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A DUAL COUPLED METHOD FOR BOUNDARY VALUE PROBLEMS OF PDE WITH COEFFICIENTS OF SMALL PERIOD*1)

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Abstract

In this paper the homogenization method is improved to develop one kind of dual coupled approximate method, which reflects both the macro-scope properties of whole structure and its loadings, and micro-scope configuration properties of composite materials. The boundary value problem of woven membrane is considered, the dual asymptotic expression of the exact solution is given, and its approximation and error estimation are discussed. Finally the numerical example shows the effectiveness of this dual coupled method.

1. Introduction

The mechanical performance analysis of the structures made of woven composite material and periodically perforated material is often encountered in the modern engineering analysis. Since this kind of composite material has periodically basic configurations, the static analysis of the structures made from this composite material leads to the boundary value problem of elliptic PDE with periodic coefficients, for example, the equilibrium problem of woven membrane under traverse loadings can be expressed in the boundary value problem of two dimension two order elliptic PDE as follows:

$$\text{(P)} \qquad \begin{cases} -\dfrac{\partial}{\partial x_i}(a_{ij}^{\epsilon}(x)\dfrac{\partial u_\epsilon}{\partial x_j}) = f(x), x \in \Omega \\ u_\epsilon|_{\partial\Omega} = 0 \end{cases}$$

Ω is shown in Figure 1, x represents both global coordinates of the structure and macro-scope properties of its geometry and loadings, ϵ is the length of basic configuration of composite material which is shown in Figure 2, and $a_{ij}^{\epsilon}(x)$ has periodicity, symmetry and ellipticity. Let $y = \dfrac{x}{\epsilon}$ and $a_{ij}(y) = a_{ij}^{\epsilon}(x)$, and then $a_{ij}(y)$ has periodicity with

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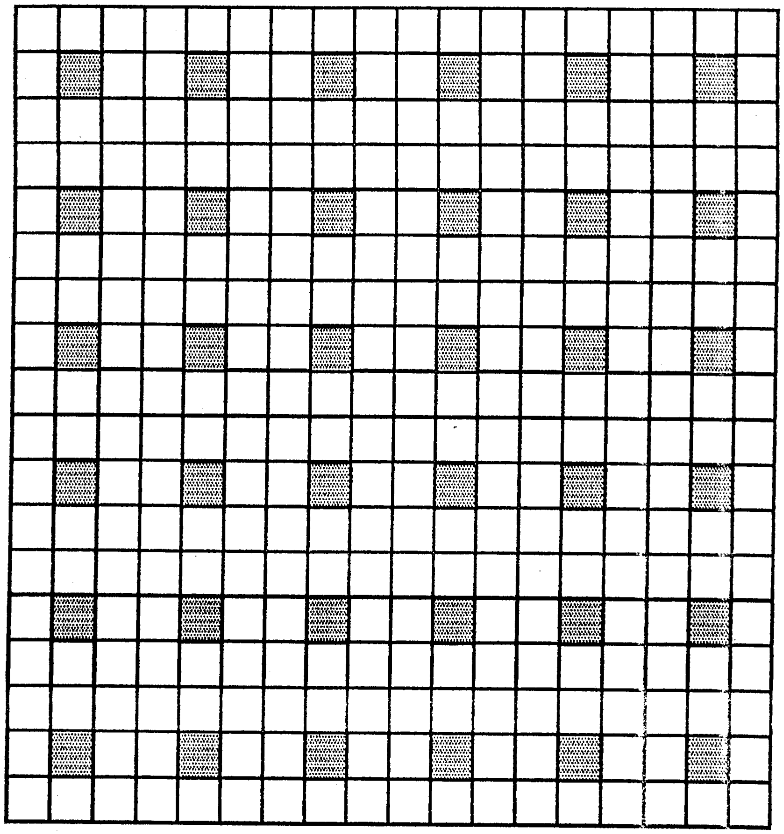

Figure 1. Weaved membrane

length 1. Engineers are often concerned with the stress state in some basic configuration due to stress concentration, and then most of breakages of structures happen locally. In order to obtain accurate stress results in the basic configuration the whole structure must be partitioned into very small meshes using finite element method, this leads to very large scale computation.

For this kind of problems of elliptic PDE, A. Bensoussan, J.L. Lions and G. Papanicolaou[1] proposed one kind of homogenization methods. The solution $u_\epsilon(x)$ of problem (P) is asymptotically expanded in dual (x, y) form as follows:

$$u_\epsilon(x) = u_0(x) + \epsilon u_1(x, y) + \epsilon^2 u_2(x, y) + \cdots, \tag{1.1}$$

where $u_0(x)$ is called as homogenization solution, and represents global mechanical and physical properties of structure, and $u_i(x, y)$ reflects both global mechanical behavior and the effect of micro-configuration of composite material. Formally the solution $u_\epsilon(x)$ is considered as one homogenization solution plus a series of relative periodic functions with high order coefficients ϵ^i.

In [1] the main results of homogenization method achieved are following:

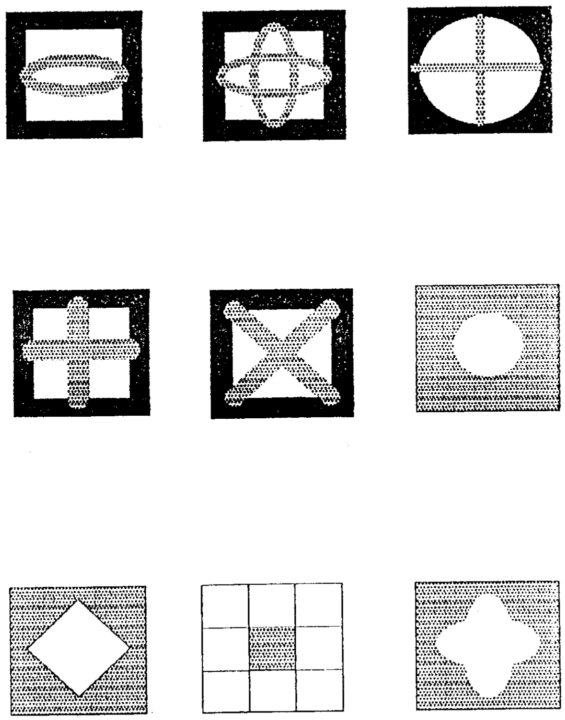

Figure 2. Some basic configurations

1. $u_1(x,y) = \dfrac{\partial u_0}{\partial x_k} w_k(y) + c(x)$, w_k exists uniquely such that,

$$\begin{cases} \exists\, w_k \in V = \{v \in H^1(\Omega), v \text{ has periodicity with length } 1\} \\ \widetilde{w_k(y)}\Big|_{\partial Y} = 0 \\ \displaystyle\int_Y a_{ij}(y) \frac{\partial w_k}{\partial y_j} \frac{\partial v}{\partial y_i} dy = \int_Y \frac{\partial a_{ik}}{\partial y_i} v dy, \forall v \in V. \end{cases}$$

where Y is the subdomain of basic configuration, and $\sim$ donates homogenization operator, $\tilde{\Phi} = \displaystyle\int_Y \Phi dy$.

