Journal of Computational Mathematics
Vol.30, No.1, 2012, 14–23.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1110-m11si08

LOW-RANK TENSOR STRUCTURE OF SOLUTIONS TO ELLIPTIC PROBLEMS WITH JUMPING COEFFICIENTS*

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Abstract

We study the separability properties of solutions to elliptic equations with piecewise constant coefficients in $\mathbb{R}^d$, $d \geq 2$. The separation rank of the solution to diffusion equation with variable coefficients is presented.

Mathematics subject classification: 65F30, 65F50, 65N35, 65F10.
Key words: Structured matrices, Elliptic operators, Poisson equation, Matrix approximations, Low-rank matrices, Tensors, Canonical decomposition.

1. Introduction

In this paper, we study the separability properties of solutions to elliptic equations with piecewise constant coefficients. By a separable decomposition of a multivariate function, we mean its representation or approximation by a sum of the products of univariate functions. The separability properties of the Laplace operator inverse and hence of the solution to Poisson equation were estimated in [1–4]. In what following, a point to study is the dependence on structure of the diffusion coefficient.

To fix the idea, we first consider a model elliptic boundary value problem in two dimensions,

$$-\nabla(a\nabla u) = f, \quad \text{in} \quad \Omega = [0,1]^2, \tag{1.1a}$$

$$u|_{\partial\Omega} = 0, \tag{1.1b}$$

with an assumption that f is represented by a piecewise smooth tensor decomposition

$$f(x,y) = \sum_{k=1}^{r_f} f_k^{(1)}(x)\, f_k^{(2)}(y), \tag{1.2}$$

* Received February 26, 2011 / Revised version received August 5, 2011 / Accepted August 20, 2011 / Published online January 9, 2012 /

and the diffusion coefficient $a(x,y)$ is a piecewise constant function on cells of a tensor grid in Ω. In the case of an $M \times M$ tensor tiling, the reciprocals $1/a$ on these cells comprise a matrix of the form

$$B = \begin{bmatrix} 1/a_{11} & \cdots & 1/a_{1M} \\ \vdots & \ddots & \vdots \\ 1/a_{M1} & \cdots & 1/a_{MM} \end{bmatrix} \tag{1.3}$$

with the notation

$$r_{1/a} = \mathrm{rank} B.$$

Clearly, the function $1/a$ has the same separable form,

$$1/a(x,y) = \sum_{l=1}^{r_{1/a}} b_l^{(1)}(x) \cdot b_l^{(2)}(y) = \sum_{l=1}^{r_{1/a}} \frac{1}{a_l^{(1)}(x)} \cdot \frac{1}{a_l^{(2)}(y)}, \tag{1.4}$$

which can be shown by a constant spline interpolation. Given $\varepsilon > 0$, we approximate u by a separable decomposition

$$u_{r_u} = \sum_{k=1}^{r_u} u_k^{(1)}(x)\, u_k^{(2)}(y), \tag{1.5}$$

so that $\|u - u_{r_u}\|_{L^\infty} \le \varepsilon$.

In this paper we investigate how r_u depends on ε, $r_{1/a}$, M and r_f. Straightforward analysis in the continuous case gives the following rank estimation,

$$r_u = \mathcal{O}(M^2 r_v),$$

where r_v is the maximal ε-rank of the solution in each domain generated by the $M \times M$ tiling. Notice that r_v depends weakly on a, since in each domain the solution satisfies just the Poisson equation: $-a\Delta u = f$.

In the 3D or higher dimensional case we formulate the problem in a similar way. Consider

$$-\nabla(a\nabla u) = f, \quad \text{in} \quad \Omega = [0,1]^d, \tag{1.6a}$$

$$u|_{\partial\Omega} = 0, \tag{1.6b}$$

and assume a separability property for the right-hand side,

$$f(\mathbf{x}) = \sum_{k=1}^{r_f} f_k^{(1)}(x_1) \cdots f_k^{(d)}(x_d), \tag{1.7}$$

and the reciprocal diffusion coefficient,

$$1/a(\mathbf{x}) = \sum_{l=1}^{r_{1/a}} b_l^{(1)}(x_1) \cdots b_l^{(d)}(x_d) = \sum_{l=1}^{r_{1/a}} \frac{1}{a_l^{(1)}(x_1)} \cdots \frac{1}{a_l^{(d)}(x_d)}. \tag{1.8}$$

Now for given $\varepsilon > 0$, we approximate u by a separable decomposition

$$u_{r_u} = \sum_{k=1}^{r_u} u_k^{(1)}(x_1) \cdots u_k^{(d)}(x_d), \tag{1.9}$$

so that $\|u - u_{r_u}\|_{L^\infty} \le \varepsilon$. Such a decomposition is crucial for the numerical solution of the problem. Suppose we discretize the problem on the grid with n points in each spatial direction.

