) + (\lambda_e \Delta t_c)^2(\frac{\partial^2(\bar{e}_h^0 - e_h^0)}{\partial x_1 \partial x_2}, \frac{\partial^2 z_h}{\partial x_1 \partial x_2}) = 0, \\ \forall z_h \in N_h^1, \tag{20b}$$

$$(\bar{p}_h^n - p_h^n, z_h) + \lambda_p \Delta t_c(\nabla(\bar{p}_h^n - p_h^n), \nabla z_h) + (\lambda_p \Delta t_c)^2(\frac{\partial^2(\bar{p}_h^n - p_h^n)}{\partial x_1 \partial x_2}, \frac{\partial^2 z_h}{\partial x_1 \partial x_2}) = 0, \\ \forall z_h \in N_h^{n+1}, \tag{21a}$$

$$(\bar{p}_h^0 - p_h^0, z_h) + \lambda_p \Delta t_c(\nabla(\bar{p}_h^0 - p_h^0), \nabla z_h) + (\lambda_p \Delta t_c)^2(\frac{\partial^2(\bar{p}_h^0 - p_h^0)}{\partial x_1 \partial x_2}, \frac{\partial^2 z_h}{\partial x_1 \partial x_2}) = 0, \\ \forall z_h \in N_h^1, \tag{21b}$$

$$(\rho(\overline{T}_h^n - T_h^n), w_h) + \lambda_T \Delta t_c(\rho \nabla(\overline{T}_h^n - T_h^n), \nabla w_h) \\ +(\lambda_T \Delta t_c)^2(\rho \frac{\partial^2(\overline{T}_h^n - T_h^n)}{\partial x_1 \partial x_2}, \frac{\partial^2 w_h}{\partial x_1 \partial x_2}) = 0, \forall w_h \in M_h^{n+1}, \tag{22a}$$

$$(\rho(\overline{T}_h^0 - T_h^0), w_h) + \lambda_T \Delta t_c (\rho \nabla (\overline{T}_h^0 - T_h^0), \nabla w_h) + (\lambda_T \Delta t_c)^2 (\rho \frac{\partial^2 (\overline{T}_h^0 - T_h^0)}{\partial x_1 \partial x_2}, \frac{\partial^2 w_h}{\partial x_1 \partial x_2}) = 0, \forall w_h \in M_h^{n+1}, \tag{22b}$$

where (20), (21) and (22) are the generalized L^2 projection. When $N_h^{n+1} \neq N_h^n, M_h^{n+1} \neq M_h^n$, we need these auxiliary projections. λ_e, λ_p and λ_T are positive constants. λ_e, λ_p and λ_T are chosen to satisfy $\lambda_e \geq \frac{1}{2} D_e^*$, $\lambda_p \geq \frac{1}{2} D_p^*$ and $\lambda_T \geq \frac{1}{2} \rho_*^{-1}$, respectively.

The finite element schemes with moving meshes of electron and hole concentrations equations:

$$\begin{aligned}
&(\frac{e_h^{n+1} - \hat{e}_h^n}{\Delta t_c}, z_h) + (D_e \nabla \bar{e}_h^n, \nabla z_h) + \lambda_e (\nabla (e_h^{n+1} - \bar{e}_h^n), \nabla z_h) \\
&\quad + \lambda_e^2 \Delta t_c (\frac{\partial^2 (e_h^{n+1} - \bar{e}_h^n)}{\partial x_1 \partial x_2}, \frac{\partial^2 z_h}{\partial x_1 \partial x_2}) \\
&= \alpha (\mu_e \bar{e}_h^n (\hat{p}_h^n - \hat{e}_h^n + N), z_h) + (\bar{e}_h^n E \underline{U}_h^{n+1} \cdot \nabla \mu_e, z_h) \\
&\quad - (R_1 (\hat{e}_h^n, \hat{p}_h^n, \bar{T}_h^n), z_h), \quad \forall z_h \in N_h^{n+1},
\end{aligned} \tag{23}$$

$$\begin{aligned}
&(\frac{p_h^{n+1} - \hat{p}_h^n}{\Delta t_c}, z_h) + (D_p \nabla \bar{p}_h^n, \nabla z_h) + \lambda_p (\nabla (p_h^{n+1} - \bar{p}_h^n), \nabla z_h) \\
&\quad + \lambda_p^2 \Delta t_c (\frac{\partial^2 (p_h^{n+1} - \bar{p}_h^n)}{\partial x_1 \partial x_2}, \frac{\partial^2 z_h}{\partial x_1 \partial x_2}) \\
&= -\alpha (\mu_p \bar{p}_h^n (\hat{p}_h^n - \hat{e}_h^n + N), z_h) - (\bar{p}_h^n E \underline{U}_h^{n+1} \cdot \nabla \mu_p, z_h) \\
&\quad - (R_2 (\hat{e}_h^n, \hat{p}_h^n, \bar{T}_h^n), z_h), \quad \forall z_h \in N_h^{n+1},
\end{aligned} \tag{24}$$

where $\hat{e}_h^n = \bar{e}_h^n (\hat{X}_e^n)$, $\hat{X}_e^n = X + \mu_e E \underline{U}_h^{n+1} \Delta t_c$, $\hat{p}_h^n = \bar{p}_h^n (\hat{X}_p^n)$, $\hat{X}_p^n = X - \mu_p E \underline{U}_h^{n+1} \Delta t_c$.

The finite element scheme with moving meshes of heat conduction equation:

$$\begin{aligned}
&(\rho \frac{T_h^{n+1} - \bar{T}_h^n}{\Delta t_c}, w_h) + (\nabla \bar{T}_h^n, \nabla w_h) + \lambda_T (\rho \nabla (T_h^{n+1} - \bar{T}_h^n), \nabla w_h) \\
&\quad + \lambda_T^2 \Delta t_c (\rho \frac{\partial^2 (T_h^{n+1} - \bar{T}_h^n)}{\partial x_1 \partial x_2}, \frac{\partial^2 w_h}{\partial x_1 \partial x_2}) \\
&= -([(D_p \nabla \bar{p}_h^n - \mu_p \bar{p}_h^n E \underline{U}_h^{n+1}) \\
&\quad - (D_e \nabla \bar{e}_h^n + \mu_p \bar{e}_h^n E \underline{U}_h^{n+1})] \cdot E \underline{U}_h^{n+1}, w_h), \quad \forall w_h \in M_h^{n+1}.
\end{aligned} \tag{25}$$

For electron concentration equation (23), if $e_h^{n+1} = \sum_{\lambda,\beta} \xi_{\alpha\beta}^{n+1} \varphi_\alpha \psi_\beta, \bar{e}_h^n = \sum_{\alpha,\beta} \bar{\xi}_{\alpha\beta}^n \varphi_\alpha \varphi_\beta$, take $z_h = \varphi_\alpha \psi_\beta$ as the test function, multiply (23) by $2\Delta t_c$, then (23) can be written in the form:

$$\begin{aligned}
&\sum_{\alpha,\beta} (\xi_{\alpha\beta}^{n+1} - \bar{\xi}_{\alpha\beta}^n)(\varphi_\alpha \otimes \psi_\beta, \varphi_\alpha \otimes \psi_\beta) + \lambda_e \Delta t_c \sum_{\lambda,\beta} (\xi_{\alpha\beta}^{n+1} - \bar{\xi}_{\alpha\beta}^n) \{ (\varphi'_\alpha \otimes \psi_\beta, \varphi'_\alpha \otimes \psi_\beta) \\
&\quad + (\varphi_\alpha \otimes \psi'_\beta, \varphi_\alpha \otimes \psi'_\beta) \} + (\lambda_e \Delta t_c)^2 \sum_{\alpha,\beta} (\xi_{\alpha\beta}^{n+1} - \bar{\xi}_{\alpha\beta}^n)(\varphi'_\alpha \otimes \psi'_\beta, \varphi'_\alpha \otimes \psi'_\beta) = \Delta t_c F^n,
\end{aligned}$$