2. As $\epsilon \to 0$, the homogenization solution $u_0(x)$ weakly converges to solution $u_\epsilon(x)$, and $u_0(x)$ satisfies

$$\begin{cases} a^*_{ik} = [a_{ij}(y)(\delta_{jk} + \dfrac{\partial w_k(y)}{\partial y_j})]^\sim \\ -\dfrac{\partial}{\partial x_i}(a^*_{ik} \dfrac{\partial u_0}{\partial x_k}) = f(x), x \in \Omega \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

The results of homogenization method only show how to determine homogenization solution $u_0(x)$ and how it approximates $u_\epsilon(x)$ as $\epsilon \to 0$. In practical engineering analysis, however, ϵ can't be regared as zero and is a finite value. For this case, how using asymptotic expression (1.1) approaches exact solution $u_\epsilon(x)$ of problem (P) ? How many terms in (1.1) should be considered to obtain enough approximate solution $u_n(x)$, and how to define and evaluate every $u_i(x,y)$? What about the error estimation of the approximate solution $u_n(x)$? We will discuss all these problems in this paper. At first we improve dual asymptotic expression (1.1) in macro-scope and micro-scope properties of composite material structure, and show out a definite problem for every $u_i(x,y)$ to form a dual coupled method. The approximation and error estimation of $u_n(x)$ to $u_\epsilon(x)$ are shown. Finally our algorithm is applied to the equilibrium problem of woven membrane, and relative numerical results are also shown out in the last section of this paper.

2. The Macro-Micro Coupled Method

First it shows out that for problem (P), if $f(x)$ belongs to $L_2(\Omega)$ and does not belong in $C^\infty(\Omega)$, then there exsists an operator J_τ such that $J_\tau(f) \in C^\infty(\Omega)$, and $\|J_\tau f - f\|_{L^2} \to 0$ as $\tau \to 0$.

And then for problem (P) we construct the following problem:

$$\begin{cases} -\dfrac{\partial}{\partial x_i}(a_{ij}^\epsilon(x)\dfrac{\partial U_\epsilon}{\partial x_j}) = J_\tau f(x) & x \in \Omega \\ U_\epsilon|_{\partial\Omega} = 0. \end{cases}$$

Obviously from the regularity theorem of solution it follows that

$$\|U_\epsilon - u_\epsilon\|_{H^2} \le C\,\|J_\tau f - f\|_{L^2} \to 0, (\tau \to 0).$$

Therefore it only needs to consider the problem with smooth $f(x)$. For short, we first consider the problem of one dimension.

2.1. The dual expression for one dimension problem

The problem of one dimension is following:

$$(\mathrm{P}^1) \qquad \begin{cases} -\dfrac{\partial}{\partial x}(a^\epsilon(x)\dfrac{\partial u_\epsilon}{\partial x}) = f(x) & x \in (0,1) \\ u_\epsilon(0) = u_\epsilon(1) = 0 \end{cases}$$

where $a^\epsilon(x)$ satisfies ellipticity and suppose that

1. $a^\epsilon(x)$ has periodicity with length ϵ, let $y = \dfrac{x}{\epsilon}$, so $a(y)$ is periodic function with length 1.

2. $[\dfrac{1}{\epsilon}] = \dfrac{1}{\epsilon}$. (Suppose it just for the boundary simplicity.)

According to (1.1), we suppose that $u_\epsilon(x)$ can be expressed with separated macro-scope variable x and micro-scope variable y,

$$u_\epsilon(x) = u_0(x) + \epsilon u_1(x)v_1(y) + \epsilon^2 u_2(x)v_2(y) + \cdots, \qquad (2.1-1)$$

and $u_0(x)$ and $v_i(y)$ satisfy the boundary conditions as follows:

$$u_0(0) = u_0(1) = 0, u_0(x) \text{ is smooth.} \qquad (2.1-2)$$

$$v_i(0) = v_i(1) = 0, v_i(y) \text{ has periodicity with length 1.} \qquad (2.1-3)$$

From derivatives of parametric variables it follows that

$$\begin{aligned}
\frac{du_\epsilon}{dx} &= \frac{du_0}{dx} + \epsilon\Big(\frac{du_1}{dx}v_1 + \epsilon^{-1}u_1\frac{dv_1}{dy}\Big) + \epsilon^2\Big(\frac{du_2}{dx}v_2 + \epsilon^{-1}u_2\frac{dv_2}{dy}\Big)\\
&\quad + \epsilon^3\Big(\frac{du_3}{dx}v_3 + \epsilon^{-1}u_3\frac{dv_3}{dy}\Big) + \epsilon^4\Big(\frac{du_4}{dx}v_4 + \epsilon^{-1}u_4\frac{dv_4}{dy}\Big) + \cdots\\
&= \Big(\frac{du_0}{dx} + u_1\frac{dv_1}{dy}\Big) + \epsilon\Big(\frac{du_1}{dx}v_1 + u_2\frac{dv_2}{dy}\Big) + \epsilon^2\Big(\frac{du_2}{dx}v_2 + u_3\frac{dv_3}{dy}\Big)\\
&\quad + \epsilon^3\Big(\frac{du_3}{dx}v_3 + u_4\frac{dv_4}{dy}\Big) + \cdots.
\end{aligned}$$

