Journal of Computational Mathematics, Vol.16, No.1, 1998, 63–80.

SOME ESTIMATES WITH NONCONFORMING ELEMENTS IN DOMAIN DECOMPOSITION ANALYSIS*1

Jin-sheng Gu
(*Department of mathematics, Capital Normal University, Beijing, China*)
Xian-cheng Hu
(*Department of Applied Mathematics, Tsinghua University, Beijing, China*)

Abstract

Some essential estimates, especially the so–called extension theorems, are established in this paper, for the nonconforming finite elements with their continuity at the vertices or the edge midpoints of the elements of the quasi–uniform mesh. As in the conforming discrete cases, these estimates play key roles in the theoretical analysis of the nonoverlap domain decomposition methods for the solving of second order self–adjoint elliptic problems discretized by the nonconforming finite element methods.

Key words: Nonconforming finite element, extension theorem, domain decomposition, elliptic problem.

1. Introduction

For simplicity of the exposition, we consider the elliptic boundary value problem on a bounded open polygonal domain $\Omega \subset \Re^2$

$$u \in H^1(\Omega): \begin{cases} a(u,v) = (f,v), & \forall\ v \in H_0^1(\Omega) \\ u = g, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where

$$a(u,v) = \int_\Omega \nabla u \nabla v, \quad (f,v) = \int_\Omega fv, \quad f \in H^{-1}(\Omega), \quad g \in H^{\frac{1}{2}}(\partial\Omega).$$

It is well–known that (1.1) has a unique solution $u \in H^1(\Omega)$ (cf.[7, 15, 16]).

Suppose that $\Omega_h = \{e\}$ is a quasi–uniform mesh of Ω, i.e., Ω_h satisfies

$$\sup_{e\in\Omega_h} \inf_{B_r \supset e} r \le ch, \quad \inf_{e\in\Omega_h} \sup_{B_r \subset e} r \ge Ch, \tag{1.2}$$

where e, a triangle or a quadrilater, represents the typical element in Ω_h, B_r is a region bounded by the circle of radius r, $h = \max\limits_{e\in\Omega_h} h_e$ is the mesh parameter and

* Received June 23, 1995.
1) This work was supported by the National Natural Science Foundation of China and the Tsinghua University Fund for Science.

$h_e = \inf_{B_r \supset e} r$. Here and later, c and C, with or without subscript, denote generic positive constants independent of h. Let V_h be the finite element space on Ω_h and π_h be the corresponding interpolation operator. V_h can be the space of Wilson elements[5], Carey membrane elements[4] or Wilson–like elements[14], which are continuous at the vertices of each $e \in \Omega_h$. Also, V_h can be the space of Crouzeix–Raviart elements[6] or quartic rectangular elements[13], which are continuous at the edge midpoints of each $e \in \Omega_h$. For briefness, the former is called the nonconforming elements of the first kind and the latter is called the nonconforming elements of the second kind. V_h can be written in the following general form

$$\begin{aligned} V_h = \{v : v|_e \ \text{ is a polynomial of finite order, } v \text{ is continuous at the} \\ \text{vertices (edge midpoints) of } e, \quad \forall\, e \in \Omega_h\}, \\ V_h^0 = \{v \in V_h :\ v(x) = 0,\ \forall \text{ interpolation point } x \in \partial\Omega\}. \end{aligned}$$

Denote $A(w,v) = \sum_{e\in\Omega_h} \int_e \nabla w \nabla v$, $|v|_{1,\Omega,h} = \sqrt{A(v,v)}$. Obviously, $A(\cdot,\cdot), |\cdot|_{1,\Omega,h}$ are the inner product of V_h^0 and its induced norm respectively. The nonconforming finite element discrete problem of (1.1) is

$$u_h \in V_h : \begin{cases} A(u_h, v) = (f, v), & \forall\, v \in V_h^0 \\ u_h(x) = g(x), & \forall \text{ interpolation point } x \in \partial\Omega \end{cases} \tag{1.3}$$

With the development of parallel computers, domain decomposition methods have recently become an important focus in the field of computational mathematics. By now, all kinds of domain decomposition algorithms have been developed to solve the algebraic system of equations arising from the discretization of (1.1) via the conforming finite element methods. It is noted that several fundamental inequalities, especially the so–called extension theorems play key roles in the theoretical analysis of those nonoverlap domain decomposition algorithms (substructuring methods)[2,3,20]. Therefore, when considering the nonconforming finite element discrete problem (1.3), we should establish those inequalities in V_h correspondingly. For this purpose, the conforming interpolation operator I_h is introduced to act as a bridge between V_h and the piecewise linear continuous finite element space where many inequalities have already been constructed[2,5,18]. Since the regularity of the solution u of (1.1) depends on the domain Ω (cf.[7, 15, 16]), we investigate advanced error estimations of the nonconforming approximate solution u_h of (1.1) under weaker assumption on the regularity. In this way, we eventually establish a series of essential estimates in V_h, some of which are the extension theorems[8,10], the Poincaré inequalities and the maximum norm estimate.