Let

$$C_{x_1} = (\int_{a_1}^{b_1} \varphi_{\alpha_1}\varphi_{\alpha_2} dx_1), \quad C_{x_2} = (\int_{c_1}^{d_1} \psi_{\beta_1}\psi_{\beta_2} dx_2),$$
$$A_{x_1} = (\int_{a_1}^{b_1} \varphi'_{\alpha_1}\varphi'_{\alpha_2} dx_1), \quad A_{x_2} = (\int_{c_1}^{d_1} \psi'_{\beta_1}\psi'_{\beta_2} dx_2).$$

Then we have

$$(C_{x_1} + \lambda_e \Delta t_c A_{x_1}) \otimes (C_{x_2} + \lambda_e \Delta t_c A_{x_2})(\xi^{n+1} - \bar{\xi}^n) = \Delta t_c F^n, \qquad (23)'$$

where

$$F^n_{\alpha\beta} = \alpha(\mu_e \bar{e}^n_h(\hat{\bar{p}}^n_h - \hat{\bar{e}}^n_h + N), \psi_\alpha \otimes \psi_\beta) + (\bar{e}^n_h E\underline{U}^{n+1}_h \cdot \nabla\mu_e, \varphi_\alpha \otimes \psi_\beta)$$
$$-(R_1(\hat{\bar{e}}^n_n, \hat{\bar{p}}^n_n, \bar{T}^n_n), \varphi_\alpha \otimes \varphi_\beta) + (\frac{1}{\Delta t_c}(\hat{\bar{e}}^n_h - \bar{e}^n_h), \varphi_\alpha \otimes \psi_\beta).$$

$(23)'$ shows that equations (23) can be solved by the alternating-direction method, that is, by solving a series of one-dimensional equations two times in succession.

Similarly, we point out that equations (24) and (25) can be solved by alternating-direction method.

The algorithm for a time step is as follows. Firstly, by the initial approximation e^0_h, p^0_h, T^0_h, we can obtain $\Psi_{h,0}$ from eq. (13). Secondly, from schemes (20)∼(25), we can obtain $(e^1_h, p^1_h, T^1_h), (e^2_h, p^2_h, T^2_h), \cdots, (e^j_h, p^j_h, T^j_h)$. Next, by $(e_{h,1}, p_{h,1}, T_{h,1})$, we can obtain $\Psi_{h,1}$. From eqs. (20)∼(25) we can obtain $(e^{j+1}_h, p^{j+1}_h, T^{j+1}_h)$, $(e^{j+2}_h, p^{j+2}_h, T^{j+2}_h), \cdots, (e_{h,2}, p_{h,2}, T_{h,2})$. In this way, we can calculate continuously, so that a complete time step can be taken. Finally, because of the positive definite condition, we can obtain only one solution for the problem.

4. Some auxiliary elliptic projections

For convergence analysis we introduce some auxiliary elliptic projections, where constants β_e, β_p and β_T are chosen to ensure the coerciveness of bilinear forms.

Let $\tilde{\Psi}_h = \tilde{\Psi} : J = (0, \bar{T}] \to V_h$ which satisfies:

$$(\nabla(\Psi - \tilde{\Psi}_h), \nabla\nu_h) = 0, \quad \forall \nu_h \in V_h,\ t \in J, \qquad (26)$$

Let $L^{n+1}e(t) \in N^{n+1} : J^n = (t^n, t^{n+1}] \to N^{n+1}_h$, which satisfies:

$$(D_e\nabla(e - L^{n+1}e), \nabla z_h) - \alpha(\mu_e\underline{u} \cdot \nabla(e - L^{n+1}e), z_h) + \beta_e(e - L^{n+1}e, z_h) = 0, \qquad \forall z_h \in N^{n+1}. \qquad (27)$$

Let $L^{n+1}p(t) \in N^{n+1} : J^n \to N^{n+1}_h$, which satisfies:

$$(D_p\nabla(p - L^{n+1}p), \nabla z_h) + \alpha(\mu_p\underline{u} \cdot \nabla(p - L^{n+1}p), z_h) + \beta_p(p - L^{n+1}p, z_h) = 0, \qquad \forall z_h \in N^{n+1}. \qquad (28)$$

Similarly, let $L^{n+1}T(t) \in M^{n+1}_h : J^n \to M^{n+1}_h$, which satisfies:

$$(\nabla(T - L^{n+1}T), \nabla\omega_h) + \beta_T(T - L^{n+1}T, \omega_h) = 0, \quad \forall \omega_h \in M^{n+1}_h. \qquad (29)$$

In addition, we assign the initial value:

$$e^0_h = L^1e(0), \quad p^0_h = L^1p(0), \quad T^0_h = L^1T(0). \qquad (30)$$

Let $\theta = \Psi - \tilde{\Psi}_h, \eta = \Psi_h - \tilde{\Psi}_h, \xi^{n+1}_e = e^{n+1}_h - L^{n+1}e^{n+1}, \zeta^{n+1}_e = e^{n+1} - L^{n+1}e^{n+1}, \bar{\xi}^n_e = \bar{e}^n_h - L^{n+1}e^n, \bar{\zeta}^n_e = e^n - L^{n+1}e^n, \xi^{n+1}_p = p^{n+1}_h - L^{n+1}p^{n+1}, \zeta^{n+1}_p = p^{n+1} - L^{n+1}p^{n+1}, \bar{\xi}^n_p = p^n_h - L^{n+1}p^n, \bar{\zeta}^n_p = p^n - L^{n+1}p^n, \pi^{n+1} = T^{n+1}_h - L^{n+1}T^{n+1}, \sigma^{n+1} = T^{n+1} - L^{n+1}T^{n+1}, \bar{\pi}^n = \bar{T}^n - L^{n+1}T^n, \bar{\sigma}^n = T^n - L^{n+1}T^n$.

The result from Galerkin methods [25−28]:

$$\|\theta\|_0 + h_\psi\|\theta\|_1 \le M\|\Psi\|_{r+1}h_\psi^{r+1}, \quad t \in J, \tag{31a}$$

$$\begin{cases} \|\zeta_e\|_0 + h_e\|\zeta_e\|_1 \le M\|e\|_{k+1}h_c^{k+1}, \quad t \in J^n, \\ \left\|\bar{\zeta}_e^n\right\|_0 + h_c\left\|\bar{\zeta}_e^n\right\|_1 \le M\|e^n\|_{k+1}h_c^{k+1}, \end{cases} \tag{31b}$$

$$\begin{cases} \|\zeta_p\|_0 + h_c\|\zeta_p\|_1 \le M\|p\|_{k+1}h_c^{k+1}, \quad t \in J^n, \\ \left\|\bar{\zeta}_p^n\right\|_0 + h_c\left\|\bar{\zeta}_p^n\right\|_1 \le M\|p^n\|_{k+1}h_c^{k+1}, \end{cases} \tag{31c}$$