Let

$$\begin{aligned}
P_0 &= a(y)\Big(\frac{du_0}{dx} + u_1\frac{dv_1}{dy}\Big).\\
P_1 &= a(y)\Big(\frac{du_1}{dx}v_1 + u_2\frac{dv_2}{dy}\Big).\\
P_2 &= a(y)\Big(\frac{du_2}{dx}v_2 + u_3\frac{dv_3}{dy}\Big).\\
P_3 &= a(y)\Big(\frac{du_3}{dx}v_3 + u_4\frac{dv_4}{dy}\Big).\\
&\cdots\cdots.
\end{aligned}$$

It follows that

$$\begin{aligned}
a\frac{du_\epsilon}{dx} &= P_0 + \epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 P_3 + \cdots.\\
\frac{d}{dx}\Big(a\frac{du_\epsilon}{dx}\Big) &= \epsilon^{-1}\frac{\partial P_0}{\partial y} + \epsilon^0\Big(\frac{\partial P_0}{\partial x} + \frac{\partial P_1}{\partial y}\Big) + \epsilon^1\Big(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y}\Big) + \epsilon^2\Big(\frac{\partial P_2}{\partial x} + \frac{\partial P_3}{\partial y}\Big) + \cdots.
\end{aligned}$$

If

$$\begin{cases}
\dfrac{\partial P_0}{\partial y} = 0 & (2.2-1)\\
\dfrac{\partial P_0}{\partial x} + \dfrac{\partial P_1}{\partial y} = -f(x) & (2.2-2)\\
\dfrac{\partial P_1}{\partial x} + \dfrac{\partial P_2}{\partial y} = 0 & (2.2-3)\\
\dfrac{\partial P_2}{\partial x} + \dfrac{\partial P_3}{\partial y} = 0 & (2.2-4)\\
\cdots\cdots,
\end{cases}$$

and then the problem (P^1) is satisfied. Considering the homogeneous boundary condition of $u_0(x)$ and $v_i(y)$, $u_\epsilon(x)$ defined by (2.1) is the formal solution of problem (P^1).

For (2.2-1), if let $u_1(x) = \dfrac{du_0}{dx}$, then (2.2-1) is changed to

$$-\frac{d}{dy}\Big(a(y)\frac{dv_1}{dy}\Big) = \frac{da}{dy}.$$

Considering the homogeneous boundary condition of $v_1(y)$ in (2.1-3), and then $v_1(y)$ exists and is unique.

By using the homogenization operator on both sides of (2.2-2), it follows that

$$\begin{aligned}
&-\frac{\partial \tilde{P}_0}{\partial x} = f(x),\\
&-\frac{\partial}{\partial x}\Big(a(\frac{du_0}{dx} + \frac{du_0}{dx}\frac{dv_1}{dy})\Big)^{\sim} = f(x),\\
&a^*\frac{d^2u_0}{dx^2} = f(x).
\end{aligned}$$

where $a^* = (a + av_1')^{\sim}$.
Considering the homogeneous boundary condition of $u_0(x)$, $u_0(x)$ exists and is unique, and smooth inside [0,1].

And from (2.2-2) we have

$$\begin{aligned}
-u_2(x)(av_2')' &= f(x) + au_0'' + au_0''v_1' + u_0''(av_1)'\\
&= u_0''(-a^* + a + av_1' + (av_1)').
\end{aligned}$$

If let $u_2(x) = u_0^{(2)}$, then

$$-(av_2')' = -a^* + a + av_1' + (av_1)'.$$

Considering the homogeneous boundary condition of $v_2(y)$, and then $v_2(y)$ exists and is unique.

From (2.2-3), we have

$$-u_3(av_3')' = u_0^{(3)}\{(av_1 + av_2' + (av_2)'\}.$$

If let $u_3(x) = u_0^{(3)}$, then

$$-(av_3')' = av_1 + av_2' + (av_2)'.$$

Considering homogeneous boundary condition of $v_3(y)$, and then $v_3(y)$ exists and is unique.

$\cdots\cdots$

Let us sum up all above, it follows that :

Theorem 2.1. *There exists an dual expression*

$$u_0(x)+\sum_{i=1}^{\infty}\epsilon^i u_0^{(i)}(x)v_i(y), \tag{2.3}$$

where $u_0(x), v_i(y)(i=1,2,\cdots)$ *satisfy the following*

$$\begin{cases} -a^*(u_0)''=f(x),\\ -(a(y)v_1')'=a(y)',\\ -(a(y)v_2')'=-a^*+a+av_1'+(av_1)',\\ -(a(y)v_i')'=av_{i-2}+av_{i-1}'+(av_{i-1})', \forall i\geq 3. \end{cases} \tag{2.4}$$

and homogeneous boundary condition (2.1-2), (2.1-3), *and then the dual expression* (2.3) *is a formal solution of the problem* (P^1).

2.2. The dual expression in high dimension problem

It is obvious that the above process can be extended to high dimension problem (2, 3 dimension).

The high dimension problem can be described as

$$(\mathrm{P}^{\mathrm{N}}) \qquad \begin{cases} -\dfrac{\partial}{\partial x_i}(a_{ij}^{\epsilon}(x)\dfrac{\partial u_\epsilon}{\partial x_j})=f(x), x\in\Omega\\ u_\epsilon|_{\partial\Omega}=0. \end{cases}$$

where $a_{ij}^{\epsilon}(x)=a_{ij}(\frac{x}{\epsilon})$ satisfies ellipticity, and suppose that:

1. $a_{ij}^{\epsilon}(x)$ has periodicity with length ϵ. Let $y_i=\dfrac{x_i}{\epsilon}$, and then $a_{ij}(y)$ has periodicity with length 1.

2. $[\frac{1}{\epsilon}]=\frac{1}{\epsilon}$.

