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NONOVERLAPPING DOMAIN DECOMPOSITION METHOD WITH MIXED ELEMENT FOR ELLIPTIC PROBLEMS*1)

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Abstract

In this paper we consider the nonoverlapping domain decomposition method based on mixed element approximation for elliptic problems in two dimentional space. We give a kind of discrete domain decomposition iterative algorithm using mixed finite element, the subdomain problems of which can be implemented parallelly. We also give the existence, uniqueness and convergence of the approximate solution.

1. Introduction

Domain decomposition as a new method of computational mathematics, was developed since the development of parallel computers and multiprocessor supercomputers. Using domain decomposition we can decrease the scale of the problem and implement the sub-problems on parallel computer. From a technical point of view most of domain decomposition methods considered so far have been dealing with finite element methods. In [1, 2] Zhang and Huang have given a kind of nonoverlapping domain decomposition procedure with piecewise linear finite element approximation.

Since the advantage of mixed element method in dealing with some engineering problems when accurate approximates to the first derivatives of the solution of the elliptic problem is required, such as numerical simulation in oil recovery, as early as 1988, Glowinski and Wheeler[3] have given a domain decomposition conjugate gradient algorithm with mixed element.

In this paper we give a kind of nonoverlapping domain decomposition algorithm with mixed finite element, which can be implemented in parallel computer. We also give the existence, uniqueness and convergence analysis. Finally we give the numerical examples.

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2. Domain Decomposition Algorithm

Without loss of generality we consider the following problem:

$$\begin{cases} -\triangle u = f, & \text{in}\Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on}\partial\Omega. \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded domain and can be decomposed into two polygonal domains, n is the unit vector in outer normal direction, $f \in L^2(\Omega)$ satisfying

$$\int_\Omega f dx = 0. \tag{2}$$

We decomposition Ω into nonoverlapping subdomains Ω_1, Ω_2 such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and Ω_1, Ω_2 are two polygonal domains, $\partial\Omega_1 \cap \partial\Omega_2$ is a striaight line. Let

$$\Gamma = \partial\Omega_1 \cap \partial\Omega_2 \cap \Omega; \Gamma_i = \partial\Omega \cap \partial\Omega_i (i = 1, 2).$$

then $\partial\Omega_i = \Gamma \cup \Gamma_i (i = 1, 2)$.

Let $(\cdot,\cdot)$ denote the innerproduct on $L^2(\Omega)$ or $(L^2(\Omega))^2, (\cdot,\cdot)_i$ denote the innerproduct on $L^2(\Omega_i)$ or $(L^2(\Omega_i))^2, \sigma = -\bigtriangledown u$, then we can derive the mixed formulation of (1): Find $(\sigma, u) \in H^0(\text{div}; \Omega) \times L^2(\Omega)$ such that

$$\begin{cases} (\sigma, q) - (\text{div}q, u) = 0, & \forall q \in H^0(\text{div}, \Omega), \\ (\text{div}\sigma, w) = (f, w), & \forall w \in L^2(\Omega), \end{cases} \tag{3}$$

where

$$H(\text{div}; \Omega) = \{v : v \in (L^2(\Omega))^2, \text{div}v \in L^2(\Omega)\},$$

$$H^0(\text{div}; \Omega) = \{v : v \in H^0(\text{div}, \Omega), v \cdot n|_{\partial\Omega} = 0\}.$$

Under the condition of

$$\int_{\Omega_1} u dx = 0, \tag{4}$$

the problem (3), or (1), has a unique solution.

Let $\Omega_h = \{k\}$ denote the quasi-uniform trianglation of Ω based on the discretization of Ω_1, Ω_2, with elements of size h. We choose $Q_h \times M_h \subset H^0(\text{div}, \Omega) \times L^2(\Omega)$ as the lowest order Raviart-Thomas mixed element space. Then the mixed element solution of problem (3), $(\sigma_h, u_h) \in Q_h \times M_h$, satisfying

$$\begin{cases} (\sigma_h, q) - (\text{div}q, u_h) = 0, & \forall q \in Q_h, \\ (\text{div}\sigma_h, v) = (f, v), & \forall v \in M_h. \end{cases} \tag{5}$$

$$\int_{\Omega_h} u_1 dx = 0. \tag{6}$$

The solution of (5)-(6) is unique and we have[5]

$$\|\sigma_h - \sigma\| + \|u_h - u\| \le Ch, \tag{7}$$

where C denotes a generic constant independent of h.