Then the solution might be represented as a d-dimensional tensor with n^d entries, and the so-called "curse of dimensionality" arises [5, 6]. The approximation (1.9) is then a reduction of degrees of freedom using the canonical approximation of a tensor [7–10]. This problem is ill-posed in general [6, 11–13], so the rank estimates provide us with important practical information.

The main result is the rank bound

$$\mathcal{O}(M^{d-1}r_v)$$

for a separable approximation of the solution. However, the numerical experiments point to a better estimate like $\mathcal{O}(r_{1/a}r_v)$.

The rest of the paper is organized as follows. In the section 2 we prove a theorem on the rank estimate for continuous functions. In the section 3 we present numerical experiments in the 2D case showing that the rank of the solution depends on the rank of the reciprocal coefficient rather than of the number of subdomains.

2. Continuous Case Analysis

We can split the initial problem (1.1) into the following two ones:

- Poisson equation in the whole domain with the scaled right-hand side:

$$-\Delta u_I = \frac{1}{a} f, \quad \text{in} \quad \Omega, \tag{2.1a}$$

$$u_I|_{\partial\Omega} = 0; \tag{2.1b}$$

- Laplace equation in each domain $\Omega_{i,j}$ of a constant values of a with nonhomogeneous Dirichlet boundary conditions:

$$-\Delta u_{II} = 0 \quad \text{in} \quad \Omega_{i,j}, \tag{2.2a}$$

$$u_{II}|_{\partial\Omega_{i,j}} = u|_{\partial\Omega_{i,j}} - u_I|_{\partial\Omega_{i,j}} = g(\partial\Omega_{i,j}). \tag{2.2b}$$

Then, $u_I + u_{II} = u$.

Theorem 2.1. *Suppose that a 2D problem* (1.1) *has a separable right-hand side* (1.2) *and the diffusion coefficient in the form* (1.3), (1.4). *Let* Ω_h *denote a subdomain of all points with the distance at least* h *from the interface specifying the jumps of the coefficient. Then the solution* u *can be approximated in* Ω_h *by a separable function* u_{r_u} *with the rank bound*

$$r_u \le \Big(4(M+1) + r_{1/a}r_f\Big) \cdot C|\log(\varepsilon)||\log(h)| \tag{2.3}$$

and the accuracy

$$\|u - u_{r_u}\|_{L^\infty} \le \varepsilon.$$

Proof. For each of the solutions u_I, u_{II} we can use Green's formula [14] in the corresponding domain:

$$u(x,y) = \frac{1}{\sigma_d}\left(\int\limits_{\partial\Omega}\left(\mathtt{K}(x,y,\xi,\eta)\frac{\partial u(\xi,\eta)}{\partial \mathbf{n}} - u(\xi,\eta)\frac{\partial \mathtt{K}(x,y,\xi,\eta)}{\partial \mathbf{n}}\right)d\xi d\eta + \int\limits_{\Omega}\frac{f(\xi,\eta)}{a}\mathtt{K}(x,y,\xi,\eta)d\xi d\eta\right),$$

where $\sigma_d = 2\pi$ and $\mathtt{K} = \ln(1/||\mathbf{x}-\mathbf{x_0}||)$ for the 2D case; $\mathbf{x} = (x, y)$ and $\mathbf{x_0} = (\xi, \eta)$. From [2,15] we have the following approximation for the logarithmic potential (kernel) at some distance away from the singularity:

$$\mathtt{K}(x, y, \xi, \eta) = \ln \frac{1}{||\mathbf{x}-\mathbf{x_0}||} \approx \sum_{k=1}^{r_{log}} K_k^{(1)}(x-\xi) \cdot K_k^{(2)}(y-\eta) \tag{2.4}$$