Similarly, the solution $u_\epsilon(x)$ of problem (P^N) is expressed in dual form as

$$u_\epsilon(x)=u_0(x)+\epsilon u_1(x,y)+\epsilon^2 u_2(x,y)+\cdots.$$

and then

$$\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u_\epsilon}{\partial x_j}\Big)=\epsilon^{-1}\frac{\partial P_{ij}^0}{\partial y_i}+\epsilon^0\Big(\frac{\partial P_{ij}^0}{\partial x_i}+\frac{\partial P_{ij}^1}{\partial y_i}\Big)+\epsilon^1\Big(\frac{\partial P_{ij}^1}{\partial x_i}+\frac{\partial P_{ij}^2}{\partial y_i}\Big)+\cdots$$

where

$$\begin{aligned} P_{ij}^0&=a_{ij}(y)\Big(\frac{\partial u_0}{\partial x_j}+\frac{\partial u_1}{\partial y_j}\Big),\\ P_{ij}^1&=a_{ij}(y)\Big(\frac{\partial u_1}{\partial x_j}+\frac{\partial u_2}{\partial y_j}\Big),\\ P_{ij}^2&=a_{ij}(y)\Big(\frac{\partial u_2}{\partial x_j}+\frac{\partial u_3}{\partial y_j}\Big),\\ &\cdots\cdots. \end{aligned} \tag{2.5}$$

And from problem ($\mathrm{P^N}$), if $u_0(x)$,$u_i(x,y)$, $(i=1,2,\cdots)$ satisfy the following equations

$$\begin{cases} \dfrac{\partial P_{ij}^0}{\partial y_i}=0 & (2.6-1)\\ \dfrac{\partial P_{ij}^0}{\partial x_i}+\dfrac{\partial P_{ij}^1}{\partial y_i}=-f(x) & (2.6-2)\\ \dfrac{\partial P_{ij}^1}{\partial x_i}+\dfrac{\partial P_{ij}^2}{\partial y_i}=0 & (2.6-3)\\ \cdots\cdots, \end{cases}$$

and homogeneous boundary conditions, then the solution $u_\epsilon(x)$ of problem ($\mathrm{P^N}$) is formally worked out.

For (2.3-1), if let $u_1(x,y)=w_l(y)\dfrac{\partial u_0}{\partial x_l}$, $(l=1,2,\cdots,N)$, a form with separated variables, then

$$-\frac{\partial}{\partial y_i}\Big(a_{ij}(y)\frac{\partial w_l}{\partial y_j}\Big)=\frac{\partial a_{il}}{\partial y_i},\quad y\in Y \tag{2.7}$$

where Y is micro-scope doamin of basic configuration, and if let $w_l(y)|_{\partial Y}=0$, and then $w_l(y)$ exists and is unique.

Using homogenization operator on both sides of (2.6-2), and considering the periodicity of P_{ij}^1 on y,we have

$$-\frac{\partial P_{ij}^{0\sim}}{\partial x_i}=f(x),$$
$$-\frac{\partial}{\partial x_i}\Big((a_{ij}+a_{ik}\frac{\partial w_j}{\partial y_k})^\sim\frac{\partial u_0}{\partial x_j}\Big)=f(x).$$

Define $a_{ij}^*=(a_{ij}+a_{ik}\dfrac{\partial w_j}{\partial y_k})^\sim$, and in [1] it has been proved that $\{a_{ij}^*\}$ is symmetric and positive definite . Therefore if $u_0(x)$ satisfies the following equation

$$-\frac{\partial}{\partial x_i}\Big(a_{ij}^*\frac{\partial u_0}{\partial x_j}\Big)=f(x), \tag{2.8}$$

with homogeneous boundary conditions, then $u_0(x)$ exists and is unique, and if $f(x)\in C^\infty(\Omega)$, $u_0(x)$ is also smooth inside Ω.

For (2.6-2), if let $u_2(x,y)=w_{kl}(y)\dfrac{\partial^2 u_0}{\partial x_k\partial x_l}(k,l=1,2,\cdots,N)$, and then (2.6-2) is changed into

$$\begin{aligned}&-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial w_{kl}}{\partial y_j}\frac{\partial^2 u_0}{\partial x_k\partial x_l}\Big)\\ &=f+\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u_0}{\partial x_j}\Big)+\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u_0}{\partial x_l}\frac{\partial w_l}{\partial y_j}\Big)+\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial^2 u_0}{\partial x_j\partial x_l}w_l\Big)\\ &=-\frac{\partial}{\partial x_k}\Big(a_{kl}^*\frac{\partial u_0}{\partial x_l}\Big)+\frac{\partial}{\partial x_k}\Big(a_{kl}\frac{\partial u_0}{\partial x_l}\Big)+\frac{\partial}{\partial x_k}\Big(a_{kj}\frac{\partial u_0}{\partial x_l}\frac{\partial w_l}{\partial y_j}\Big)+\frac{\partial}{\partial y_i}\Big(a_{ik}\frac{\partial^2 u_0}{\partial x_k\partial x_l}w_l\Big).\end{aligned}$$

For w_{kl} we obtain that

$$-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial w_{kl}}{\partial y_j}\Big) = -a^*_{kl} + a_{kl} + a_{kj}\frac{\partial w_l}{\partial y_j} + \frac{\partial}{\partial y_i}(a_{ik}w_l), \tag{2.9}$$

considering the homogeneous boundary condition of $w_{kl}(y)|_{\partial Y} = 0$, $w_{kl}(y)$ exists and is unique.

For (2.6-3), if let $u_3(x,y) = w_{hkl}\dfrac{\partial^3 u_0}{\partial x_h \partial x_k \partial x_l}(h,k,l = 1,2,\cdots,N)$, then (2.6-3) is changed into

$$\begin{aligned}
&-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial w_{hkl}}{\partial y_j}\frac{\partial^3 u_0}{\partial x_h\partial x_k\partial x_l}\Big)\\
&= \frac{\partial}{\partial x_i}\Big(a_{ij}(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_2}{\partial y_j})\Big) + \frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial u_2}{\partial x_j}\Big)\\
&= \frac{\partial}{\partial x_i}\Big(a_{ij}(w_l\frac{\partial^2 u_0}{\partial x_j\partial x_l} + \frac{\partial w_{kl}}{\partial y_j}\frac{\partial^2 u_0}{\partial x_k\partial x_l})\Big) + \frac{\partial}{\partial y_i}\Big(a_{ij}w_{kl}\frac{\partial^3 u_0}{\partial x_j\partial x_k\partial x_l}\Big)\\
&= \frac{\partial}{\partial x_h}\Big(a_{hk}w_l\frac{\partial^2 u_0}{\partial x_k x_l} + a_{hj}\frac{\partial w_{kl}}{\partial y_j}\frac{\partial^2 u_0}{\partial x_k\partial x_l}\Big) + \frac{\partial}{\partial y_i}\Big(a_{ih}w_{kl}\frac{\partial^3 u_0}{\partial x_h\partial x_k\partial x_l}\Big).
\end{aligned}$$

Therefore for definite h,k,l, there is

$$-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial w_{hkl}}{\partial y_j}\Big) = a_{hk}w_l + a_{hj}\frac{\partial w_{kl}}{\partial y_j} + \frac{\partial}{\partial y_i}(a_{ih}w_{kl}), \tag{2.10}$$

considering the homogeneous boundary condition of $w_{hkl}(y)|_{\partial Y} = 0$, $w_{hkl}(y)$ exists and is unique.