Let $M_i \subset L^2(\Omega_i), Q_i \subset H(\text{div};\Omega)$ be the following spaces:

$$M_i = \{v|_{\Omega_i} : v \in M_h\}, i = 1, 2, \tag{8}$$

$$Q_i = \{q|_{\Omega_i} : q \in Q_h\}, Q_i^0 = \{q \in Q_i : q \cdot n|_{\partial\Omega_i} = 0\}, i = 1, 2, \tag{9}$$

$$P_{0h} = \{q \in Q_h : q \cdot n|_{\partial k_i} = 0, \text{ if } k_i \cap \Gamma = \emptyset, i = 1, 2, 3, k \in \Omega_h\}, \tag{10}$$

where $\partial k_i, i = 1, 2, 3,$ denotes the three sides of k. It is clear that for any $q \in Q_i$, we have $q \cdot n|_{\partial\Omega_i \cap \partial\Omega} = 0$ and

$$Q_h = Q_1^0 \oplus Q_2^0 \oplus P_{0h.} \tag{11}$$

Let $T_1 = \{\tau\}$ denote the partition of Γ which is the restriction of Ω_h on Γ, $S(\Gamma) = \{v \in L^2(\Gamma) : v|_\tau = \text{constant}, \forall \tau \in T_1\}$, n_1, n_2 denotes the outer normal directions of Ω_1, Ω_2 on $\Gamma and d\pi \in P_{0h}$ be a function such that $\int_\Gamma \pi \cdot n_1 d\Gamma \ne 0, d_0 \in S(\Gamma)$ such that $\int_\Gamma d_0 d\Gamma - \int_{\Omega_1} f dx = 0$, we give the following domain decomposition mixed element procedure:

For n=0, 1, 2, ...

1) Define $\sigma_i^{2n} \in Q_i, u_i^{2n} \in M_i (i = 1, 2)$ such that

$$(\sigma_i^{2n}, q)_i - (\text{div}q, u_i^{2n})_i = 0, \forall q \in Q_i^0, i = 1, 2, \tag{12}$$

$$(\text{div}\sigma_i^{2n}, v)_i = (f, v)_i, \forall v \in M_i, i = 1, 2, \tag{13}$$

$$\sigma_i^{2n} \cdot n_1|_\Gamma = d^n, i = 1, 2, \tag{14}$$

$$\int_{\Omega_1} u_1^{2n} dx = 0, \tag{15}$$

$$\sum_{i=1}^{2}((\sigma_i^{2n}, \pi)_i - (\text{div}\pi, u_i^{2n})_i) = 0. \tag{16}$$

2) For $q \in P_{0h}$ define

$$g^n(q) = \theta_1[(\sigma_1^{2n}, q)_1 - (\text{div}q, u_1^{2n})_1] - (1 - \theta_1)[(\sigma_2^{2n}, q)_2 - (\text{div}q, u_2^{2n})_2]. \tag{17}$$

3) Define $\sigma_i^{2n+1} \in Q_i, u_i^{2n+1} \in M_i (i = 1, 2)$ such that

$$(\sigma_i^{2n+1}, q)_i - (\text{div}q, u_i^{2n+1})_i = 0, \forall q \in Q_i^0, i = 1, 2, \tag{18}$$

$$(\text{div}\sigma_i^{2n+1}, v)_i = (f, v)_i, \forall v \in M_i, i = 1, 2, \tag{19}$$

$$(\sigma_1^{2n+1}, q)_1 - (\text{div}q, u_1^{2n+1})_1 = g^n(q), \forall q \in P_{0h}, \tag{20}$$

$$(\sigma_2^{2n+1}, q)_2 - (\text{div}q, u_2^{2n+1})_2 = -g^n(q), \forall q \in P_{0h}. \tag{21}$$

4) Define

$$d^{n+1} = (\theta_2\sigma_1^{2n+1}\cdot n_1 + (1-\theta_2)\sigma_2^{2n+1}\cdot n_1)|_\Gamma \tag{22}$$

where $\theta_1, \theta_2 \in (0,1)$ are two parameters.