with the accuracy

$$\left\| \mathtt{K}(x, y, \xi, \eta) - \sum_{k=1}^{r_{log}} K_k^{(1)}(x-\xi) \cdot K_k^{(2)}(y-\eta) \right\|_{L^\infty} \leq \varepsilon$$

and the rank

$$r_{log} = \mathcal{O}(|\log\ \varepsilon|).$$

Since the coefficient a is discontinuous, the right-hand side f/a and the solution of Poisson equation have singularities in the points of discontinuity in the coefficient. Hence we can not consider these functions in that points, but only outside some neighborhood of the singularities. If the size of neighborhood is bounded by h then r_{log} is multiplied by $|\log\ h|$ [15,16].

So, consider the first part of solution u_I. From the separability properties of f, $1/a$ and $\mathtt{K}$ we have:

$$u_I(x,y) = \frac{1}{\sigma_d} \left(\int_{\partial\Omega} \mathtt{K} \frac{\partial u_I(\xi,\eta)}{\partial \mathbf{n}} d\xi d\eta + \sum_{k=1}^{r_{log}} \sum_{l=1}^{r_{1/a}} \sum_{p=1}^{r_f} \int_0^1 \frac{f_p^{(1)}(\xi)}{a_l^{(1)}(\xi)} K_k^{(1)}(x-\xi) d\xi \cdot \int_0^1 \frac{f_p^{(2)}(\eta)}{a_l^{(2)}(\eta)} K_k^{(2)}(y-\eta) d\eta \right).$$

The first term consists of 4 boundary integrals, each of them requires integration only by one variable. Applying the separability of $\mathtt{K}$, we obtain rank $4r_{log}$. The second term has maximal rank $r_f r_{1/a} r_{log}$. So, the rank of u_I is estimated by

$$r_{u_I} \leq \left(4 + r_f r_{1/a}\right) \cdot C |\log(\varepsilon)||\log(h)|.$$

As for the second term u_{II}, we use the following approach. Consider one column of cells (see Fig. 2.1).

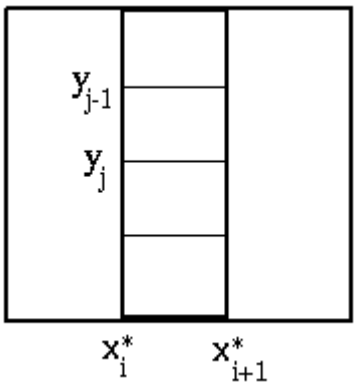

Fig 2.1. Working column in the boundary integral

Consider a solution, obtained from the boundary integral by the left boundary of this column. On each cell, i.e. $(x,y) \in \Omega_{i,j}$, $i,j = 1, \cdots, M$, we can write the following:

$$u_{II,ij}(x,y) = \frac{1}{\sigma_d} \sum_{k=1}^{r_{log}} K_k^{(1)}(x - x_i^*) \cdot \int\limits_{y_{j-1}}^{y_j} \left(\frac{\partial g(x_i^*, \eta)}{\partial y} K_k^{(2)}(y-\eta) - \frac{\partial K_k^{(2)}(y-\eta)}{\partial y} g(x_i^*, \eta) \right) d\eta,$$

where x_i^* is (fixed) x coordinate of this boundary. The solution on the whole column can be represented as follows:

$$u_{II,i}(x,y) = \sum_{j=1}^{M} \theta_i(x)\theta_j(y) u_{II,ij}(x,y), \quad i = 1, \cdots, M,$$

where $\theta_i(x)$ is a characteristic function of interval $[x_{i-1}, x_i]$. Then

$$u_{II,i} = \frac{1}{\sigma_d} \sum_{k=1}^{r_{log}} \theta_i(x) K_k^{(1)}(x - x_i^*) \cdot \sum_{j=1}^{M} \theta_j(y) \int\limits_{y_{j-1}}^{y_j} \left(\frac{\partial g(x_i^*, \eta)}{\partial y} K_k^{(2)}(y-\eta) - \frac{\partial K_k^{(2)}(y-\eta)}{\partial y} g(x_i^*, \eta) \right) d\eta.$$

So, we obtain a function of rank r_{log}. After the summation by i in the direction x, and by 3 other boundaries we obtain $4M$ r_{log}. And for the full solution

$$r_u \le \left(4M + 4 + r_{1/a} r_f \right) \cdot C |\log(\varepsilon)| |\log(h)|.$$

Theorem 2.1 is proved. □

In the higher dimensional case we formulate the problem in a similar way, as it is shown in the introduction, see (1.6)-(1.9).