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Summing up all above, we obtain the following results.

Theorem 2.2. *There exists an expression*

$$u_0(x) + \sum_{n=1}^{\infty}\frac{\partial^n u_0}{\partial x_{i_n}\cdots\partial x_{i_1}}w_{i_n\cdots i_1}(y), \tag{2.11}$$

which is the formal solution of problem ($\mathrm{P^N}$) *where* $u_0(x)$ *satisfies* (2.8), *and* $w_{kl}(y)$ *satisfy* (2.9) *and* $w_*(y)$ *satisfy*

$$-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial w_{i_n\cdots i_1}}{\partial y_j}\Big) = a_{i_n i_{n-1}}w_{i_{n-2}\cdots i_1} + a_{i_n j}\frac{\partial w_{i_{n-1}\cdots i_1}}{\partial y_j} + \frac{\partial}{\partial y_i}(a_{ii_n}w_{i_{n-1}\cdots i_1}), n \geq 3$$

and $u_0(x)$ *on* Ω *and* $w_*(y)$ *on* Y *satisfy the homogeneous boundary conditions.*

3. Approximation and Error Estimation

In practical computation the formal expression of solution $u_\epsilon(x)$ previously can not be evaluated completely. It means that only the sum of the first several terms in formal expression of $u_\epsilon(x)$ can be evaluated, it is, for one dimension, only evaluate

$$u_n(x) = u_0(x) + \sum_{i=1}^{n} \epsilon^i u_0^{(i)}(x) v_i(y), \tag{3.1}$$

for high dimension, only evaluate

$$u_m(x) = u_0(x) + \sum_{n=1}^{m} \epsilon^n \frac{\partial^n u_0(x)}{\partial x_{i_n} \cdots \partial x_{i_1}} w_{i_n \cdots i_1}. \tag{3.2}$$

The approximation and error estimation of them will be discussed following.

3.1. Approximation

For problem (P^1), if $\|a\|_{H^1}$ and $\|f\|_{L^2}$ are bounded, by Lax-Milgram theorem we have

$$\begin{aligned}
\|v_1\|_{H^1} \leq \;& C\,\|a'\|_{L^2}\,, \\
\|v_2\|_{H^1} \leq \;& C\,\|-a^* + a + av_1' + (av_1)'\|_{L^2} \leq C + C\,\|v_1\|_{H^1}\,, \\
\|v_i\|_{H^1} \leq \;& C\,\|av_{i-2} + av_{i-1} + (av_{i-1})'\|_{L^2} \\
\leq \;& C(\|v_{i-1}\|_{H^1} + \|v_{i-2}\|_{H^1}), \forall i \geq 3.
\end{aligned} \tag{3.3}$$

Following discussing proceeds in two case respectively:

(1) If $\left|u_0^{(i)}(x)\right| \leq C, \quad \forall i$, then

$$\begin{aligned}
\left\|\sum_{i=n}^{\infty} \epsilon^i u_0^{(i)}(x) v_i(y)\right\|_{H^1(\Omega)} & \leq C \sum_{i=n}^{\infty} \epsilon^i \left|v_i(y)\right|_{H^1(\Omega)} \\
& \leq C \sum_{i=n}^{\infty} \epsilon^i \Big(\int_0^1 v_i'(\frac{x}{\epsilon})^2 dx\Big)^{1/2} \\
& = C \sum_{i=n}^{\infty} \epsilon^i \epsilon^{-1} \Big(\int_0^1 v_i'(y)^2 dy\Big)^{1/2} \\
& \leq C \sum_{i=n}^{\infty} \epsilon^{i-1} \left\|v_i(y)\right\|_{H^1(Y)}
\end{aligned}$$

from (3.3) it follwos that

$$\left\|\sum_{i=n}^{\infty} \epsilon^i u_0^{(i)}(x) v_i(y)\right\|_{H^1(\Omega)} \leq C_1 (C_2 \epsilon)^{n-1}. \tag{3.4}$$

(2) If $\left|u_0^{(i)}(x)\right| \leq C\left|u_0^{(i-1)}(x)\right|, \quad \forall i$, then

$$\begin{aligned}
\left\|\sum_{i=n}^{\infty}\epsilon^i u_0^{(i)}(x)v_i(y)\right\|_{H^1(\Omega)} &\leq C\sum_{i=n}^{\infty}\epsilon^i\left|u_0^{(i)}v_i\right|_{H^1(\Omega)} \\
&\leq C\sum_{i=n}^{\infty}\epsilon^i\left|u_0^{(i)}\right||v_i|_{H^1(\Omega)} \\
&\leq C|u_0|\sum_{i=n}^{\infty}(C\epsilon)^{i-1}|v_i(y)|_{H^1(Y)},
\end{aligned}$$

from (3.3) it follows that

$$\left\|\sum_{i=n}^{\infty}\epsilon^i u_0^{(i)}(x)v_i(y)\right\|_{H^1(\Omega)} \leq C_1(C_3\epsilon)^{n-1}. \tag{3.5}$$

Therefore the following theorem is obtained:

Theorem 3.1. *Under the suppositions of small periodicity of coefficients and the above condition* (1) *or* (2)*, the approximation solution expressed in* (3.1) *converges to exact solution* $u_\epsilon(x)$ *of problem* (P^1) *by* H^1 *norm ,and* $\|u_n(x)-u_\epsilon(x)\|_{H^1(\Omega)}$ *has the same order as* $(C\epsilon)^{n-1}$.

The similar result will be got in high dimension analysis.