3. Existence and Uniqueness of Approximation Solution

Lemma 3.1. *If* $(\sigma_i, u_i) \in Q_i \times M_i$ *satisfies*

$$\begin{cases} (\sigma_i, q)_i - (\mathrm{div}q, u_i)_i = 0, & \forall q \in Q_i \\ (\mathrm{div}\sigma_i, v)_i = 0, & \forall v \in M_i. \end{cases} \tag{23}$$

then $\sigma_i = 0, u_i = 0, i = 1, 2.$

Proof. It is clear that $\sigma_i = 0, u_i = 0$ is the solution of (23). Suppose $(\sigma_i, u_i) \in Q_i \times M_i$ is anyone of the solution of (23). Let $q = \sigma_i$, $v = u_i$, then we have $(\sigma_i, \sigma_i) = 0$, that is $\sigma_i = 0$, from (23) we have that

$$(\mathrm{div}q, u_i)_i = 0, \forall q \in Q_i. \tag{24}$$

Let $q \in Q_i^0$ we derive that

$$(\mathrm{div}q, u_i) = 0, \forall q \in Q_i^0. \tag{25}$$

Using the theory of mixed element we have that

$$\|u_i\|_{L^2(\Omega_i)/R} \le \sup_{q\in Q_i^0} \frac{(\mathrm{div}q, u_i)_i}{\|q\|_{H(\mathrm{div};\Omega_i)}} = 0, \tag{26}$$

that is u_i=constant on Ω_i. From (24) we have that

$$(\mathrm{div}q, u_i)_i = u_i \int_{\Omega_i} \mathrm{div}q dx = u_i \int_\Gamma q \cdot n_i d\Gamma = 0, \forall q \in Q_i. \tag{27}$$

Selecting $q \in Q_i$ such that $\int_\Gamma q \cdot n_i d\Gamma \neq 0$, from (27) we have $u_i = 0$.

Theorem 1. *If* $d^0 \in S(\Gamma), \int_\Gamma d^0 d\Gamma - \int_{\Omega_1} f dx = 0,$ *then the system* (12)-(22) *has a unique solution* (σ_i^n, u_i^n) *for* $i = 1, 2$ *and* $n = 0, 1, 2, \cdots.$

Proof. We use the induction method to prove the theorem. We prove that

$$\int_\Gamma d^n d\Gamma - \int_{\Omega_1} f dx = 0, n = 0, 1, 2, \cdots. \tag{28}$$

It is clear that (28) holds for n=0. Suppose (28) holds for n, from (12)-(14) we know that $(\sigma_1^{2n}, u_1^{2n})$ is the mixed element solution of the following problem:

$$-\triangle u = f, x \in \Omega_1, \frac{\partial u}{\partial n}|_{\partial\Omega\cap\partial\Omega_1} = 0, \frac{\partial u}{\partial n_1}|_\Gamma = d^n, \tag{29}$$

(28) is the compatibility condition for (29). By (15) we know that $(\sigma_1^{2n}, u_1^{2n})$ is the unique solution of (12)-(14) for i=1. Since $n_2 = -n_1$, using (2) we have that

$$\int_\Gamma -d^n d\Gamma - \int_{\Omega_2} f dx = -\int_\Gamma d^n d\Gamma - (\int_\Omega f dx - \int_{\Omega_1} f dx) = 0, \tag{30}$$

which is the compatibility condition of the problem

$$-\triangle u = f, x \in \Omega_2, \frac{\partial u}{\partial n}|_{\partial\Omega\cap\partial\Omega_2} = 0, \frac{\partial u}{\partial n_2}|_\Gamma = -d^n, \tag{31}$$

$(\sigma_2^{2n}, u_2^{2n})$ is the mixed element solution of (31). From (16) we know that $(\sigma_2^{2n}, u_2^{2n})$ exists and is unique .