In this case we can prove the following separability properties of u:

Theorem 2.2. *Suppose that a d-dimensional problem* (1.6) *has a separable right-hand side* (1.7) *and the diffusion coefficient in the form* (1.8). *Let* Ω_h *denote a subdomain of all points with the distance at least h from the interface specifying the jumps of the coefficient. Then the solution u can be approximated in* Ω_h *by a separable function* u_{r_u} (1.9) *with the rank bound*

$$r_u \left(2d(M^{d-1} + 1) + r_{1/a} r_f \right) \cdot C |\log(\varepsilon)| |\log(h)|, \tag{2.5}$$

and the accuracy

$$\|u - u_{r_u}\|_{L^\infty} \le \varepsilon.$$

Proof. Green's formula in this case holds as well:

$$u(\mathbf{x}) = \frac{1}{(d-2)\sigma_d} \left(\int\limits_{\partial\Omega} \left(\mathtt{K}(\mathbf{x},\xi) \frac{\partial u(\xi)}{\partial \mathbf{n}} - u(\xi) \frac{\partial \mathtt{K}(\mathbf{x},\xi)}{\partial \mathbf{n}} \right) d\xi + \int\limits_{\Omega} \frac{f(\xi)}{a} \mathtt{K}(\mathbf{x},\xi) d\xi \right),$$

where σ_d is a surface of unitary sphere ($\sigma_d = 4\pi$ in 3D), $\mathtt{K}(\mathbf{x},\xi) = ||\mathbf{x} - \xi||^{2-d}$, and the kernel also has a low-rank approximation:

$$\mathtt{K}(\mathbf{x},\xi) = \frac{1}{||\mathbf{x} - \xi||^{d-2}} \approx \sum_{k=1}^{r_{log}} K_k^{(1)}(x_1 - \xi_1) \cdots K_k^{(d)}(x_d - \xi_d).$$

The main idea of the proof is the same, as in 2D case. The differences are:

1. Now there are 2d boundaries of d-dimensional cube;
2. Each boundary has dimension $d-1$, hence, M^{d-1} tiling.

Then, u_I is approximated with the rank $(2d+r_f r_{1/a})r_{log}$, and u_{II} with the rank $2d\ M^{d-1}\ r_{log}$. Hence the total rank is estimated as (2.5).

3. Numerical Separability Properties in 2D

In the previous section we estimated the separation rank for the continuous solution to the elliptic equation. Obviously, for the discretized problem, the same estimate (2.5) holds for the canonical rank of discrete solution tensor. However, as we will see, the best approximation of two-dimensional discrete solution might have significantly lower rank. Namely, it is proportional to the *rank* of the reciprocal coefficient $r_{1/a}$, but not to the number of the cells with constant coefficient.

We solve the equation (1.1) using the Galerkin method [17]: choose appropriate basis functions $\varphi_1(x), \cdots, \varphi_n(x)$ and find the solution as a linear combination

$$u_h(x,y) = \sum_{i_1,i_2=1}^{n} u(i_1,i_2)\varphi_{i_1}(x)\varphi_{i_2}(y),$$

with the unknown coefficients $u(i_1,i_2)$ to be obtained from a linear system

$$\begin{aligned} &\sum_{i_1,i_2=1}^{n} u(i_1,i_2)\Big(a\nabla\varphi_{i_1}(x)\varphi_{i_2}(y), \nabla\varphi_{j_1}(x)\varphi_{j_2}(y)\Big)_{L_2(\Omega)} \\ =&\Big(f, \varphi_{j_1}(x)\varphi_{j_2}(y)\Big)_{L_2(\Omega)}, \quad j_1, j_2 = 1, \cdots, n. \end{aligned} \tag{3.1}$$

Remark 3.1. Although we denote the basis functions by φ both for x and y directions (for the ease of presentation), in fact, the number of grid points and the grid cell size can be different for different directions, hence, in such case there will be different sets of basis functions $\varphi_{i_1}(x)$ and $\psi_{i_2}(y)$.