For problem (P^N), if $\|a\|_{H^1}$ and $\|f\|_{L^2}$ are bounded, by Lax-Milgram theorem we obtain that

$$\begin{aligned}
&\|w_{i_n\cdots i_1}\|_{H^1} \\
&\leq C\left\|a_{i_n i_{n-1}}w_{i_{n-2}\cdots i_1}+a_{i_n j}\frac{\partial w_{i_{n-1}\cdots i_1}}{\partial y_j}+\frac{\partial}{\partial y_i}(a_{ii_n}w_{i_{n-1}\cdots i_1})\right\|_{L^2} \\
&\leq C(\|w_{i_{n-2}\cdots i_1}\|_{L_2}+\sum_{j=1}^{N}\left\|\frac{\partial w_{i_{n-1}\cdots i_1}}{\partial y_j}\right\|_{L^2}+\sum_{i=1}^{N}\left\|\frac{\partial}{\partial y_i}w_{i_{n-1}\cdots i_1}\right\|_{L^2}) \\
&\leq C(\|w_{i_{n-2}\cdots i_1}\|_{H^1}+\|w_{i_{n-1}\cdots i_1}\|_{H^1}), \forall i\geq 3.
\end{aligned} \tag{3.6}$$

Under conditions

(1) $\left|\dfrac{\partial^n u_0}{\partial x_{i_n}\cdots\partial x_{i_1}}\right| \leq C$

or

(2) $\left|\dfrac{\partial^n u_0}{\partial x_{i_n}\cdots\partial x_{i_1}}\right| \leq C\left|\dfrac{\partial^{n-1} u_0}{\partial x_{i_{n-1}}\cdots\partial x_{i_1}}\right|$,

we obtain that

$$\left\|\sum_{n=m}^{\infty}\epsilon^n\frac{\partial^n u_0}{\partial x_{i_n}\cdots\partial x_{i_1}}w_{i_n\cdots i_1}(y)\right\|_{H^1} \leq C_1(C_2\epsilon)^{m-1}. \tag{3.7}$$

From this it follows that:

Theorem 3.2. *Under the suppositions of small periodicity of coefficients and above condition* (1) *or* (2)*, the approximation solution expressed in* (3.2) *converges to the exact solution* $u_\epsilon(x)$ *of problem* (P^N) *by* H^1 *norm ,and* $\|u_m(x) - u_\epsilon(x)\|_{H^1(\Omega)}$ *has the same order as* $(C\epsilon)^{n-1}$.

3.2. Error estimation in $H^2(\Omega)$ norm

If u_ϵ is the exact solution of problem (P^1), since

$$\begin{aligned} & \frac{d}{dx}\Big(a\frac{d(\epsilon^i u_0^{(i)} v_i)}{dx}\Big) \\ = & \frac{d}{dx}(a\epsilon^i u_0^{(i+1)} v_i + a\epsilon^{(i-1)} u_0^{(i)} v_i') \\ = & \epsilon^i u_0^{(i+2)} a v_i + \epsilon^{i-1} u_0^{(i+1)}((av_i)' + av_i') + \epsilon^{i-2} u_0^{(i)} (av_i')', \end{aligned}$$

then

$$\begin{aligned} & \frac{d}{dx}\Big(a\frac{d(u_\epsilon - u_n)}{dx}\Big) \\ = & \frac{d}{dx}(a\frac{du_\epsilon}{dx}) - \frac{d}{dx}(a\frac{du_0}{dx}) - \sum_{i=1}^{n} \frac{d}{dx}\Big(a\frac{d(\epsilon^i u_0^{(i)} v_i)}{dx}\Big) \\ = & -f - \frac{d}{dx}(a\frac{du_0}{dx}) - \sum_{i=3}^{n+2} \epsilon^{i-2} u_0^{(i)} a v_{i-2} - \sum_{i=2}^{n+1} \epsilon^{i-2} u_0^{(i)}((av_{i-1})' + av_{i-1}') \\ & - \sum_{i=1}^{n} \epsilon^{i-2} u_0^{(i)} (av_i')' \\ = & -f - \frac{d}{dx}(a\frac{du_0}{dx}) - \sum_{i=3}^{n+1} \epsilon^{i-2} u_0^{(i)} (av_{i-2} + av_{i-1}' + (av_{i-1})') \\ & - \sum_{i=1}^{n} \epsilon^{i-2} u_0^{(i)} (av_i')' - \epsilon^n u_0^{(n+2)} a v_n - u_0^{(2)}((av_1)' + av_1'). \end{aligned}$$

According to theorem 2.1, we have

$$\frac{d}{dx}\Big(a(x)\frac{d(u_\epsilon - u_n)}{dx}\Big) = \epsilon^{n-1} u_0^{(n+1)} (av_{n+1}')' - \epsilon^n u_0^{(n+2)} a v_n,$$

then $u_\epsilon - u_n$ satisfies

$$\begin{cases} -\dfrac{\partial}{\partial x}\Big(a^\epsilon(x)\dfrac{\partial(u_\epsilon - u_n)}{\partial x}\Big) = \epsilon^{n-1} u_0^{(n+1)} (av_{n+1}')' - \epsilon^n u_0^{(n+2)} a v_n, x \in \Omega \\ (u_\epsilon - u_n)|_{\partial\Omega} = 0. \end{cases} \tag{3.8}$$

By regularity theorem of solution,

$$\begin{aligned}\|u_\epsilon - u_n\|_{H^2} &\le C\left\|\epsilon^{n-1}u_0^{(n+1)}(av'_{n+1})' - \epsilon^n u_0^{(n+2)}av_n\right\|_{L^2}\\ &= C\left\|\epsilon^{n-1}u_0^{(n+1)}(-av_{n-1} - av'_n - (av_n)') - \epsilon^n u_0^{(n+2)}av_n\right\|_{L^2}\\ &\le C\epsilon^{n-1}\left\|u_0^{(n+1)}\right\|_{L^2}(\|v_{n-1}\|_{H^1} + \|v_n\|_{H^1}) + C\epsilon^n\left\|u_0^{(n+2)}\right\|_{L^2}\|v_n\|_{H^1}.\end{aligned}$$

Since $\|v_n\|_{H^1} \le C(\|v_{n-1}\|_{H^1} + \|v_{n-2}\|_{H^1})$, we have

$$\|u_\epsilon - u_n\|_{H^2} \le C(\left\|u_0^{(n+1)}\right\|_{L^2} + \epsilon\left\|u_0^{(n+2)}\right\|_{L^2})(C\epsilon)^{n-1}\|v_2\|_{H^1}.$$