Using (11) and Lemma 3.1 we can get that the system (18)-(21) has only one solution. It is clear that d^{n+1} defined by (22) belongs to $S(\Gamma)$, from (19) we can get that

$$\begin{aligned}\int_\Gamma d^{n+1} d\Gamma &= \theta_2 \int_\Gamma \sigma_1^{2n+1} \cdot n_1 d\Gamma + (1-\theta_2) \int_\Gamma \sigma_2^{2n+1} \cdot (-n_2) d\Gamma \\ &= \theta_2 \int_{\Omega_1} \text{div}\sigma_1^{2n+1} dx - (1-\theta_2) \int_{\Omega_2} \text{div}\sigma_2^{2n+1} dx \\ &= \theta_2 \int_{\Omega_1} f dx - (1-\theta_2) \int_{\Omega_2} f dx = \int_{\Omega_1} f dx.\end{aligned}$$

That is (28) holds for $(n+1)$. The proof is completed.

4. Some Lemmas

Let $\pi \in P_{0h}$ be the same as in (16). For $v_1 \in L^2(\Omega_1), v_2 \in L^2(\Omega_2)$, define $v = [v_1, v_2] \in L^2(\Omega)$ such that $v|_{\Omega_1} = v_1, v|_{\Omega_2} = v_2$. We give the following notation:

$$W_i = \{(q, v) \in Q_i \times M_i : \text{div}q = 0, (q, \widetilde{q})_i - (\text{div}\widetilde{q}, v)_i = 0, \forall \widetilde{q} \in Q_i^0\}, i = 1, 2, \tag{32}$$

$$N = \{(q, v) = ([q_1, q_2], [v_1, v_2]) : (q_i, v_i) \in W_i; (q, \widetilde{q}) - (\text{div}\widetilde{q}, v) = 0, \forall \widetilde{q} \in P_{0h}\}, \tag{33}$$

$$\begin{aligned}V = \{(q, v) = ([q_1, q_2], [v_1, v_2]) : (q_i, v_i) \in W_i, i = 1, 2; (q_1 - q_2) \cdot n_1|_\Gamma = 0; \\ \int_{\Omega_1} v_1 dx = 0; \sum_{i=1}^{2}((q_i, \pi)_i - (\text{div}\pi, v_i)_i) = 0\},\end{aligned} \tag{34}$$

For $q \in H(\text{div}; \Omega_i)$, let

$$\|q\|_i^2 = (q, q)_i, i = 1, 2 \tag{35}$$

it is clear that

Lemma 4.1. *For $(q, v) \in W_i$ we have that*

$$\|q\|_i = \|q\|_{H(\text{div};\Omega_i)}, i = 1, 2. \tag{36}$$

Lemma 4.2. *For* $(q,v)=([q_1,q_2],[v_1,v_2])\in N,(\widetilde{q},\widetilde{v})=([\widetilde{q}_1,\widetilde{q}_2],[\widetilde{v}_1,\widetilde{v}_2])\in V$ *we have that*

$$(q,\widetilde{q})=(q_1,\widetilde{q}_1)_1+(q_2,\widetilde{q}_2)_2=0 \tag{37}$$

Proof. From $(\widetilde{q},\widetilde{v})\in V$ we have that $(\widetilde{q}_1-\widetilde{q}_2)\cdot n_1|_\Gamma=0,\widetilde{q}\in Q_h$. Using (11) we know that there exist $\widetilde{q}_{11}\in Q_1^0,\widetilde{q}_{22}\in Q_2^0,g=[g_1,g_2]\in P_{0h}$ such that

$$\widetilde{q}=[\widetilde{q}_1,\widetilde{q}_2]=[\widetilde{q}_{11},0]+[0,\widetilde{q}_{22}]+[g_1,g_2]. \tag{38}$$

From the definition of N and W_i we have that

$$(q_1,g_1)_1+(q_2,g_2)_2-(\text{div}g_1,v_1)_1-(\text{div}g_2,v_2)_2=0, \tag{39}$$

$$\text{div}\widetilde{q}_i=\text{div}(\widetilde{q}_{ii}+g_i)=0,i=1,2, \tag{40}$$

$$(q_i,\widetilde{q}_{ii})_i-(\text{div}\widetilde{q}_{ii},v_i)_i=0,i=1,2. \tag{41}$$

Summing (39) and (41), by (40) we know that (37) holds.