We can write (3.1) in the following form:

$$AU = F,$$

where

$$A = \left[\Big(a\nabla\varphi_{i_1}(x)\varphi_{i_2}(y), \nabla\varphi_{j_1}(x)\varphi_{j_2}(y)\Big)_{L_2(\Omega)}\right],$$

$$F = \left[\Big(f, \varphi_{j_1}(x)\varphi_{j_2}(y)\Big)_{L_2(\Omega)}\right] = \sum_{k=1}^{r_f}\left[\Big(f_k^{(1)}, \varphi_{j_1}(x)\Big)_{L_2(0,1)}\right] \otimes \left[\Big(f_k^{(2)}, \varphi_{j_2}(y)\Big)_{L_2(0,1)}\right].$$

Let us gather coefficients $u(i_1,i_2)$ into a matrix $U = [u(i_1,i_2)] \in \mathbb{R}^{n\times n}$ and decompose it using the SVD:

$$u(i_1,i_2) = \sum_{k=1}^{n} \sigma_k U^{(1)}_{i_1,k} U^{(2)}_{i_2,k},$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are the singular values, and $U^{(1)}_{i_1,k}$, $U^{(2)}_{i_2,k}$ are the kth singular vectors. In order to obtain a reduced representation for the solution, we can truncate this sum keeping only the summands with a certain number of senior singular values and neglecting the summands with smaller singular values. In this way we arrive at an approximation to U of a lower rank $U_{r_u} = [u_{r_u}(i_1, i_2)]$:

$$u_{r_u}(i_1, i_2) = \sum_{k=1}^{r_u} \sigma_k \; U^{(1)}_{i_1,k} \; U^{(2)}_{i_2,k}.$$

Given an accuracy parameter ε, we can choose r_u so that the estimate $||U - U_{r_u}|| \leq \varepsilon$ is guaranteed to hold with a minimal possible r_u. Then, it is easy to derive that

$$\hat{u}_{r_u}(x, y) = \sum_{i_1,i_2=1}^{n} u_{r_u}(i_1, i_2)\varphi_{i_1}(x)\varphi_{i_2}(y) = \sum_{k=1}^{r_u} \sigma_k \left(\sum_{i_1=1}^{n} U^{(1)}_{i_1,k}\varphi_{i_1}(x) \right) \left(\sum_{i_2=1}^{n} U^{(2)}_{i_2,k}\varphi_{i_2}(y) \right)$$

approximates $u_h(x, y)$ with accuracy $\mathcal{O}(\varepsilon)$.

In the numerical examples below, we are interested to find relations between r_u and ε, $r_{1/a}$, r_f, and their dependence on n. In the following we assume that a has constant values on $M \times M$ cells. We take piecewise linear hat elements as basis functions $\varphi_i(x)$ on the uniform grid.

1. Dependence on ε and n (Table 3.1).

Table 3.1: r_u versus ε and n; $r_{1/a} = 1$; $M = 8$.

	$\log_{10}(1/\varepsilon)$						
n	4	5	6	7	8	9	10
16	2	4	5	5	6	7	7
32	3	5	5	7	7	9	9
64	2	4	4	6	6	9	9
128	2	4	5	6	8	10	11
256	2	4	5	6	8	10	12
512	3	4	5	7	8	11	13
1024	3	4	6	8	9	12	14

We can deduce that practical dependence is of the form

$$r_u(\varepsilon) = C \cdot \log(1/\varepsilon). \tag{3.2}$$

If we make a linear fit of $r_u(|\log(\varepsilon)|)$ for $n = 1024$, using the least squares method, the dependence is $r_u = 1.86 \cdot \log(1/\varepsilon) - 5$. Also we can see that if the approximation tolerance ε is greater than the discretization error $\mathcal{O}(1/n^2)$, then r_u does not depend on n (e.g., see the column with $\varepsilon = 10^{-5}$).