Now we obtain the following conclusion:

Theorem 3.3. *If $\|a\|_{H^1}, \|f\|_{L^2}, \left\|u_0^{(n+1)}\right\|_{L^2}$ and $\left\|u_0^{(n+2)}\right\|_{L^2}$ are bounded, then the difference between the exact solution of problem* (P^1) *and the approximation solution composed from the first $n+1$ terms $u_0 + \sum_{i=1}^{n}\epsilon^i u_0^{(i)} v_i$ has the order $(C\epsilon)^{n-1}$ by H^2 norm.*

For high dimension problem, similarly, $u_\epsilon - u_m$ satisfies homogeneous boundary condition and

$$\begin{aligned}-\frac{\partial}{\partial x_i}(a_{ij}{}^\epsilon(x)\frac{\partial(u_\epsilon - u_m)}{\partial x_j}) = \;& \epsilon^{m-1}\frac{\partial^{m+1}u_0}{\partial x_{i_{m+1}}\cdots\partial x_{i_1}}\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial w_{i_{m+1}\cdots i_1}}{\partial y_j})\\ &-\epsilon^m\frac{\partial^{m+2}u_0}{\partial x_{i_{m+2}}\cdots\partial x_{i_1}}a_{i_{m+2}i_{m+1}}w_{i_m\cdots i_1}.\end{aligned}$$

And then

$$\|u_\epsilon - u_m\|_{H^2} \le C\Big(\left\|\frac{\partial^{m+1}u_0}{\partial x_{i_{m+1}}\cdots\partial x_{i_1}}\right\|_{L^2} + \epsilon\left\|\frac{\partial^{m+2}u_0}{\partial x_{i_{m+2}}\cdots\partial x_{i_1}}\right\|_{L^2}\Big)(C\epsilon)^{m-1}\|w_{i_2i_1}\|_{H^1}.$$

Theorem 3.4. *If $\left\|\frac{\partial^{m+1}u_0}{\partial x_{i_{m+1}}\cdots\partial x_{i_1}}\right\|_{L^2}, \left\|\frac{\partial^{m+2}u_0}{\partial x_{i_{m+2}}\cdots\partial x_{i_1}}\right\|_{L^2}, \|a\|_{H^1}$ and $\|f\|_{L^2}$ are bounded, then the difference between the exact solution of problem* (P^{N}) *and approximation solution $u_0(x) + \sum_{n=1}^{m}\epsilon^n\frac{\partial^n u_0}{\partial x_{i_n}\cdots\partial x_{i_1}}w_{i_n\cdots i_1}$ has the order $(C\epsilon)^{m-1}$ by H^2 norm.*

From previous results it is easy to see that if the period of PDE's coefficients is so small that $C\epsilon \le 1$, the formal solution in theorem 2.1 and Theorem 2.2 are the exact solution of problem (P^1) and (P^{N}) respectively, and then $u_n(x)$ expressed in (3.1) and $u_m(x)$ in (3.2) are approximate solution of them respectively, and the approximate is improved rapidly when the number of the terms included raises.

4. Numerical Experiment

Using previous methods we have made some numerical experiment for model problem of woven membrane

$$\begin{cases} -\dfrac{\partial}{\partial x_i}(a_{ij}^{\epsilon}(x)\dfrac{\partial u_\epsilon}{\partial x_j}) = f(x_1, x_2), \qquad x \in \Omega = [0,1]\times[0,1] \\ u_\epsilon|_{\partial\Omega} = 0 \end{cases} \tag{4.1}$$

where

$$a_{ij}^{\epsilon}(x) = \begin{cases} \lambda\delta_{ij}, & \text{the other} \\ \delta_{ij}, & \text{in shadow} \end{cases}$$

and $u_0(x_1, x_2) = x_1x_2(1-x_1)(1-x_2)$ satisfies homogenization equation, $f(x_1, x_2)$ is designated and the problem is now definite.

According to our method, we need to compute

$$\begin{aligned} u_4(x) = \quad & u_0(x) + \epsilon\Big\{\frac{\partial u_0}{\partial x_1}w_1 + \frac{\partial u_0}{\partial x_2}w_2\Big\} \\ & +\epsilon^2\Big\{\frac{\partial^2 u_0}{\partial x_1\partial x_1}w_{11} + \frac{\partial^2 u_0}{\partial x_1\partial x_2}w_{12} + \frac{\partial^2 u_0}{\partial x_2\partial x_1}w_{21} + \frac{\partial^2 u_0}{\partial x_2\partial x_2}w_{22}\Big\} \\ & +\epsilon^3\Big\{\frac{\partial^3 u_0}{\partial x_1\partial x_1\partial x_1}w_{111} + \frac{\partial^3 u_0}{\partial x_1\partial x_1\partial x_2}w_{112} + \frac{\partial^3 u_0}{\partial x_1\partial x_2\partial x_1}w_{121} + \frac{\partial^3 u_0}{\partial x_1\partial x_2\partial x_2}w_{122} \\ & +\frac{\partial^3 u_0}{\partial x_2\partial x_1\partial x_1}w_{211} + \frac{\partial^3 u_0}{\partial x_2\partial x_1\partial x_2}w_{212} + \frac{\partial^3 u_0}{\partial x_2\partial x_2\partial x_1}w_{221} + \frac{\partial^3 u_0}{\partial x_2\partial x_2\partial x_2}w_{222}\Big\} \\ & +\epsilon^4\Big\{\frac{\partial^4 u_0}{\partial x_1\partial x_1\partial x_2\partial x_2}w_{1122} + \frac{\partial^4 u_0}{\partial x_2\partial x_2\partial x_1\partial x_1}w_{2211} + \frac{\partial^4 u_0}{\partial x_1\partial x_2\partial x_1\partial x_2}w_{1212} \\ & +\frac{\partial^4 u_0}{\partial x_2\partial x_1\partial x_2\partial x_1}w_{2121} + \frac{\partial^4 u_0}{\partial x_1\partial x_2\partial x_2\partial x_1}w_{1221} + \frac{\partial^4 u_0}{\partial x_2\partial x_1\partial x_1\partial x_2}w_{2112}\Big\}. \end{aligned}$$