Let $S_h(\Omega)\in H_0^1(\Omega)$ denote the space of continuous, piecewise linear functions, $X_h=\{q\in Q_h:\text{div}q=0\}$. We have[4]

Lemma 4.3. *If* $q\in X_h$ *then there exists a scaler function* $\psi\in S_h(\Omega)$ *such that*

$$q=\text{curl}\psi=(\frac{\partial\psi}{\partial x_2},-\frac{\partial\psi}{\partial x_1}).$$

Conversely, if $\psi\in S_h(\Omega)$, *then* $\text{curl}\psi\in X_h$.

Lemma 4.4. *There exist two constant* σ,τ , *independent of* h *such that*

$$\sup_{(q,v)\in V}\frac{\|q_2\|_2^2}{\|q_1\|_1^2}\le\sigma,\ \sup_{(q,v)\in V}\frac{\|q_1\|_1^2}{\|q_2\|_2^2}\le\tau. \tag{42}$$

Proof. From $(q,v)\in V$ and the definition of $V,W_i,i=1,2$ we have that $q\in Q_h$, $\text{div}q=0,q\in X_h$. Using Lemma 4.3 we have that there exists $\psi_0\in S_h(\Omega)$ such that

$$q=[q_1,q_2]=\text{curl}\psi_0,$$

$$\|q_1\|_1^2=\int_{\Omega_1}q_1\cdot q_1dx=\int_{\Omega_1}(\text{curl}\psi_0,\text{curl}\psi_0)dx=\int_{\Omega_1}|\bigtriangledown\psi_0|^2dx. \tag{43}$$

Similarly $\|q_2\|_2^2=\int_{\Omega_2}|\bigtriangledown\psi_0|^2dx.$

Let $S_h^0(\Omega_i)\subset H_0^1(\Omega_i)$ denote the restriction of $S_h(\Omega)$ on $\Omega_i,i=1,2$, when $\omega\in S_h^0(\Omega_i)$, it is clear that

$$\begin{cases}\text{divcurl}\omega=0,\\ \text{curl}\omega\cdot n|_{\partial\Omega_i}=-\dfrac{\partial\omega}{\partial\tau}|_{\partial\Omega_i}=0.\end{cases}$$

where $\dfrac{\partial}{\partial \tau}$ denotes the tangential derivative, therefore curl$\omega \in Q_i^0$. Since $(q, v) \in V$, $q = \text{curl}\psi_0$, using the definition of W_i and V, we have that

$$\begin{aligned}(\nabla\psi_0, \nabla\omega)_i &= (\text{curl}\psi_0, \text{curl}\omega)_i = (q, \text{curl}\omega)_i \\ &= (q, \text{curl}\omega)_i - (\text{divcurl}\omega, v)_i \\ &= 0, \forall \omega \in S_h^0(\Omega_i), i = 1, 2.\end{aligned}$$

That means ψ_0 satisfies the condition of Theorem 1.2 of [2], from which we know that there exists a constant σ, independent of h such that

$$\sup_{\psi \in S_h} \frac{\displaystyle\int_{\Omega_2} |\bigtriangledown \psi|^2 dx}{\displaystyle\int_{\Omega_1} |\bigtriangledown \psi|^2 dx} \leq \sigma, \tag{44}$$

therefore

$$\sup_{(q,v)\in V} \frac{\|q_2\|_2^2}{\|q_1\|_1^2} \leq \sup_{\psi \in S_h} \frac{\displaystyle\int_{\Omega_2} |\bigtriangledown \psi_0|^2 dx}{\displaystyle\int_{\Omega_1} |\bigtriangledown \psi_0|^2 dx} \leq \sigma. \tag{45}$$

Similarly we can prove the second inequality.