The remainder of this paper is organized as follows: Sect.2 gives advanced error estimations of (1.3). Sect.3 introduces the conforming interpolation operator I_h and analyses its properties. Sect.4 describes and proves some essential estimates in V_h to conclude the paper.

For the length of the present paper, we omit here their applications to the theoretical analysis of nonoverlap domain decomposition methods for the solving of (1.3), which can be referred to [8, 9, 11, 12].

2. Advanced Error Estimations

Theorem 2.1. *Let u, u_h be the solutions of* (1.1) *and* (1.3) *respectively. Then*

$$|u-u_h|_{1,\Omega,h} \le c\Big\{ \inf_{v\in V_h^*} |u-v|_{1,\Omega,h} + \sup_{w\in V_h^0\setminus\{0\}} \frac{|E(u,w)|}{|w|_{1,\Omega,h}} \Big\}, \tag{2.1}$$

where

$$E(u,w) \triangleq A(u,w) - (f,w) = \sum_{e\in\Omega_h} \int_{\partial e} \frac{\partial u}{\partial \mathbf{n}} w, \tag{2.2}$$

$$V_h^* = \{v \in V_h :\ v(x) = u(x),\ \forall \text{ interpolation point } x \in \partial\Omega\},$$

$\mathbf{n} = (\nu_1, \nu_2)^T$ *represents the unit outward normal vector of* e.

Theorem 2.1 is in fact the variant of the second Strang lemma[5] in the nonhomogeneous boundary value case. Its proof is trivial, so we omit it here.

Let L_e be the linear (bilinear) interpolation operator on e with the vertices of e as its interpolation points. For any measurable set z, we define the mean value operator $M_z : L^2(z) \to \Re$ by

$$\forall\ v \in L^2(z), \quad M_z v \in \Re, \quad M_z v = \frac{1}{\mathrm{meas}(z)} \int_z v.$$

Let $\hat{e}$ be the reference element, which is a square or an isosceles right triangle with $O(1)$ as its area. If there exists an invertible affine mapping

$$x = F_{\hat{e}}(\hat{x}) = B\hat{x} + b : \quad \hat{e} \to e \tag{2.3}$$

such that $e = F_{\hat{e}}(\hat{e})$, then we say that e is affine equivalent to $\hat{e}$. Here, $B \in \Re^{2\times 2}$ is nonsingular. For any function v defined on e, let $\hat{v}$ be the corresponding function defined on $\hat{e}$ such that $\hat{v}(\hat{x}) = (v \cdot F_{\hat{e}})(\hat{x})$, $\forall\ \hat{x} \in \hat{e}$.

Lemma 2.2. *Let $u \in H^{1+\varepsilon}(\Omega)$ be the solution of* (1.1) ($\varepsilon > 0$). *Then*

1) *For the first kind nonconforming element space V_h, $\forall\ w \in V_h^0$, we have*

$$E(u,w) \le c\Big\{ \sum_{e\in\Omega_h} \sum_{i=1}^{2} \int_{\partial e} \Big| \frac{\partial u}{\partial x_i} - M_e\Big(\frac{\partial u}{\partial x_i}\Big)\Big|^2 \Big\}^{\frac12} \Big(\sum_{e\in\Omega_h} \int_{\partial e} |w - L_e w|^2 \Big)^{\frac12}; \tag{2.4}$$

2) *For the second kind nonconforming element space V_h, $\forall\ w \in V_h^0$, we have*

$$E(u,w) \le c\Big\{ \sum_{e\in\Omega_h} \sum_{i=1}^{2} \int_{\partial e} \Big| \frac{\partial u}{\partial x_i} - M_e\Big(\frac{\partial u}{\partial x_i}\Big)\Big|^2 \Big\}^{\frac12} \Big(\sum_{e\in\Omega_h} \int_{\partial e} |w - M_e w|^2 \Big)^{\frac12}. \tag{2.5}$$

Proof. Let's prove (2.4) first. $\forall\ w \in V_h^0$, $L_e w$ is a linear (bilinear) function on e, and for each $e \in \Omega_h$, we have $L_e w$, which results in a piecewise linear (bilinear) function on Ω_h, denoted $L_e w$ still. Obviously, $L_e w \in H^1(\Omega)$, thus $E(u, L_e w) = 0$, $E(u,w) = E(u, w - L_e w)$.