Considering the symmetry of the problem, we can delete the zero terms and incoorperate the equal terms, and then only need to compute

$$\begin{aligned} u_4(x) \quad &= u_0(x) - 2\epsilon^2(x_1 + x_2 - x_1^2 - x_2^2)w_{11}(y_1, y_2) + \\ & 4\epsilon^3(x_2 + x_1 - 1)w_{211}(y_1, y_2) + 8\epsilon^4 w_{2211}(y_1, y_2), \end{aligned}$$

where $w_{11}(y), w_{211}(y), w_{2211}(y)$ satisfy homogeneous conditions on the boundary of basic configuration Y and the following equations respectively

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\Big(a_{ij}(y)\dfrac{\partial w_{11}}{\partial y_j}\Big) = -\tilde{a_{11}} + a_{11}, \\ -\dfrac{\partial}{\partial y_i}\Big(a_{ij}(y)\dfrac{\partial w_{211}}{\partial y_j}\Big) = 2a_{11}\dfrac{\partial w_{11}}{\partial y_2}, \\ -\dfrac{\partial}{\partial y_i}\Big(a_{ij}(y)\dfrac{\partial w_{2211}}{\partial y_j}\Big) = a_{11}w_{11} + 2a_{11}\dfrac{\partial w_{211}}{\partial y_2}. \end{cases}$$

Using FEM $u_0(x)$ is solved in the whole domain Ω, and $w_{11}(y)$, $w_{211}(y)$, $w_{2211}(y)$ on one basic configuration are solved.

The following table shows the numerical results on $\epsilon = 1/3, x_2 = 0.5, \lambda = 6$, and Figure 3 shows the correspondent curves.

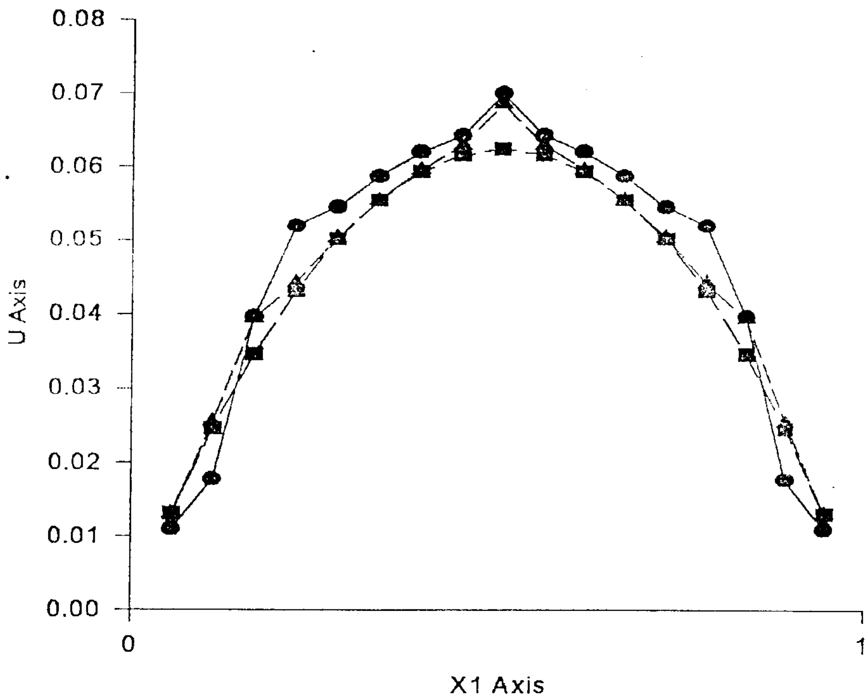
x_1	Direct solution	Homogenizaion solution	Macro-micro coupled solution
1/18	0.11004E-01	0.13117E-01	0.13042E-01
2/18	0.17737E-01	0.24691E-01	0.25594E-01
3/18	0.39724E-01	0.34722E-01	0.39513E-01
4/18	0.52016E-01	0.43210E-01	0.44304E-01
5/18	0.54571E-01	0.50154E-01	0.50418E-01
6/18	0.58789E-01	0.55556E-01	0.55556E-01
7/18	0.62069E-01	0.59414E-01	0.59699E-01
8/18	0.64305E-01	0.61728E-01	0.63014E-01
9/18	0.70099E-01	0.62500E-01	0.68659E-01

From the numerical results and solving process it follows that using dual coupled method described previously the precision of the numerical results can be greatly improved after adding the effect of $w_*(y)$ to the homogenization solution $u_0(x)$. As all periodical solutions $w_*(y)$ are evaluated on same basic configuration, the global stiffness matrix is assembled and decomposed only one time, and then the nodal loading corresponding to $w_*(y)$ can be evaluated recurrently, and then forward substitution and backward substitution are performed for every FE equations of $w_*(y)$. Therefore additional amount of computing is very small. In comparison with directly refining FE meshes, the dual coupled method in this paper has very small computing amount and high accuracy, and thus it is available method.

Obviously the above method can be extended to mechanical analysis of other structure with periodical configuration.

This paper only discussed the dual coupled method and its approximation analysis theoretically. The numerical analysis is dealt with in rough. There are several problems in numerical analysis on this method, which have not been discussed, while this method is used to analyse practical problems, such as, how to evaluate higher order derivatives $u_0^{(i)}(x)$, how to calculate the right sides of equation for every $w_{i_n \cdots i_1}(y)$, and what about their approximation, and how to deal with Newman boundary problam, and if the domain Ω contains some incomplete basic configurations and how to treat the problem, and so on. All of these problems need to be discussed in other paper.

Finally an important remark is about O.A. Oleinik's book[2]. When we finish this paper, we are very pleased to find that O.A. Oleinik has also used almost similar method, while there are some obvious differences between our paper and her book. In one hand, we give a relatively simple mathematical expression and practical numerical method based on FEM, therefore numerical effect is clearly shown. In the other hand, O.A. Oleinik's work concentrates on detailed and beautiful mathematical analysis, which will help to deepen our future research on this field.



■ Homogenization solution
● Refined FEM solution
▲ Macro-Micro Coupled Method solution

Figure 3. Numerical Results

References

[1] A. Bensoussan, J.L. Lions, Asymptotic Analysis For Periodic Structures, North Holland, 1978.

[2] O.A. Oleinik, Mathematical Problems in Elasticity and Homogenization, North Holland, 1992.