$$\begin{cases} \|\sigma\|_0 + h_T\|\sigma\|_1 \le M\|T\|_{l+1}h_T^{l+1}, \quad t \in J^n, \\ \|\bar{\sigma}^n\|_0 + h_T\|\bar{\sigma}^n\|_1 \le M\|T^n\|_{l+1}h_T^{l+1}. \end{cases} \tag{31d}$$

$$\left\|\frac{\partial\varsigma_e}{\partial t}\right\|_0 + h_c\left\|\frac{\partial\varsigma_e}{\partial t}\right\|_1 \le M\{\|e\|_{k+1} + \left\|\frac{\partial e}{\partial t}\right\|_{k+1}\}h_c^{k+1}, \quad t \in J^n, \tag{32a}$$

$$\left\|\frac{\partial\varsigma_p}{\partial t}\right\|_0 + h_c\left\|\frac{\partial\varsigma_p}{\partial t}\right\|_1 \le M\{\|p\|_{k+1} + \left\|\frac{\partial p}{\partial t}\right\|_{k+1}\}h_c^{k+1}, \quad t \in J^n, \tag{32b}$$

$$\left\|\frac{\partial\sigma}{\partial t}\right\|_0 + h_T\left\|\frac{\partial\sigma}{\partial t}\right\|_1 \le M\{\|T\|_{k+1} + \left\|\frac{\partial T}{\partial t}\right\|_{k+1}\}h_T^{l+1}, \quad t \in J^n. \tag{32c}$$

5. Convergence analysis

Theorem Suppose that the exact solution of problems (1)∼(6) satisfies smooth condition:
$\Psi \in L^\infty(J;W^{r+1}(\Omega)), e,p \in L^\infty(J;W^{k+1}(\Omega))$, $\dfrac{\partial^2 e}{\partial\tau_e^2}, \dfrac{\partial^2 p}{\partial\tau_p^2} \in L^\infty(J;L^\infty(\Omega))$, $T \in L^\infty(J;W^{l+1}(\Omega))$, $\dfrac{\partial^2 T}{\partial t^2} \in L^\infty(J;L^\infty(\Omega))$. Adopt characteristic finite element alternating-direction schemes with moving meshes (20)∼(25) computation. Suppose $r \ge 0$, $k \ge 1$, $l \ge 1$, and the spatial and time discretization satisfy the following relations:

$$\Delta t_c = O(h_c^2) = O(h_p^2),\ h_\psi^{r+1} = o(h_c) = o(h_T),\ h_c^{k+1} = o(h_T),\ h_T^{l+1} = o(h_c). \tag{33}$$

Then, the following error estimate holds:

$$\begin{aligned} &\|\Psi - \Psi_h\|_{\bar{L}_\infty(J;L^2(\Omega))} + \|e - e_h\|_{\bar{L}_\infty(J;L^2(\Omega))} + \|p - p_h\|_{\bar{L}_\infty(J;L^2(\Omega))} \\ &\quad + \|T - T_h\|_{\bar{L}_\infty(J;L^2(\Omega))} \le M\{\Delta t_c + h_\psi^{r+1} + h_c^{k+1} + h_T^{l+1}\}, \end{aligned} \tag{34}$$

where $\|g\|_{\bar{L}\infty(J;X)} = \sup\limits_{n\Delta t_c \le T}\|g^n\|_X$, constant M depends on ψ, e, p, T and their derivatives.

Proof. Firstly, we consider the estimate of the electric potential equation. Subtracting (13) from (9a) $(t = t_m)$ and using (26) $(t = t_m)$, we obtain

$$\|\nabla\theta_m\|_0 \le M\{\|\xi_{e,m}\|_0 + \|\xi_{p,m}\|_0 + h_\Psi^{r+1}\}. \tag{35}$$

Secondly, consider the electron concentration equations. Subtracting (23) from (9b) ($t = t^{n+1}$) and using (27) ($t = t^{n+1}$), we obtain

$$\begin{aligned}
&(\frac{\xi_e^{n+1} - \bar{\xi}_e^n}{\Delta t_c}, z_h) + (D_e \nabla \bar{\xi}_e^n, \nabla z_h) + \lambda_e (\nabla(\xi_e^{n+1} - \bar{\xi}_e^n), \nabla z_h) \\
&\quad + \lambda_e^2 \Delta t_c (\frac{\partial^2(\xi_e^{n+1} - \bar{\xi}_e^n)}{\partial x_1 \partial x_2}, \frac{\partial^2 z_h}{\partial x_1 \partial x_2}) \\
&= ([\frac{\partial e^{n+1}}{\partial t} - \mu_e E \underline{U}_h^{n+1} \cdot \nabla e^{n+1}] - \frac{e^{n+1} - \hat{e}^n}{\Delta t_c}, z_h) \\
&\quad + (\frac{\hat{\bar{\xi}}_e^n - \bar{\xi}_e^n}{\Delta t_c}, z_h) + (\frac{\zeta_e^{n+1} - \hat{\bar{\zeta}}_e^n}{\Delta t_c}, z_h) + \beta_e(\zeta_e^{n+1}, z_h) \\
&\quad + \alpha(\mu_e [e^{n+1}(e^{n+1} - p^{n+1} + N) - \bar{e}_h^n (\hat{\bar{p}}_h^n - \hat{\bar{e}}_h^n + N)], z_h) \\
&\quad + ([e^{n+1} \cdot \underline{u}^{n+1} - \bar{e}_h^n E \underline{U}_h^{n+1}] \cdot \nabla \mu_e, z_h) \\
&\quad + (R_1(\hat{\bar{e}}_h^n, \hat{\bar{p}}_h^n, \bar{T}_h^n) - R_1(e^{n+1}, p^{n+1}, T^{n+1}), z_h) \\
&\quad + \lambda_e (\nabla(\zeta_e^{n+1} - \bar{\zeta}_e^n) - \nabla(e^{n+1} - e^n), \nabla z_h) \\
&\quad + \lambda_e^2 \Delta t_c (\frac{\partial^2(\zeta_e^{n+1} - \bar{\zeta}_e^n)}{\partial x_1 \partial x_2} - \frac{\partial^2(e^{n+1} - e^n)}{\partial x_1 \partial x_2}, \frac{\partial^2 z_h}{\partial x_1 \partial x_2}), \quad \forall z_h \in N_h^{n+1},
\end{aligned} \tag{36}$$

where $\hat{e}^n = e^n(\hat{X}_e^n), \hat{X}_e^n = X + \mu_e E \underline{U}_h^{n+1} \Delta t_c, \hat{\bar{\xi}}_e^n = \bar{\xi}_e^n(\hat{X}_e^n), \cdots$.

For the remainder of the proof, we let k denote the largest index such that $t_{k-1} < t^L$, if t^L is a electric potential time level, then $t^L = t_k$.