2. Dependence on $r_{1/a}$ (Table 3.2).

Now the least squares linear fitting gives a dependence $r_u = 13.95 \cdot r_{1/a} + 7.96$ (for $\varepsilon = 10^{-10}$). Thus,

$$r_u(r_{1/a}) = C \cdot r_{1/a}. \tag{3.3}$$

3. Dependence on M (Table 3.3).

In this example we use randomly generated values in the closed interval $[1, 7]$ for rank-1 a. We see that for sufficiently large M ($M > 4$), the rank r_u does not depend on M. As a matter

Table 3.2: r_u versus ε and $r_{1/a}$; $M = 8$; $n = 256$.

	$\log_{10}(1/\varepsilon)$						
$r_{1/a}$	4	5	6	7	8	9	10
1	3	4	6	8	9	12	14
2	5	8	14	21	28	34	41
3	5	8	14	20	30	37	47
4	7	13	22	35	45	56	67
5	8	17	31	46	60	73	85
6	8	17	30	46	65	80	93
7	11	19	34	54	72	91	107
8	11	23	41	60	81	96	112

Table 3.3: r_u versus ε and M; $r_{1/a} = 1$; $n = 256$.

	$\log_{10}(1/\varepsilon)$			
M	4	5	8	11
2	2	3	7	12
3	2	4	9	16
4	3	4	11	17
8	3	5	12	18
12	4	5	12	19
16	3	5	11	18
32	3	5	11	18

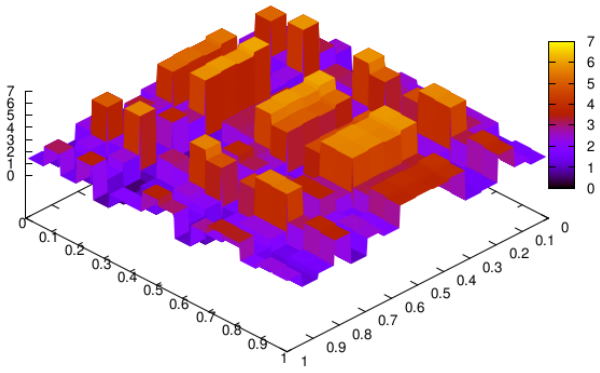

Fig 3.1. Randomly filled coefficient a with rank 1 and 16×16 domain splitting

of fact, if the rank $r_{1/a}$ is fixed, then r_u becomes a constant, no matter whatever big jumps and high oscillations in a might occur (see Fig. 3.1).

In this examples we take a separable function f with $r_f = 1$, but the same results are observed as well with $r_f > 1$. Consequently, from Eqs. (3.2)-(3.3) we observe an estimate of

the form

$$r_u \le C \cdot r_{1/a} \cdot \log(1/\varepsilon). \tag{3.4}$$

Thus the experimental rank of the solution on the uniform grid depends on $r_{1/a}$.

4. Conclusion

We presented an estimate of the separation (canonical) rank of the solution to diffusion equation with variable coefficient. This result is based on the known estimates of the separation rank for the Poisson equation with the constant coefficient in $\mathbb{R}^d$. As the structuring property of the coefficient, the number of cells with different values of a is included in the theoretical rank bound. This result can be applied for the discrete solution as well, with a discretization scheme on tensor grids which possess the approximation property. But the best approximation to discrete solution usually has essentially lower rank. To obtain this result theoretically, a special approach is required. The estimate (3.4) is going to be proved (under additional constraint to the separation of $1/a$) in the forthcoming paper.

Another part of work is the usage of more robust tensor formats, for example, the *Tensor Train (TT)* format [18, 19] and Quantics-TT [20, 21]. The stable linear operations and rank truncation in the formats allow to keep all the data in TT representation during the whole iterative solution process. As the TT ranks are less or equal to the canonical rank, the estimate (2.5) can be applied here straightforwardly. However, usually the bound (2.5) provides significantly overestimated ranks. So the application of TT/QTT formats to elliptic equations is to be considered in a separate paper.

Acknowlegments. This work was supported by the RFBR grants 11-01-12137, 11-01-00549-a, 09-01-91332 (joint with DFG), the Government Contracts П940, П1112, 14.740.11.0345 and Priority Research Grants of the Presidium and of the Department of Mathematical Sciences of the Russian Academy of Sciences.

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