Lemma 4.5. *For function* $(q, v) = ([q_1, q_2], [v_1, v_2])$ *we have that*

$$\sup_{(q,v)\in N} \frac{\|q_2\|_2^2}{\|q_1\|_1^2} \leq \tau, \ \sup_{(q,v)\in N} \frac{\|q_1\|_1^2}{\|q_2\|_2^2} \leq \sigma, \tag{46}$$

where σ, τ *are the same as those in Lemma* 4.4.

Proof. For$(q, v) \in N$ define $(q_2', v_2') \in Q_2 \times M_2$ such that $q_2' \cdot n_1|_\Gamma = q_1 \cdot n_1|_\Gamma$ and

$$(q_1, \pi)_1 + (q_2', \pi)_2 - (\text{div}\pi, \quad v_1 - |\Omega_1|^{-1} \int_{\Omega_1} v_1 dx)_1 - (\text{div}\pi, v_2')_2 = 0, \tag{47}$$

$$(q_2', \widetilde{q})_2 - (\text{div}\widetilde{q}, v_2')_2 = 0, \qquad \forall \widetilde{q} \in Q_2^0, \tag{48}$$

$$(\text{div}q_2', \widetilde{v})_2 = 0, \qquad \forall \widetilde{v} \in M_2. \tag{49}$$

then$([q_1, q_2'], [v_1 - |\Omega_1|^{-1} \int_{\Omega_1} v_1 dx, v_2']) \in V$. Using Lemma 4.2 we have

$$\|q_1\|_1^2 = (q_1, q_1) = -(q_2, q_2')_2 \leq \|q_2\|_2 \|q_2'\|_2,$$

$$\frac{\|q_1\|_1^2}{\|q_2\|_2^2} = \frac{\|q_1\|_1^4}{\|q_1\|_1^2 \cdot \|q_2\|_2^2} = \frac{\|q_2\|_2^2 \cdot \|q_2'\|_2^2}{\|q_1\|_1^2 \cdot \|q_2\|_2^2} = \frac{\|q_2'\|_2^2}{\|q_1\|_1^2} \leq \sigma. \tag{50}$$

Then the second inequality in (46) holds. Similarly we can prove the first one.

5. Convergence Analyses

For the solution (σ_h, u_h) of (5)-(6) it is easy to see that

$$(\sigma_h, q)_i - (\text{div} q, u_h)_i = 0, \forall q \in Q_i^0, i = 1, 2, \tag{51}$$

$$(\text{div}\sigma_h, v)_i = (f, v)_i, \forall q \in M_i, i = 1, 2, \tag{52}$$

$$(\sigma_h, q)_1 - (\text{div} q, u_h)_1 + (\sigma_h, q)_2 - (\text{div} q, u_h)_2 = 0, \forall q \in P_{oh}, \tag{53}$$

Let $q = \pi \in P_{oh}$ in (5) we have that

$$((\sigma_h, \pi)_1 - (\text{div}\pi, u_h)_1) + ((\sigma_h, \pi)_2 - (\text{div}\pi, u_h)_2) = 0. \tag{54}$$

Let $\pi_i^n = \sigma_i^n - \sigma_h, e_i^n = u_i^n - u_h, x \in \Omega_i; \pi^n = [\pi_1^n, \pi_2^n], e^n = [e_1^n, e_2^n]$. Using (51)-(54) and (12)-(22) we get that

1) $\pi_i^{2n} \in Q_i, e_i^{2n} \in M_i, i = 1, 2$

$$(\pi_i^{2n}, q)_i - (\text{div} q, e_i^{2n})_i = 0, \forall q \in Q_i^0, \tag{55}$$

$$(\text{div}\pi_i^{2n}, v)_i = 0, \forall v \in M_i, \tag{56}$$

$$\pi_i^{2n} \cdot n_1 = \theta_2 \pi_1^{2n-1} \cdot n_1 + (1 - \theta_2)\pi_2^{2n-1} \cdot n_1, \tag{57}$$

$$\int_\Omega e_1^{2n} dx = 0, \tag{58}$$

$$\sum_{i=1}^{2}((\pi_i^{2n}, \pi)_i - (\text{div}\pi, u_i^{2n})_i) = 0. \tag{59}$$