On the other hand, one–by–one analysis shows that

$$\int_{\partial e} \nu_i (w - L_e w) = 0, \quad i = 1, 2. \tag{2.6}$$

Furthermore, we get (2.4) by the Schwarz inequality.

Now, let's take the Carey membrane elements as an example to prove (2.6). Let (x_i, y_i), $i = 1, 2, 3$ be the vertices of the triangle element e. F_i denotes the opposite edge of (x_i, y_i) whose length is l_i. Let λ_i be the corresponding area coordinate. The unit outward normal vector $\mathbf{n} = (\nu_1, \nu_2)^T$ of e is

$$\nu_1 = \frac{y_{i+2} - y_{i+1}}{l_i}, \quad \nu_2 = -\frac{x_{i+2} - x_{i+1}}{l_i}, \quad \text{on } F_i,\ i = 1, 2, 3, \tag{2.7}$$

where $x_4 = x_1$, $y_4 = y_1$, $x_5 = x_2$, $y_5 = y_2$ for notational convenience. Obviously, in order to see (2.6) is right for the Carey elements, it suffices to show

$$\int_{\partial e} \nu_i (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) = 0, \quad i = 1, 2. \tag{2.8}$$

Note that, from (2.7), it is easy to obtain

$$\begin{aligned}
\sum_{i=1}^{3} \int_{F_i} \nu_1 (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) &= \sum_{i=1}^{3} \int_{F_i} \nu_1 \lambda_{i+1}\lambda_{i+2} \\
&= \sum_{i=1}^{3} \frac{y_{i+2} - y_{i+1}}{l_i} \int_0^{l_i} \frac{s}{l_i} \frac{l_i - s}{l_i} ds = \sum_{i=1}^{3} \frac{1}{6}(y_{i+2} - y_{i+1}) = 0.
\end{aligned}$$

Thus, (2.8) is true for $i = 1$. Similarly, (2.8) is true for $i = 2$.

By now, we get (2.4).

Next, we prove (2.5). For the Crouzeix–Raviart elements, let F be the edge of the triangle element $e \in \Omega_h$. Obviously, $\int_F \nu_i w = 0$, $i = 1, 2$, $\forall\ F \subset \partial\Omega$; if F is the common edge of e_1, e_2, then it follows from the linearity of w on F and the continuity of w at the midpoint of F that $\int_F \nu_i w_F = 0$, $i = 1, 2$. Here, w_F is the jump of w on F. Therefore, we have

$$\begin{aligned}
E(u, w) &= \sum_{e \in \Omega_h} \int_{\partial e} \frac{\partial u}{\partial \mathbf{n}} w = \sum_{e \in \Omega_h} \sum_{F \subset \partial e} \int_F \frac{\partial u}{\partial \mathbf{n}} w = \sum_{e \in \Omega_h} \sum_{F \subset \partial e} \int_F \Big[\frac{\partial u}{\partial \mathbf{n}} - M_F\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\Big] w \\
&= \sum_{e \in \Omega_h} \sum_{F \subset \partial e} \int_F \Big[\frac{\partial u}{\partial \mathbf{n}} - M_F\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\Big](w - M_F w) \\
&\le \sum_{e \in \Omega_h} \Big\{ \sum_{F \subset \partial e} \int_F \Big|\frac{\partial u}{\partial \mathbf{n}} - M_F\Big(\frac{\partial u}{\partial \mathbf{n}}\Big)\Big|^2 \Big\}^{\frac{1}{2}} \Big(\sum_{F \subset \partial e} \int_F |w - M_F w|^2 \Big)^{\frac{1}{2}} \\
&\le \sum_{e \in \Omega_h} \Big\{ \sum_{F \subset \partial e} \sum_{i=1}^{2} \int_F \Big|\frac{\partial u}{\partial x_i} - M_e\Big(\frac{\partial u}{\partial x_i}\Big)\Big|^2 \Big\}^{\frac{1}{2}} \Big(\sum_{F \subset \partial e} \int_F |w - M_e w|^2 \Big)^{\frac{1}{2}},
\end{aligned}$$

which implies that (2.5) holds.

For the quartic rectangular elements, let F_i, $i = 1, 2, 3, 4$, be the four edges of $e \in \Omega_h$ and a_i be their corresponding midpoints. l_i is the length of F_i. a_5 denotes the centroid of e. There exists an invertible affine mapping $x = F_{\hat{e}}(\hat{x}) : \hat{e} \to e$, $\hat{e}$ is the reference element as shown in Fig.2.1.