We need to introduce the induction hypothesis:

$$||\nabla \Psi_{h,m}|| \leq M, \quad 0 \leq m \leq k-1. \tag{37}$$

Take $z_h = \xi^{n+1}$ as the test function. Then estimate the terms on the left-side of (36).

$$(\frac{\xi_e^{n+1} - \bar{\xi}_e^n}{\Delta t_c}, \xi_e^{n+1}) \geq \frac{1}{2\Delta t_c} \{ \|\xi_e^{n+1}\|_0^2 - \|\bar{\xi}_e^n\|_0^2 \}, \tag{38a}$$

$$\begin{aligned}
&(D_e \nabla \bar{\xi}_e^n, \nabla \xi_e^{n+1}) + \lambda_e (\nabla \xi_e^{n+1} - \bar{\xi}_e^n), \nabla \xi_e^{n+1}) \\
&\quad = \lambda_e (\nabla \xi_e^{n+1}, \nabla \xi_e^{n+1}) + ([D_e - \lambda_e] \nabla \bar{\xi}_e^n, \nabla \xi_e^{n+1}) \\
&\quad \geq \lambda_e \|\nabla \xi_e^{n+1}\|_0^2 - \frac{1}{2} \max_X |D_e - \lambda_e| \{ \|\nabla \bar{\xi}_e^n\|_0^2 + \|\nabla \xi_e^{n+1}\| \} \\
&\quad = \frac{1}{2} \{ 2\lambda_e - \max_X |D_e - \lambda_e| \} \|\nabla \xi_e^{n+1}\|_0^2 - \frac{1}{2} \max_X |D_e - \lambda_e| \|\nabla \bar{\xi}_e^n\|_0^2.
\end{aligned} \tag{38b}$$

Notice that $\lambda_e \geq \max_X |D_e - \lambda_e|$, and we have

$$\lambda_e^2 \Delta t_c (\frac{\partial^2(\xi_e^{n+1} - \bar{\xi}_e^n)}{\partial x_1 \partial x_2}, \frac{\partial^2 \xi_e^{n+1}}{\partial x_1 \partial x_2}) \geq \frac{\lambda_e^2 \Delta t_c}{2} \{ \left\| \frac{\partial^2 \xi_e^{n+1}}{\partial x_1 \partial x_2} \right\|_0^2 - \left\| \frac{\partial^2 \bar{\xi}_e^n}{\partial x_1 \partial x_2} \right\|_0^2 \}. \tag{38c}$$

Multiply (36) by $2\Delta t_c$, then estimate the terms on the left-hand side of (36).

$$\begin{aligned}
&2\Delta t_c\{(\frac{\xi_e^{n+1}-\bar{\xi}_e^n}{\Delta t_c},\xi_e^{n+1})+(D_e\nabla\bar{\xi}_e^n,\nabla\xi_e^{n+1})\\
&\quad+\lambda_e(\nabla(\xi_e^{n+1}-\bar{\xi}_e^n),\nabla\xi_e^{n+1})\\
&\quad+\lambda_e^2\Delta t_c(\frac{\partial^2(\xi_e^{n+1}-\bar{\xi}_e^n)}{\partial x_1\partial x_2},\frac{\partial^2\xi_e^{n+1}}{\partial x_1\partial x_2})\}\\
&\geq\left\|\xi_e^{n+1}\right\|_0^2-\left\|\bar{\xi}_e^n\right\|_0^2+2\Delta t_c\{2\lambda_e-\max_X|D_e-\lambda_e|\}\left\|\nabla\xi_e^{n+1}\right\|_0^2\\
&\quad-\Delta t_c\max_X|D_e-\lambda_e|\left\|\nabla\bar{\xi}_e^n\right\|_0^2+(\lambda_e\Delta t_c)^2\{\left\|\frac{\partial^2\xi_e^{n+1}}{\partial x_1\partial x_2}\right\|_0^2-\left\|\frac{\partial^2\bar{\xi}_e^n}{\partial x_1\partial x_2}\right\|_0^2\}.
\end{aligned}\tag{39}$$

We now estimate the terms on the right-hand side of (36). By the induction hypothesis (37) and negative norm estimate [17,18,23,24], we have

$$\begin{aligned}
&\left\|[\frac{\partial e^{n+1}}{\partial t}-\mu_e E\underline{U}_h^{n+1}\cdot\nabla e^{n+1}]-\frac{e^{n+1}-\hat{e}^n}{\Delta t_c}\right\|_0^2\\
&\quad\leq\int_\Omega\Phi_{h,e}^2\Delta t_c\left|\int_{(\hat{X},t^n)}^{(X,t^{n+1})}\frac{\partial^2 e}{\partial\tau^2}d\tau\right|^2dX\\
&\quad\leq\Delta t_c\left\|\Phi_{h,e}^3\right\|_{0,\infty}\int_\Omega\int_{(\hat{X},t^n)}^{(X,t^{n+1})}\left|\frac{\partial^2 e}{\partial\tau^2}\right|^2d\tau dX\\
&\quad\leq M\Delta t_c\int_\Omega\int_{t^n}^{t^{n+1}}\left|\frac{\partial^2 e}{\partial\tau^2}\right|^2dtdX,
\end{aligned}$$

where $\Phi_{h,e}=[1+\mu_e^2|EU_h^{n+1}|^2]^{1/2}$.

Then, we have:

$$\begin{aligned}
&2([\tfrac{\partial e^{n+1}}{\partial t}-\mu_e E\underline{U}_h^{n+1}\cdot\nabla e^{n+1}]-\tfrac{e^{n+1}-\hat{e}^n}{\Delta t_c},\xi_e^{n+1})\Delta t_c\\
&\quad\leq M\{(\Delta t_c)^2\textstyle\int_\Omega\int_{t^n}^{t^{n+1}}\left|\tfrac{\partial^2 e}{\partial\tau^2}\right|^2dtdX+\left\|\xi_e^{n+1}\right\|_0^2\Delta t_c\},
\end{aligned}\tag{40a}$$

$$2(\frac{\hat{\bar{\xi}}_e^n-\bar{\xi}_e^n}{\Delta t_c},\xi_e^{n+1})\Delta t_c\leq M\left\|\bar{\xi}_e^n\right\|_0^2\Delta t_c+\varepsilon\left\|\nabla\xi_e^{n+1}\right\|_0^2\Delta t_c,\tag{40b}$$

$$\begin{aligned}
&2\{(\tfrac{\zeta_e^{n+1}-\hat{\bar{\zeta}}_e^n}{\Delta t_c},\xi_e^{n+1})+\lambda_e(\zeta_e^{n+1},\xi_e^{n+1})\}\Delta t_c\\
&\quad\leq M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+\|\xi_{e,m}\|_0^2\\
&\qquad+\|\xi_{p,m}\|_0^2+\left\|\xi_e^{n+1}\right\|^2\}\Delta t_c+\varepsilon\left\|\nabla\xi_e^{n+1}\right\|_0^2\Delta t_c,
\end{aligned}\tag{40c}$$

$$\begin{aligned}
&2\{\alpha(\mu_e[e^{n+1}(p^{n+1}-e^{n+1}+N)-\bar{e}_h^n(\hat{\bar{p}}_h^n-\hat{\bar{e}}_h^n+N)],\xi_e^{n+1})\\
&\quad+([e^{n+1}\underline{u}^{n+1}-\bar{e}_h^nE\underline{U}_h^{n+1}]\cdot\nabla\mu_e,\xi_e^{n+1})+(R_1(\hat{\bar{e}}_h^n,\hat{\bar{p}}_h^n,\bar{T}_h^n)\\
&\quad-R_1(e^{n+1},p^{n+1},T^{n+1}),\xi_e^{n+1})\}\Delta t_c\\
&\leq M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\|\xi_{e,m}\|_0^2+\|\xi_{p,m}\|_0^2\\
&\quad+\|\bar{\pi}^n\|_0^2+\left\|\xi_e^{n+1}\right\|_0^2\}\Delta t_c+\varepsilon\left\|\xi_e^{n+1}\right\|_0^2,
\end{aligned}\tag{40d}$$

$$\begin{aligned}
&2\lambda_e(\nabla(\zeta_e^{n+1}-\bar{\zeta}_e^n)-\nabla(e^{n+1}-e^n),\nabla\xi_e^{n+1})\Delta t_c\\
&\quad\leq M(\Delta t_c)^2\textstyle\int_{t^n}^{t^{n+1}}[\left\|\tfrac{\partial e}{\partial t}\right\|_1^2+\left\|\tfrac{\partial\varsigma_e}{\partial t}\right\|_1^2]dt+\varepsilon\left\|\nabla\xi_e^{n+1}\right\|_0^2\Delta t_c,
\end{aligned}\tag{40e}$$

$$2(\lambda_e\Delta t_c)^2(\frac{\partial^2(\zeta_e^{n+1}-\bar{\zeta}_e^n)}{\partial x_1\partial x_2}-\frac{\partial^2(e^{n+1}-e^n)}{\partial x_1\partial x_2},\frac{\partial^2\xi_e^{n+1}}{\partial x_1\partial x_2})$$
$$\leq M(\Delta t_c)^2\int_{t^n}^{t^{n+1}}[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial\varsigma_e}{\partial t}\right\|_2^2]dt+M(\Delta t_c)^3\left\|\frac{\partial^2\xi_e^{n+1}}{\partial x_1\partial x_2}\right\|_0^2. \tag{40f}$$