2) $\pi_i^{2n+1} \in Q_i, e_i^{2n+1} \in M_i, i = 1, 2$

$$(\pi_i^{2n+1}, q)_i - (\text{div} q, e_i^{2n+1})_i = 0, \forall q \in Q_i^0, \tag{60}$$

$$(\text{div}\pi_i^{2n+1}, v)_i = 0, \forall v \in M_i, \tag{61}$$

$$\begin{aligned}(\pi_1^{2n+1}, q)_1 - (\text{div} q, e_1^{2n+1})_1 &= -[(\pi_2^{2n+1}, q)_2 - (\text{div} q, e_2^{2n+1})_2] \\ = \theta_1[(\pi_1^{2n}, q)_1 - (\text{div} q, e_1^{2n})_1] &- (1 - \theta_1)[(\pi_2^{2n}, q)_2 - (\text{div} q, e_2^{2n})_2], \forall q \in P_{0h}.\end{aligned} \tag{62}$$

Let $v = \text{div}\pi_i^{2n}$ in (56), $v = \text{div}\pi_i^{2n+1}$ in (61) we have that

$$\text{div}\pi_i^{2n} = \text{div}\pi_i^{2n+1} = 0, i = 1, 2. \tag{63}$$

Similarly to the proof of Theorem 3.1 of [2], we can prove that:

Theorem 2. *When* $\theta_1 \in (1 - \frac{2(\tau + 1)}{\sigma^2\tau + \tau + 2}, 1), \theta_2 \in (1 - \frac{2(\sigma + 1)}{\tau^2\sigma + \sigma + 2}, 1)$*, there exist two constants* $k_1, k_2 \in (0, 1)$ *such that*

$$\|\pi^{2n+2}\|^2 \le k_1 k_2 \|\pi^{2n}\|^2 \le (k_1 k_2)^{n+1} \|\pi^0\|^2, n \ge 0 \tag{64}$$

$$\|\pi^{2n+1}\|^2 \leq k_1k_2\|\pi^{2n-1}\|^2 \leq (k_1k_2)^n\|\pi^1\|^2, n \geq 0 \tag{65}$$

where $\|\pi^j\| = \|\pi_1^j\|_1 + \|\pi_2^j\|_2^2$.

Proof. 1) From (55)-(59) we know that $(\pi^{2n}, e^{2n}) \in V$. Define $(\tilde{\pi}_1^{2n}, \tilde{e}_1^{2n}) \in W_1$ such that

$$(\tilde{\pi}_1^{2n}, q)_1 - (\text{div}q, \tilde{e}_1^{2n})_1 = -[(\pi_2^{2n}, q)_2 - (\text{div}q, e_2^{2n})_2], \forall q \in P_{0h} \tag{66}$$

By (33) we know that $([\tilde{\pi}_1^{2n},\ \pi_2^{2n}],\ [\tilde{e}_1^{2n},\ e_2^{2n}]) \in N$. Similarly we define $(\tilde{\pi}_2^{2n},\ \tilde{e}_2^{2n}) \in W_2$ such that $([\tilde{\pi}_1^{2n},\ \tilde{\pi}_2^{2n}],\ [\tilde{e}_1^{2n} - \alpha^{2n},\ \tilde{e}_2^{2n}]) \in V$; define $(\tilde{\pi}_2^{2n+1},\ \tilde{e}_2^{2n+1}) \in W_2$ such that $([\pi_1^{2n+1},\ \tilde{\pi}_2^{2n+1}],\ [e_1^{2n+1} - \alpha^{2n+1},\ \tilde{e}_2^{2n+1}]) \in V$; define $(\tilde{\pi}_1^{2n+1},\ \tilde{e}_1^{2n+1}) \in W_1$ such that $([\tilde{\pi}_1^{2n+1},\ \pi_2^{2n+1}],\ [e_1^{2n+1},\ e_2^{2n+1} - \alpha]) \in V$, where the constants $\alpha^{2n}, \alpha^{2n+1}, \alpha$ are determined by using the definition of V.