Fig. 2.1

It is easy to see that

$$E(u, w) = \sum_{e\in\Omega_h} \int_{\partial e} \frac{\partial u}{\partial \mathbf{n}} w = \sum_{e\in\Omega_h} \sum_{i=1}^{4} \int_{F_i} \frac{\partial u}{\partial \mathbf{n}} w = \sum_{e\in\Omega_h} \sum_{i=1}^{4} \int_{F_i} \frac{\partial u}{\partial \mathbf{n}} \Big(w - w(a_i)\Big). \qquad (2.9)$$

By the affine mapping $x = F_{\hat{e}}(\hat{x}) : \hat{e} \to e$, we obtain that

$$\int_{F_i} \nu_j \Big(w - w(a_i)\Big) = \frac{l_i}{2} \int_{\hat{F}_i} \nu_j \Big(\hat{w} - \hat{w}(\hat{a}_i)\Big), \; j = 1, 2. \qquad (2.10)$$

It follows from Fig.2.1 and [13] that

$$\int_{\hat{F}_2} \nu_1 \Big(\hat{w} - \hat{w}(\hat{a}_2)\Big) = 0, \qquad \int_{\hat{F}_4} \nu_1 \Big(\hat{w} - \hat{w}(\hat{a}_4)\Big) = 0,$$

$$\hat{w} - \hat{w}(\hat{a}_1) = \frac{\hat{w}(\hat{a}_2) - \hat{w}(\hat{a}_4)}{2} \hat{x}_2 + \frac{\hat{w}(\hat{a}_2) + \hat{w}(\hat{a}_4) - 2\hat{w}(\hat{a}_5)}{2} \phi(\hat{x}_2), \quad \text{on } \hat{F}_1,$$

where $\phi(t) = \frac{1}{2}(5t^4 - 3t^2)$. Since $\int_{-1}^{1} \hat{x}_2 d\hat{x}_2 = 0$, $\int_{-1}^{1} \phi(\hat{x}_2) d\hat{x}_2 = 0$, we have $\int_{\hat{F}_1} \nu_1 \Big(\hat{w} - \hat{w}(\hat{a}_1)\Big) = 0$. In the same manner, it is easy to get $\int_{\hat{F}_3} \nu_1 \Big(\hat{w} - \hat{w}(\hat{a}_3)\Big) = 0$. Therefore, by (2.10), we obtain that $\sum_{i=1}^{4} \int_{F_i} \nu_1 \Big(w - w(a_i)\Big) = 0$.

Similarly, $\sum_{i=1}^{4} \int_{F_i} \nu_2 \Big(w - w(a_i)\Big) = 0$ can be established.

Furthermore, it follows from (2.9) and the Schwarz inequality that

$$E(u, w) = \sum_{e\in\Omega_h} \sum_{i=1}^{4} \int_{F_i} \Big[\sum_{j=1}^{2} \nu_j \Big(\frac{\partial u}{\partial x_j} - M_e\Big(\frac{\partial u}{\partial x_j}\Big)\Big)\Big] \Big(w - w(a_i)\Big)$$

$$\leq \sum_{e\in\Omega_h} \Big(\sum_{i=1}^{4}\sum_{j=1}^{2} \int_{F_i} \Big| \frac{\partial u}{\partial x_j} - M_e\Big(\frac{\partial u}{\partial x_j}\Big)\Big|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{4} \int_{F_i} |w - w(a_i)|^2\Big)^{\frac{1}{2}}$$

$$\leq \sum_{e\in\Omega_h} \Big(\sum_{i=1}^{4}\sum_{j=1}^{2} \int_{F_i} \Big| \frac{\partial u}{\partial x_j} - M_e\Big(\frac{\partial u}{\partial x_j}\Big)\Big|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{4} \int_{F_i} |w - M_e w|^2\Big)^{\frac{1}{2}},$$

from which (2.5) follows. Hence the proof of the lemma is complete. □

Lemma 2.3. *For $\varepsilon \in (0,1)$, we have*

$$\|M_e w\|_{0,e} \leq \|w\|_{0,e}, \quad \forall\ w \in L^2(e) \tag{2.11}$$

$$\|w - M_e w\|_{0,e} \leq c h_e^{\varepsilon} \|w\|_{\varepsilon,e},\ \forall\ w \in H^{\varepsilon}(e) \tag{2.12}$$

Proof. By the Schwarz inequality, it is easy to get (2.11). We next prove (2.12). Note that $M_e c = c$, $\forall\, c \in \Re$, thus the error estimate of the finite element interpolation[5] yields

$$\|w - M_e w\|_{0,e} \leq c h_e |w|_{1,e} \leq c h_e \|w\|_{1,e}, \quad \forall\ w \in H^1(e)$$