For error equation (36), using (38)∼(40), we can obtain:

$$\begin{aligned}
&(1-3M\Delta t_c)\left\|\xi_e^{n+1}\right\|_0^2-(1+2M\Delta t)\left\|\bar{\xi}_e^n\right\|_0^2\\
&+\Delta t_c\{2\lambda_e-\max_X|D_e(X)-\lambda_e|-\lambda_e\varepsilon\}\left\|\nabla\xi_e^{n+1}\right\|_0^2\\
&-\Delta t_c\max_X|D_e(X)-\lambda_e|\left\|\nabla\bar{\xi}_e^n\right\|_0^2+\\
&(1-M\Delta t_c)(\lambda_e\Delta t_c)^2\left\|\frac{\partial^2\xi_e^{n+1}}{\partial x_1\partial x_2}\right\|_0^2-(\lambda_e\Delta t_c)^2\left\|\frac{\partial^2\bar{\xi}_e^n}{\partial x_1\partial x_2}\right\|_0^2\\
&\leq M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}\\
&+\|\xi_{e,m}\|_0^2+\|\xi_{p,m}\|^2+\left\|\bar{\xi}_e^n\right\|_0^2+\left\|\bar{\xi}_p^n\right\|_0^2+\|\bar{\pi}^n\|_0^2\}\Delta t_c\\
&+M(\Delta t_c)^2\int_{t^n}^{t^{n+1}}[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial\varsigma_e}{\partial t}\right\|_2^2]dt.
\end{aligned} \tag{41}$$

We use $\|\xi_n^e\|_0$ to estimate $\left\|\bar{\xi}_e^n\right\|_0$, and from (20a) we have

$$\begin{aligned}
&(\bar{\xi}_e^n-\xi_e^n,\nabla z_h)+\lambda_e\Delta t_c(\nabla(\bar{\xi}_e-\xi_e^n),\nabla z_h)+(\lambda_e\Delta t_c)^2(\frac{\partial^2(\bar{\xi}_e^n-\xi_e^n)}{\partial x_1\partial x_2},\frac{\partial^2 z_h}{\partial x_1\partial x_2})\\
&=(\bar{\varsigma}_e^n-\zeta_e^n,\nabla z_h)+\lambda_e\Delta t_c(\nabla(\bar{\zeta}_e-\zeta_e^n),\nabla z_h)+(\lambda_e\Delta t_c)^2(\frac{\partial^2(\bar{\varsigma}_e^n-\varsigma_e^n)}{\partial x_1\partial x_2},\frac{\partial^2 z_h}{\partial x_1\partial x_2}),
\end{aligned} \tag{42}$$

Take $z_h=\bar{\xi}_e^n$, and we can obtain

$$\begin{aligned}
&(1-\varepsilon)[\left\|\bar{\xi}_e^n\right\|_0^2+\lambda_e\Delta t_c\left\|\nabla\bar{\xi}_e^n\right\|_0^2+(\lambda_e\Delta t_c)^2\left\|\frac{\partial^2\bar{\xi}_e^n}{\partial x_1\partial x_2}\right\|_0^2]\\
&\leq\|\xi_e^n\|_0^2+\lambda_e\Delta t_c\|\nabla\xi_e^n\|_0^2]+(\lambda_e\Delta t_c)^2\left\|\frac{\partial^2\xi_e^n}{\partial x_1\partial x_2}\right\|_0^2+\frac{1}{\varepsilon}[\left\|\bar{\zeta}_e^n-\zeta_e^n\right\|\\
&+\lambda_e\Delta t_c\left\|\nabla(\bar{\zeta}_e^n-\zeta_e^n)\right\|_0^2+(\lambda_e\Delta t_c)\left\|\frac{\partial^2(\bar{\zeta}_e^n-\zeta_e^n)}{\partial x_1\partial x_2}\right\|_0^2].
\end{aligned} \tag{43}$$

For the hole concentration equation (24), we can obtain the following error estimates:

$$\begin{aligned}
&(1-3M\Delta t_c)\left\|\xi_p^{n+1}\right\|_0^2-(1+2M\Delta t_c)\left\|\bar{\xi}_p^n\right\|_0^2\\
&\quad+\Delta t_c\{2\lambda_p-\max_X|D_p(x)-\lambda_p|-\lambda_p\varepsilon\}\left\|\nabla\xi_e^{n+1}\right\|_0^2\\
&\quad-\Delta t_c\max_X|D_p(x)-\lambda_p|\left\|\nabla\bar{\xi}_p^n\right\|_0^2+(1-M\Delta t_c)(\lambda_p\Delta t_c)^2\left\|\frac{\partial^2\xi_p^{n+1}}{\partial x_1\partial x_2}\right\|_0^2\\
&\quad-(\lambda_p\Delta t_c)^2\left\|\frac{\partial^2\bar{\xi}_p}{\partial x_1\partial x_2}\right\|_0^2\\
&\le M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\|\xi_{e,m}\|_0^2+\|\xi_{p,m}\|_0^2\\
&\quad+\left\|\bar{\xi}_e^n\right\|_0^2+\left\|\bar{\xi}_p^n\right\|_0^2+\|\bar{\pi}^n\|_0^2\}\Delta t_c+M(\Delta t_c)^2\int_{t^n}^{t^{n+1}}[\left\|\frac{\partial p}{\partial t}\right\|_2^2+\left\|\frac{\partial\varsigma_p}{\partial t}\right\|_2^2]dt.
\end{aligned}\tag{44}$$

$$\begin{aligned}
&(1-\varepsilon)[\left\|\bar{\xi}_p^n\right\|_0^2+\lambda_p\Delta t_c\left\|\nabla\bar{\xi}_p^n\right\|_0^2+(\lambda_p\Delta t_c)^2\left\|\frac{\partial^2\bar{\xi}_p^n}{\partial x_1\partial x_2}\right\|_0^2]\\
&\quad\le\left\|\xi_p^n\right\|_0^2+\lambda_p\Delta t_c\left\|\nabla\xi_p^n\right\|_0^2+(\lambda_p\Delta t_c)^2\left\|\frac{\partial^2\xi_p^n}{\partial x_1\partial x_2}\right\|_0^2+\frac{1}{\varepsilon}[\left\|\bar{\zeta}_p^n-\zeta_p^n\right\|_0^2\\
&\qquad+\lambda_p\Delta t_c\left\|\nabla(\bar{\zeta}_p^n-\zeta_p^n)\right\|_0^2+(\lambda_p\Delta t_c)^2\left\|\frac{\partial^2(\bar{\zeta}_p^n-\zeta_p^n)}{\partial x_1\partial x_2}\right\|_0^2].
\end{aligned}\tag{45}$$