2) From (61)-(64) we know taht $\tilde{\tilde{\pi}}_1^{2n+1} = \pi_1^{2n+1} - (\theta_1\pi_1^{2n} + (1-\theta_1)\tilde{\pi}_1^{2n}) \in Q_1$, $\tilde{\tilde{e}}_1^{2n+1} = e_1^{2n+1} - (\theta_1 e_1^{2n} + (1-\theta_1)\tilde{e}_1^{2n}) \in M_1$,

$$(\tilde{\tilde{\pi}}_1^{2n+1}, q)_1 - (\text{div}q, \tilde{\tilde{e}}_1^{2n+1})_1 = 0, \quad \forall q \in Q_1,$$
$$(\text{div}\tilde{\tilde{\pi}}_1^{2n+1}, v)_1 = 0, \quad \forall v \in M_1.$$

By Lemma 3.1 we have $\tilde{\tilde{\pi}}_1^{2n+1} = 0$, $[\pi_1^{2n+1}, \pi_2^{2n}] = \theta_1[\pi_1^{2n}, \pi_2^{2n}] + (1-\theta_1)[\tilde{\pi}_1^{2n}, \pi_2^{2n}]$. Using the method of Zhang and Huang[2] and Lemma 4.2-Lemma 4.4 we can prove that

$$\|\pi^{2n+1}\|_1^2 \leq k_1\|\pi^{2n}\|_1^2 \tag{67}$$

$$k_1 = [(1-\theta_1)^2(\sigma^2\tau + \tau + 2) - 2(1-\theta_1)(\tau+1) + \tau]\tau^{-1}. \tag{68}$$

Similarly we can get that

$$\|\pi^{2n+2}\|_1^2 \leq k_2\|\pi_1^{2n+1}\|_1^2, \tag{69}$$

$$\|\pi^{2n+2}\|_2^2 \leq k_2\|\tilde{\pi}_2^{2n+1}\|_2^2, \|\tilde{\pi}^{2n+1}\|_2^2 \leq k_1\|\pi_2^{2n}\|_2^2, \tag{70}$$

$$k_2 = [(1-\theta_2)^2(\sigma\tau^2 + \sigma + 2) - 2(1-\theta_2)(\sigma+1) + \sigma]\sigma^{-1} \tag{71}$$

From (68)-(70) we know that (64) holds. Similarly we can prove that (65) holds.

Remark: When $\theta_1 = \dfrac{\sigma^2\tau + 1}{\sigma^2\tau + \tau + 2}, \theta_2 = \dfrac{\tau^2\sigma + 1}{\tau^2\sigma + \sigma + 2}$ we have $k_1 = \dfrac{\sigma^2\tau^2 - 1}{\sigma^2\tau^2 + \tau^2 + 2\tau}$, $k_2 = \dfrac{\sigma^2\tau^2 - 1}{\tau^2\sigma^2 + \sigma^2 + 2\sigma}$, which take their minimum values. In practical computation, the values of θ_1, θ_2 can be determined by using of the computational results(see [2], [6]).

6. Numerical Example

Consider the following problem

$$-\triangle u = f, rmin\Omega, \frac{\partial u}{\partial n} = g, \text{on}\partial\Omega. \tag{72}$$

where $\Omega = (0, 2\pi) \times (0, \pi)$, f, g have been chosen in such a way that the exact solution is $u = \sin x \sin y$. We use a uniform mesh with $10 \times 5, 20 \times 10$ elementary squares($h = 0.2\pi$ and $h = 0.1\pi$ respectively). Decompose the domain into $\Omega_1 = (0, 0.8\pi) \times (0, \pi), \Omega_2 = (0.8\pi, 2\pi) \times (0, \pi)$, using the algorithm given in Section 2. The error E_σ, E_u, denoting L^2 error for σ and u respectively, are depicted in the following table.

	$h = 0.1\pi$		$h = 0.2\pi$	
	E_σ	E_u	E_σ	E_u
n=0	2.82		2.78	
n=1	0.437	0.07	0.235	0.047
n=2	0.093	0.017	0.088	0.021
n=3	0.017	0.007	0.051	0.008

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