It follows from (2.11) that $\|w - M_e w\|_{0,e} \leq c\|w\|_{0,e}$, $\forall\ w \in L^2(e)$. Therefore, $I - M_e : L^2(e) \to L^2(e)$ and $I - M_e : H^1(e) \to L^2(e)$ are bounded linear operators. Since $H^{\varepsilon}(e)$ is the interpolation space between $L^2(e)$ and $H^1(e)$, (2.12) can be established by the interpolation theorem of Sobolev spaces[1,15]. □

Lemma 2.4. *If e is affine equivalent to the reference element $\hat{e}$ $\Big(1 > \varepsilon \geq \dfrac{1}{2}\Big)$, then*

$$\int_{\partial e} w^2 \leq c\{h_e^{-1}\|w\|_{0,e}^2 + h_e^{2\varepsilon-1}|w|_{\varepsilon,e}^2\},\ \forall\ w \in H^{\varepsilon}(e)$$

where $|w|_{\varepsilon,e}^2 \triangleq \displaystyle\int_e\int_e \frac{|w(x)-w(y)|^2}{|x-y|^{2+2\varepsilon}} dxdy.$

Proof. In the reference element $\hat{e}$, the trace theorem[1,15] yields

$$\int_{\partial\hat{e}} (\hat{w})^2 \leq c\|\hat{w}\|_{\varepsilon,\hat{e}}^2 \leq c\{\|\hat{w}\|_{0,\hat{e}}^2 + |\hat{w}|_{\varepsilon,\hat{e}}^2\}.$$

It follows from the affine equivalence of e and $\hat{e}$ that

$$\|\hat{w}\|_{0,\hat{e}}^2 \leq c h_e^{-2}\|w\|_{0,e}^2, \quad |\hat{w}|_{\varepsilon,\hat{e}}^2 \leq c h_e^{2\varepsilon-2}|w|_{\varepsilon,e}^2, \quad \int_{\partial\hat{e}} (\hat{w})^2 \geq c h_e^{-1}\int_{\partial e} w^2.$$

With above inequalities, we see that Lemma 2.4 holds. □

Lemma 2.5. *Let $u \in H^{1+\varepsilon}(\Omega)$ be the solution of* (1.1) $\Big(1 > \varepsilon \geq \dfrac{1}{2}\Big)$. *We have*

$$E(u,w) \leq c h^{\varepsilon}\|u\|_{H^{1+\varepsilon}(\Omega)}|w|_{1,\Omega,h}, \quad \forall\ w \in V_h^0$$

Proof. (2.12) gives

$$\Big\|\frac{\partial u}{\partial x_i} - M_e\Big(\frac{\partial u}{\partial x_i}\Big)\|_{0,e} \leq c h_e^{\varepsilon}\Big\|\frac{\partial u}{\partial x_i}\Big\|_{\varepsilon,e} \leq c h_e^{\varepsilon}\|u\|_{1+\varepsilon,e},$$

$$\Big|\frac{\partial u}{\partial x_i}-M_e\Big(\frac{\partial u}{\partial x_i}\Big)\Big|_{\varepsilon,e}=\Big|\frac{\partial u}{\partial x_i}\Big|_{\varepsilon,e}\le\|u\|_{1+\varepsilon,e}.$$

Combining them with Lemma 2.4, we obtain

$$\begin{aligned}&\sum_{e\in\Omega_h}\sum_{i=1}^{2}\int_{\partial e}\Big|\frac{\partial u}{\partial x_i}-M_e\Big(\frac{\partial u}{\partial x_i}\Big)\Big|^2\\ \le& c\sum_{e\in\Omega_h}\sum_{i=1}^{2}\Big\{h_e^{-1}\Big\|\frac{\partial u}{\partial x_i}-M_e\Big(\frac{\partial u}{\partial x_i}\Big)\Big\|_{0,e}^2+h_e^{2\varepsilon-1}\Big|\frac{\partial u}{\partial x_i}-M_e\Big(\frac{\partial u}{\partial x_i}\Big)\Big|_{\varepsilon,e}^2\Big\}\\ \le& c\sum_{e\in\Omega_h}\sum_{i=1}^{2}\Big\{h_e^{-1}h_e^{2\varepsilon}\|u\|_{1+\varepsilon,e}^2+h_e^{2\varepsilon-1}\|u\|_{1+\varepsilon,e}^2\Big\}\le ch^{2\varepsilon-1}\|u\|_{H^{1+\varepsilon}(\Omega)}^2.\end{aligned}\tag{2.13}$$

On the other hand, It follows from the interpolation error estimates and the inverse inequalities[5] that

$$\begin{aligned}&\|w-L_ew\|_{0,e}\le ch_e^2|w|_{2,e}\le ch_e|w|_{1,e},\quad \|w-M_ew\|_{0,e}\le ch_e|w|_{1,e},\\ &|w-L_ew|_{1,e}\le ch_e|w|_{2,e}\le c|w|_{1,e},\quad |w-M_ew|_{1,e}\le c|w|_{1,e}.\end{aligned}$$