Similarly, for the heat conduction equation, we have the following error estimates:

$$\begin{aligned}
&(1-3M\Delta t_c)\left\|\rho^{\frac12}\pi^{n+1}\right\|_0^2-(1+2M\Delta t_c)\left\|\rho^{\frac12}\bar{\pi}^n\right\|_0^2\\
&\quad+\Delta t_c\{2\lambda_T-|1-\lambda_T|-\lambda_T\varepsilon\}\left\|\rho^{\frac12}\nabla\pi^{n+1}\right\|_0^2-\Delta t_c|1-\lambda_T|\left\|\rho^{\frac12}\nabla\bar{\pi}^n\right\|_0^2\\
&\quad+(1-M\Delta t_c)(\lambda_T\Delta t_c)^2\left\|\rho^{\frac12}\frac{\partial^2\pi^{n+1}}{\partial x_1\partial x_2}\right\|_0^2-(\lambda_T\Delta t_c)^2\left\|\rho^{\frac12}\frac{\partial^2\bar{\pi}^n}{\partial x_1\partial x_2}\right\|_0^2\\
&\le M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\|\bar{\pi}^n\|_0^2+\left\|\bar{\xi}_e^n\right\|_0^2+\left\|\bar{\xi}_p^n\right\|_0^2+\|\xi_{e,m}\|_0^2\\
&\quad+\|\xi_{p,m}\|_0^2\}\Delta t_c+M(\Delta t_c)^2\int_{t^n}^{t^{n+1}}[\left\|\frac{\partial T}{\partial t}\right\|_2^2+\left\|\frac{\partial\sigma}{\partial t}\right\|_2^2]dt,
\end{aligned}\tag{46}$$

$$\begin{aligned}
&(1-\varepsilon)[\left\|\rho^{\frac12}\bar{\pi}^n\right\|_0^2+\lambda_T\Delta t_c\left\|\rho^{\frac12}\nabla\bar{\pi}^n\right\|_0^2+(\lambda_T\Delta t_c)^2\left\|\rho^{\frac12}\frac{\partial^2\bar{\pi}^n}{\partial x_1\partial x_2}\right\|_0^2]\\
&\le\left\|\rho^{\frac12}\pi^n\right\|_0^2+\lambda_T\Delta t_c\left\|\rho^{\frac12}\nabla\pi^n\right\|_0^2+(\lambda_T\Delta t_c)^2\left\|\rho^{\frac12}\frac{\partial^2\pi^n}{\partial x_1\partial x_2}\right\|_0^2+\frac{1}{\varepsilon}[\left\|\rho^{\frac12}(\bar{\sigma}^n-\sigma^n)\right\|_0^2\\
&\quad+\lambda_T\Delta t_c\left\|\rho^{\frac12}\nabla(\bar{\sigma}^n-\sigma^n)\right\|_0^2+(\lambda_T\Delta t_c)^2\left\|\rho^{\frac12}\frac{\partial^2(\bar{\sigma}^n-\sigma^n)}{\partial x_1\partial x_2}\right\|_0^2.
\end{aligned}\tag{47}$$

Let $K = 3M\Delta t_c$ be suitably small so that $K\Delta t_c \leq \frac{1}{2}$. From (41), (44) and (46) we can obtain:

$$\begin{aligned}
&(1-K\Delta t_c)\{\,\|\xi_e^{n+1}\|_0^2 + \|\xi_p^{n+1}\|_0^2 + \|\pi^{n+1}\|_0^2 + \lambda_e\Delta t_c\|\nabla\xi_e^{n+1}\|_0^2 \\
&\quad +\lambda_p\Delta t_c\|\nabla\xi_p^{n+1}\|_0^2 + \lambda_T\Delta t_c\|\nabla\pi^{n+1}\|_0^2 + (\lambda_c\Delta t_c)^2\left\|\frac{\partial^2\xi_e^{n+1}}{\partial x_1\partial x_2}\right\|_0^2 \\
&\quad +(\lambda_c\Delta t_c)^2\left\|\frac{\partial^2\xi_p^{n+1}}{\partial x_1\partial x_2}\right\|_0^2 + (\lambda_c\Delta t_c)^2\left\|\frac{\partial^2\pi^{n+1}}{\partial x_1\partial x_2}\right\|_0^2\} \\
&\quad -(1+K\Delta t)\{\,\|\bar{\xi}_e^n\|_0^2 + \|\bar{\xi}_p^n\|_0^2 + \|\bar{\pi}^n\|_0^2 + \lambda_e\Delta t_c\|\nabla\bar{\xi}_e^n\|_0^2 \\
&\quad +\lambda_c\Delta t_c\|\nabla\bar{\xi}_p^n\|_0^2\lambda_T\Delta t_c\|\nabla\bar{\pi}^n\|_0^2 \\
&\quad +(\lambda_c\Delta t_c)^2\left\|\frac{\partial^2\bar{\xi}_e^n}{\partial x_1\partial x_2}\right\|_0^2 + (\lambda_c\Delta t_c)^2\left\|\frac{\partial^2\bar{\xi}_p^n}{\partial x_1\partial x_2}\right\|_0^2 + (\lambda_T\Delta t_c)^2\left\|\frac{\partial^2\bar{\pi}^n}{\partial x_1\partial x_2}\right\|_0^2\} \\
&\leq M\{\,(\Delta t_c)^2 + h_\psi^{2(r+1)} + h_c^{2(k+1)} + h_T^{2(l+1)} + \|\xi_{e,m}\|_0^2 + \|\xi_{p,m}\|_0^2\}\Delta t \\
&\quad +M(\Delta t_c)^2\int_{t^n}^{t^{n+1}}[\left\|\frac{\partial e}{\partial t}\right\|_2^2 + \left\|\frac{\partial p}{\partial t}\right\|_2^2 + \left\|\frac{\partial T}{\partial t}\right\|_2^2 + \left\|\frac{\partial \varsigma_e}{\partial t}\right\|_2^2 + \left\|\frac{\partial \varsigma_p}{\partial t}\right\|_2^2 + \left\|\frac{\partial \sigma}{\partial t}\right\|_2^2]dt.
\end{aligned} \tag{48}$$

We use $\|\xi_e^n\|_0$, $\|\xi_p^n\|_0$, and $\|\pi^n\|_0$ to estimate $\|\bar{\xi}_e^n\|_0$, $\|\bar{\xi}_p^n\|_0$, and $\|\bar{\pi}^n\|_0$. For (48), by (43), (45) and (47) we can write it in the following form:

$$\begin{aligned}
&\frac{1-K\Delta t_c}{1+K\Delta t_c}(1-\varepsilon)\{\,\|\xi_e^{n+1}\|_0^2 + \|\xi_p^{n+1}\|_0^2 + \|\pi^{n+1}\|_0^2 + \lambda_e\Delta t_c\|\nabla\xi_e^{n+1}\|_0^2 \\
&\quad +\lambda_p\Delta t_c\|\nabla\xi_p^{n+1}\|_0^2 + \lambda_T\Delta t_c\|\nabla\pi^{n+1}\|_0^2 + (\lambda_e\Delta t_c)^2\left\|\frac{\partial^2\xi_e^{n+1}}{\partial x_1\partial x_2}\right\|_0^2 \\
&\quad +(\lambda_p\Delta t_c)^2\left\|\frac{\partial^2\xi_p^{n+1}}{\partial x_1\partial x_2}\right\|_0^2 + (\lambda_T\Delta t_c)^2\left\|\frac{\partial^2\pi^{n+1}}{\partial x_1\partial x_2}\right\|_0^2\} \\
&\quad -\{\,\|\xi_e^n\|_0^2 + \|\xi_p^n\|_0^2 + \|\pi^n\|_0^2 + \lambda_e\Delta t_c\|\nabla\xi_e^n\|_0^2 + \lambda_p\Delta t_c\|\nabla\xi_p^n\|_0^2 + \lambda_T\Delta t_c\|\nabla\pi^n\|_0^2 \\
&\quad +(\lambda_e\Delta t_c)^2\left\|\frac{\partial^2\xi_e^n}{\partial x_1\partial x_2}\right\|_0^2 + (\lambda_c\Delta t_c)^2\left\|\frac{\partial^2\xi_p^n}{\partial x_1\partial x_2}\right\|_0^2 + (\lambda_T\Delta t_c)^2\left\|\frac{\partial^2\pi^n}{\partial x_1\partial x_2}\right\|_0^2\} \\
&\leq M(1-\varepsilon)\{\,(\Delta t_c)^2 + h_\psi^{2(r+1)} + h_c^{2(k+1)} + h_T^{2(l+1)} + \|\xi_{e,m}\|_0^2 + \|\xi_{p,m}\|_0^2\}\Delta t_c \\
&\quad +\frac{M}{\varepsilon}\{\,h_\psi^{2(r+1)} + h_c^{2(k+1)} + h_T^{2(l+1)}\} + M(1-\varepsilon)(\Delta t_c)^2\int_{t^n}^{t^{n+1}}[\left\|\frac{\partial e}{\partial t}\right\|_2^2 + \left\|\frac{\partial p}{\partial t}\right\|_2^2 \\
&\quad +\left\|\frac{\partial T}{\partial t}\right\|_2^2 + \left\|\frac{\partial \varsigma_e}{\partial t}\right\|_2^2 + \left\|\frac{\partial \varsigma_p}{\partial t}\right\|_2^2 + \left\|\frac{\partial \sigma}{\partial t}\right\|_2^2]dt.
\end{aligned} \tag{49}$$

(49) is the estimate for the moving mashes in the ease of $N_h^{n+1} \neq N_h^n, M_h^{n+1} \neq M_h^n$. When $N_h^{n+1} = N_h^n,\ M_h^{n+1} = M_h^n$, then $\bar{e}_h^n = e_h^n,\ \bar{p}_h^n = p_h^n,\ \bar{T}_h^n = T_h^n$, and we have

the following estimates :

$$\begin{aligned}
&\frac{(1-K\Delta t_c)}{(1+K\Delta t_c)}\{\,\|\xi_e^{n+1}\|_0^2+\|\xi_p^{n+1}\|_0^2+\|\pi^{n+1}\|_0^2+\lambda_e\Delta t_c\|\nabla\xi_e^{n+1}\|_0^2\\
&\quad+\lambda_p\Delta t_c\|\nabla\xi_p^{n+1}\|_0^2+\lambda_T\Delta t_c\|\nabla\pi^{n+1}\|_0^2+(\lambda_e\Delta t_c)^2\left\|\frac{\partial^2\xi_p^{n+1}}{\partial x_1\partial x_2}\right\|_0^2\\
&\quad+(\lambda_p\Delta t)^2\left\|\frac{\partial^2\xi_p^{n+1}}{\partial x_1\partial x_2}\right\|_0^2+(\lambda_T\Delta t)^2\left\|\frac{\partial^2\pi^{n+1}}{\partial x_1\partial x_2}\right\|_0^2\}-\{\,\|\xi_e^n\|^2+\|\xi_p^n\|^2\\
&\quad+\|\sigma^n\|_0^2+\lambda_e\Delta t_c\|\nabla\xi_e^n\|_0^2+\lambda_p\Delta t_c\|\nabla\xi_p^n\|_0^2+\lambda_T\Delta t_c\|\nabla\pi^n\|_0^2\\
&\quad+(\lambda_e\Delta t_c)^2\left\|\frac{\partial^2\xi_e^n}{\partial x_1\partial x_2}\right\|_0^2+(\lambda_p\Delta t_c)^2\left\|\frac{\partial^2\xi_p^n}{\partial x_1\partial x_2}\right\|_0^2+(\lambda_T\Delta t_c)^2\left\|\frac{\partial^2\pi^n}{\partial x_1\partial x_2}\right\|_0^2\}\\
&\le M\{\,(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\|\xi_{e,m}\|_0^2+\|\xi_{p,m}\|_0^2\}\Delta t_c\\
&\quad+M(\Delta t_c)^2\int_{t^n}^{t^{n+1}}[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial p}{\partial t}\right\|_2^2+\left\|\frac{\partial T}{\partial t}\right\|_2^2+\left\|\frac{\partial \varsigma_e}{\partial t}\right\|_2^2+\left\|\frac{\partial \varsigma_p}{\partial t}\right\|_2^2+\left\|\frac{\partial \sigma}{\partial t}\right\|_2^2]dt.
\end{aligned} \tag{50}$$

Let us suppose that in the whole process of computation, the meshes have been moved R times, which corresponds to estimate (49), and for the remaining $L-R$ times, the meshes are not moved, which corresponds to estimate (50). Without loss of generality, we make the following arrangement:

$$\begin{aligned}
&\frac{(1-K\Delta t_c)}{(1+K\Delta t_c)}\{\,\|\xi_e^1\|_0^2+\|\xi_p^1\|_0^2+\|\pi^1\|_0^2+\cdots\}-\{\,\|\xi_e^0\|_0^2+\|\xi_p^0\|_0^2+\|\pi^0\|_0^2+\cdots\,\}\\
&\le M\{\,(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\|\xi_{e,0}\|_0^2+\|\xi_{p,0}\|_0^2+\cdots\,\}\Delta t_c\\
&\quad+M(\Delta t_c)^2\int_{t^0}^{t^1}\left[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial \varsigma_e}{\partial t}\right\|_2^2+\cdots\right]dt.
\end{aligned} \tag*{$(51)_1$}$$

$$\begin{aligned}
&(1-\varepsilon)\frac{(1-K\Delta t_c)}{(1+K\Delta t_c)}\{\,\|\xi_e^2\|_0^2+\|\xi_p^2\|_0^2+\|\pi^2\|_0^2+\cdots\,\}\\
&\quad-\{\,\|\xi_e^1\|_0^2+\|\xi_p^1\|_0^2+\|\pi^1\|_0^2+\cdots\,\}\\
&\le M(1-\varepsilon)\{\,(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\cdots\,\}\Delta t\\
&\quad+\frac{M}{\varepsilon}\{\,h_\psi^{2(k+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}\,\}\\
&\quad+M(1-\varepsilon)(\Delta t_c)^2\int_{t^1}^{t^2}\left[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial \varsigma_e}{\partial t}\right\|_2^2+\cdots\right]dt.
\end{aligned} \tag*{$(51)_2$}$$

$$\frac{(1-K\Delta t_c)}{(1+K\Delta t_c)}\{\,\|\xi_e^3\|_0^2+\|\xi_p^3\|_0^2+\|\pi^3\|_0^2+\cdots\}-\{\,\|\xi_e^2\|_0^2+\|\xi_p^2\|_0^2+\|\pi^2\|_0^2+\cdots\}$$
$$\le M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\cdots\}$$
$$+M(\Delta t_c)^2\int_{t^2}^{t^3}\left[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial\zeta_e}{\partial t}\right\|_2^2+\cdots\right]dt. \tag{51$_3$}$$

$$(1-\varepsilon)\frac{(1-K\Delta t_c)}{(1+K\Delta t_c)}\{\,\|\xi_e^4\|_0^2+\|\xi_p^4\|_0^2+\|\pi^4\|_0^2+\cdots\}$$
$$-\{\,\|\xi_e^3\|_0^2+\|\xi_p^3\|_0^2+\|\pi^3\|_0^2+\cdots\}$$
$$\le M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\cdots\} \tag{51$_4$}$$
$$+\frac{M}{\varepsilon}\{h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}\}$$
$$+M(1-\varepsilon)(\Delta t_c)^2\int_{t^3}^{t^4}\left[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial\varsigma_e}{\partial t}\right\|_2^2+\cdots\right]dt.$$