Furthermore, Lemma 2.4 indicates

$$\begin{aligned}\sum_{e\in\Omega_h}\int_{\partial e}|w-L_ew|^2&\le c\sum_{e\in\Omega_h}\{h_e^{-1}\|w-L_ew\|_{0,e}^2+h_e|w-L_ew|_{1,e}^2\}\\ &\le c\sum_{e\in\Omega_h}h_e|w|_{1,e}^2\le ch|w|_{1,\Omega,h}^2.\end{aligned}\tag{2.14}$$

Similarly, we have

$$\sum_{e\in\Omega_h}\int_{\partial e}|w-M_ew|^2\le ch|w|_{1,\Omega,h}^2.\tag{2.15}$$

Lemma 2.5 follows from (2.4), (2.5), (2.13), (2.14) and (2.15). □

Lemma 2.6. *Suppose that $\hat\pi: H^1(\hat e)\to L^2(\hat e)$ is the linear operator on the reference element $\hat e$, which satisfies that there exists a positive constant $c_1=c_1(\hat\pi)$, such that*

$$\|\hat\pi\hat v\|_{0,\hat e}\le c_1\|\hat v\|_{1,\hat e},\ \forall\ \hat v\in H^1(\hat e)\tag{2.16}$$

$$\hat\pi\hat c=\hat c,\quad \forall\ \hat c\in\Re.\tag{2.17}$$

Suppose that the element e is affine equivalent to $\hat e$ and π_e is a linear mapping on e defined by

$$\widehat{\pi_e v}=\hat\pi\hat v,\quad \forall\ v\in H^1(e)\tag{2.18}$$

Then there exists a positive constant $c_2=c_2(\hat\pi,\hat e)$, such that

$$\|v-\pi_ev\|_{0,e}\le c_2h_e|v|_{1,e},\ \forall\ v\in H^1(e)$$

Proof. $\forall\ \hat c\in\Re$, (2.16) and (2.17) yield

$$\|\hat v-\hat\pi\hat v\|_{0,\hat e}=\|(\hat v+\hat c)-\hat\pi(\hat v+\hat c)\|_{0,\hat e}\le(1+c_1)\|\hat v+\hat c\|_{1,\hat e},\ \forall\ \hat v\in H^1(\hat e)$$

Furthermore, it follows from Theorem 3.1.1[5] that

$$\|\hat{v}-\hat{\pi}\hat{v}\|_{0,\hat{e}} \le (1+c_1)\inf_{\hat{c}\in\Re}\|\hat{v}+\hat{c}\|_{1,\hat{e}} \le c_3|\hat{v}|_{1,\hat{e}},\ \forall\ \hat{v}\in H^1(\hat{e})$$

On the other hand, (2.18) implies $\hat{v}-\hat{\pi}\hat{v}=\hat{v}-\widehat{\pi_e v}$, hence Theorem 3.1.2[5] gives

$$\|v-\pi_e v\|_{0,e} \le c|\det B|^{\frac{1}{2}}\|\hat{v}-\hat{\pi}\hat{v}\|_{0,\hat{e}}, \quad |\hat{v}|_{1,\hat{e}} \le c\|B\||\det B|^{-\frac{1}{2}}|v|_{1,e},$$

where $\det B$ is the determinant of matrix B in the affine equivalence of e and $\hat{e}$. $\|B\|$ represents the Euclidean norm of B in $\Re^2$. It follows from Theorem 3.1.3[5] that $\|B\| \le h_e/\hat{\rho}$, where $\hat{\rho} \overset{\triangle}{=} \sup\limits_{B_r\subset\hat{e}} r$. Obviously, $\hat{\rho}=O(1)$.

With the above facts in mind, we end the proof of Lemma 2.6. □

Remark 2.1. By the imbedding theorem[1], the Schwarz inequality and the trace theorem, it is easy to see that (2.16) is true for finite element interpolation operators.

Let $P(e)$ be the polynomial space on e of finite order. Define the L^2 projection operator $Q_e : L^2(e) \to P(e)$ as follows

$$(Q_e v, w)_{L^2(e)} = (v, w)_{L^2(e)}, \quad \forall\ w \in P(e) \tag{2.19}$$

Lemma 2.7. *If the element e is affine equivalent to the reference element $\hat{e}$, then we have*

$$|Q_e v|_{1,e} \le c|v|_{1,e}, \quad \forall\ v\in H^1(e) \tag{2.20}$$

$$\|v-Q_e v\|_{0,e} \le ch_e|v|_{1,e}, \quad \forall\ v\in H^1(e) \tag{2.21}$$

Proof. Analogously to (2.19), we can define the L^2 projection operator on $\hat{e}$, which is denoted by $Q_{\hat{e}}$. It is easy to see that

$$\widehat{Q_e v} = Q_{\hat{e}}\hat{v},$$

which implies that (2.20) is equivalent to

$$|Q_{\hat{e}}\hat{v}|_{1,\hat{e}} \le c|\hat{v}|_{1,\hat{e}}, \quad \forall\ \hat{v}\in H^1(\hat{e}) \tag{2.22}$$

It follows from the definition of $Q_{\hat{e}}$ and the Schwarz inequality that

$$\|Q_{\hat{e}}\hat{v}\|_{0,\hat{e}} \le \|\hat{v}\|_{0,\hat{e}}. \tag{2.23}$$

Note that the norms of the finite dimensional space $P(\hat{e})$ are equivalent, thus we obtain

$$|Q_{\hat{e}}\hat{v}|_{1,\hat{e}} \le c\|Q_{\hat{e}}\hat{v}\|_{0,\hat{e}} \le c\|\hat{v}\|_{0,\hat{e}} \le c\|\hat{v}\|_{1,\hat{e}},$$
$$|Q_{\hat{e}}\hat{v}|_{1,\hat{e}} = |Q_{\hat{e}}(\hat{v}+\hat{c})|_{1,\hat{e}} \le c\|\hat{v}+\hat{c}\|_{1,\hat{e}},\ \forall\ \hat{c}\in\Re$$

Furthermore, Theorem 3.1.1[5] yields

$$|Q_{\hat{e}}\hat{v}|_{1,\hat{e}} \le c\inf_{\hat{c}\in\Re}\|\hat{v}+\hat{c}\|_{1,\hat{e}} \le c|\hat{v}|_{1,\hat{e}},$$

which is (2.22). Therefore, (2.20) holds.

(2.23) implies that

$$\|\hat{v} - Q_{\hat{e}}\hat{v}\|_{0,\hat{e}} \le 2\|\hat{v}\|_{0,\hat{e}} \le 2\|\hat{v}\|_{1,\hat{e}},$$
$$\|\hat{v} - Q_{\hat{e}}\hat{v}\|_{0,\hat{e}} = \|(\hat{v}+\hat{c}) - Q_{\hat{e}}(\hat{v}+\hat{c})\|_{0,\hat{e}} \le 2\|(\hat{v}+\hat{c})\|_{1,\hat{e}},\ \forall\ \hat{c}\in\Re$$

Applying Theorem 3.1.2[5], Theorem 3.1.3[5] and Theorem 3.1.1[5], we have

$$\|v - Q_e v\|_{0,e} \le ch_e\|\hat{v} - Q_{\hat{e}}\hat{v}\|_{0,\hat{e}} \le ch_e \inf_{\hat{c}\in\Re}\|\hat{v}+\hat{c}\|_{1,\hat{e}} \le ch_e|\hat{v}|_{1,\hat{e}} \le ch_e|v|_{1,e}.$$

Hence, (2.21) is established. □

Lemma 2.8. *Let $\hat{e}, \hat{\pi}, e, \pi_e$ be those in Lemma 2.6. $0 \le \varepsilon < 1$. If π_e satisfies*

$$|v - \pi_e v|_{1,e} \le ch_e^k\|v\|_{k+1,e}, \quad \forall\ v \in H^{k+1}(e) \tag{2.24}$$

for some integer $k \ge 1$, then

$$|v - \pi_e v|_{1,e} \le ch_e^{\varepsilon}\|v\|_{\varepsilon+1,e}, \quad \forall\ v \in H^{\varepsilon+1}(e) \tag{2.25}$$

Proof. It follows from the inverse inequality, Lemma 2.6 and Lemma 2.7 that

$$\begin{aligned}|v - \pi_e v|_{1,e} &\le |v - Q_e v|_{1,e} + |Q_e v - \pi_e v|_{1,e} \le c\{|v|_{1,e} + h_e^{-1}\|Q_e v - \pi_e v\|_{0,e}\}\\ &\le c\{|v|_{1,e} + h_e^{-1}\|Q_e v - v\|_{0,e} + h_e^{-1}\|v - \pi_e v\|_{0,e}\} \le c|v|_{1,e}.\end{aligned}$$

Thus (2.25) is true for $\varepsilon = 0$. Furthermore, with (2.24) and the interpolation theorem of Sobolev spaces[1,15], we know that (2.25) is true for $\varepsilon > 0$. □