$\cdots\cdots$ $\cdots\cdots$

$$\frac{(1-K\Delta t_c)}{(1+K\Delta t_c)}\{\,\|\xi_e^q\|_0^2+\|\xi_p^q\|_0^2+\|\pi^q\|_0^2+\cdots\}$$
$$-\{\,\|\xi_e^{q-1}\|_0^2+\|\xi_p^{q-1}\|_0^2+\|\pi^{q-1}\|_0^2+\cdots\}$$
$$\le M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\|\xi_{e,[q/j]}\|_0^2+\|\xi_{p,[q/j]}\|_0^2\} \tag{51$_q$}$$
$$+M(\Delta t_c)^2\int_{t^{q-1}}^{t^q}\left[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial\zeta e}{\partial t}\right\|_2^2+\cdots\right]dt.$$

$$(1-\varepsilon)\frac{(1-K\Delta t_c)}{(1+K\Delta t_c)}\{\,\|\xi_e^{q+1}\|_0^2+\|\xi_p^{q+1}\|_0^2+\|\pi^{q+1}\|_0^2+\cdots\}$$
$$-\{\,\|\xi_e^q\|_0^2+\|\xi_p^q\|_0^2+\|\pi^q\|_0^2+\cdots\}$$
$$\le M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\|\xi_{e,[(q+1)/j]}\|_0^2+\|\xi_{p,[(q+1)/j]}\|_0^2\}$$
$$+\frac{M}{\varepsilon}\{h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}\}+M(\Delta t_c)^2\int_{t^q}^{t^{q+1}}\left[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial\varsigma_e}{\partial t}\right\|_2^2+\cdots\right]dt. \tag{51$_{q+1}$}$$

$\cdots\cdots$ $\cdots\cdots$

$$\frac{(1-K\Delta t_c)}{(1+K\Delta t_c)}\{\,\|\xi_e^L\|_0^2+\|\xi_p^L\|_0^2+\|\pi^L\|_0^2+\cdots\}$$
$$-\{\,\|\xi_e^{L-1}\|_0^2+\|\xi_p^{L-1}\|_0^2+\|\pi^{L-1}\|_0^2+\cdots\}$$
$$\le M\{(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}+\cdots\}$$
$$+M(\Delta t_c)^2\int_{t^{L-1}}^{t^L}\left[\left\|\frac{\partial e}{\partial t}\right\|_2^2+\left\|\frac{\partial\varsigma_e}{\partial t}\right\|_2^2+\cdots\right]dt. \tag{51$_L$}$$

Sum them up according to

$$
\begin{aligned}
&(51)_1+(\frac{1-K\Delta t_c}{1+K\Delta t_c})(51)_2+(1-\varepsilon)(\frac{1-K\Delta t_c}{1+K\Delta t_c})^2(51)_3\\
&+(1-\varepsilon)(\frac{1-K\Delta t_c}{1+K\Delta t_c})^3(51)_4+\cdots+(1-\varepsilon)^R(\frac{1-K\Delta t_c}{1+K\Delta t_c})^{L-1}(51)_L.
\end{aligned}
\tag{52}
$$

Suppose $r\geq 1, k\geq 1, l\geq 1$, and we can obtain

$$
\begin{aligned}
&(1-\varepsilon)^R(\frac{1-K\Delta t_c}{1+K\Delta t_c})^L\{\,\|\xi_e^L\|_0^2+\|\xi_p^L\|_0^2+\|\pi^L\|_0^2+\lambda_e\Delta t_c\|\nabla\xi_e^L\|_0^2\\
&+\lambda_p\Delta t_c\|\nabla\xi_p^L\|_0^2+\lambda_T\Delta t_c\|\nabla\pi^n\|_0^2+(\lambda_e\Delta t_c)^2\left\|\frac{\partial^2\xi_e^L}{\partial x_1\partial x_2}\right\|_0^2+(\lambda_p\Delta t_c)^2\left\|\frac{\partial^2\xi_p^L}{\partial x_1\partial x_2}\right\|_0^2\\
&+(\lambda_T\Delta t_c)^2\left\|\frac{\partial^2\pi^L}{\partial x_1\partial x_2}\right\|_0^2\}-\{\,\|\xi_e^0\|_0^2+\|\xi_p^0\|^2+\|\pi^0\|_0^2+\cdots\}\\
&\leq M\{\,(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}\\
&\quad+\sum_{m=1}^{[L/j]}[\|\xi_{e,m}\|_0^2+\|\xi_{p,m}\|_0^2]\Delta t_c\}+\frac{R}{\varepsilon}\{\,h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}\}.
\end{aligned}
\tag{53}
$$

Taking $\varepsilon=\dfrac{1}{1+R}$, in particular, we have $(1-\varepsilon)^R=(1+\dfrac{1}{R})^R\leq e$. And we notice that

$$
(\frac{1+K\Delta t_c}{1-K\Delta t_c})^L=(1+\frac{2K\Delta t_c}{1-K\Delta t_c})^L\leq(1+4K\Delta t_c)^{T/\Delta t_c}\leq e^{4KT},
$$

$\xi_e^0=\xi_p^0=\sigma^0=0$. Applying discrete Gronwall inequality, we can obtain

$$
\begin{aligned}
&\|\xi_e^L\|_0^2+\|\xi_p^L\|_0^2+\|\pi^L\|_0^2+\lambda_e\Delta t_c\|\nabla\xi_e^L\|_0^2+\lambda_p\Delta t_c\|\nabla\xi_p^L\|_0^2+\lambda_T\Delta t_c\|\nabla\pi^L\|_0^2\\
&\leq M\{\,(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}\\
&\quad+(R+1)R[h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}]\,\}.
\end{aligned}
\tag{54}
$$

Suppose that in the whole process of computation, the number of moving times R is not too large, which means that it is independent of h and Δt_c. Then we have

$$
\begin{aligned}
&\|\xi_e^L\|_0^2+\|\xi_p^L\|_0^2+\|\pi^L\|_0^2+\lambda_e\Delta t_c\|\nabla\xi_e^L\|_0^2+\lambda_p\Delta t_c\|\nabla\xi_p^L\|_0^2+\lambda_T\Delta t_c\|\nabla\pi^L\|_0^2\\
&\qquad\leq M\{\,(\Delta t_c)^2+h_\psi^{2(r+1)}+h_c^{2(k+1)}+h_T^{2(l+1)}\,\}.
\end{aligned}
\tag{55}
$$

Finally, from (35), (55) and (33) we conclude that induction hypothesis (37) holds. Based on error estimates (55) and (35), and the result (31) of projection theorem, we can come to the theorem.

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