Theorem 2.9. *Let $u \in H^{1+\varepsilon}(\Omega)$ be the solution of (1.1) $\left(\varepsilon \ge \frac{1}{2}\right)$. If u_h is the solution of (1.3), then*

$$|u - u_h|_{1,\Omega,h} \le ch^{\varepsilon}\|u\|_{H^{1+\varepsilon}(\Omega)}. \tag{2.26}$$

Proof. Note that $\pi_h:\ H^{1+\varepsilon}(\Omega) \to V_h$ is the finite element interpolation operator, which satisfies $(\pi_h w)|_e = \pi_e w$, $\forall\ e \in \Omega_h$, $\forall\ w \in H^{1+\varepsilon}(\Omega)$. Then $\pi_h u \in V_h^*$. Since (2.24) is the finite element error estimate, by (2.25) we get

$$\inf_{v\in V_h^*}|u - v|_{1,\Omega,h} \le |u - \pi_h u|_{1,\Omega,h} \le ch^{\varepsilon}\|u\|_{H^{1+\varepsilon}(\Omega)}.$$

Therefore (2.26) follows from Theorem 2.1 and Lemma 2.5. □

3. The Conforming Interpolation Operator I_h

First of all, we construct another mesh $\tilde{\Omega}_h$ of Ω based on Ω_h as follows: for the nonconforming elements of the first kind, if e is a quadrilateral, then e is divided into two triangles by connecting the opposite vertices of e as shown in Fig.3.1; for the nonconforming elements of the second kind, e is divided into several triangles by connecting the interpolation points on e as shown in Fig.3.2 and Fig.3.3. Denote

$\tilde{\Omega}_h = \{\tilde{e}\}$ where $\tilde{e} \subset e \in \Omega_h$ is a triangle. Let $S^h(\Omega)$ be the piecewise linear continuous finite element space on $\tilde{\Omega}_h$.

Fig. 3.1 Fig. 3.2 Fig. 3.3

The conforming interpolation operator $I_h : V_h \to S^h(\Omega)$ is defined as follows

$$\forall\ v \in V_h,\ I_h v \in S^h(\Omega), \quad \text{such that}$$

1) for the nonconforming elements of the first kind,

$$(I_h v)(b) = v(b), \quad \forall \text{ vertex } b \text{ of } e,\ \forall\ e \in \Omega_h$$

2) for the nonconforming elements of the second kind,

$$(I_h v)(x) = \begin{cases} v(a), & \forall \text{ interpolation point } x = a \in \overline{\Omega} \\ \dfrac{1}{2}(v(a_1) + v(a_2)), & \forall \text{ vertex } x \text{ of } e \in \Omega_h, \\ & \quad x \text{ is not the corner point of } \Omega \\ 0, & \forall \text{ corner point } x \text{ of } \Omega, \\ & \quad a_1, a_2 \in \partial\Omega, v(a_1) \cdot v(a_2) = 0 \\ v(a_1), & \forall \text{ corner point } x \text{ of } \Omega, \\ & \quad a_1, a_2 \in \partial\Omega, v(a_1) \cdot v(a_2) \neq 0 \end{cases}$$

Here, a_1, a_2 represent the midpoints of any two edges of the elements with x as their common endpoint. Generally, if possible, we select a_1, a_2 such that a_1, x, a_2 are in a line. But, when $x \in \partial\Omega$ is the vertex of some element e, we select $a_1, a_2 \in \partial\Omega$. Although there might be different way to select a_1, a_2, we always have

Theorem 3.1. *If the conforming interpolation operator I_h on V_h is defined as above, then*

$$\|v - I_h v\|_{L^2(\Omega)} \le ch|v|_{1,\Omega,h}, \quad \forall\ v \in V_h \tag{3.1}$$

$$|v - I_h v|_{1,\Omega,h} \le c|v|_{1,\Omega,h}, \quad \forall\ v \in V_h \tag{3.2}$$

$$\max_{e \in \Omega_h} \|v - I_h v\|_{L^\infty(e)} \le c|v|_{1,\Omega,h},\ \forall\ v \in V_h \tag{3.3}$$

Proof. $\forall\ v \in V_h$, if (3.1) is true, then the inverse inequalities yield

$$|v - I_h v|^2_{1,\Omega,h} = \sum_{e \in \Omega_h} |v - I_h v|^2_{1,e} \le \sum_{e \in \Omega_h} ch^{-2}|v - I_h v|^2_{0,e}$$

$$\le ch^{-2}\|v - I_h v\|^2_{L^2(\Omega)} \le c|v|^2_{1,\Omega,h},$$
$$\max_{e\in\Omega_h}\|v - I_h v\|_{L^\infty(e)} \le \|v - I_h v\|_{L^\infty(e_0)} \le ch^{-1}\|v - I_h v\|_{0,e_0} \le c|v|_{1,\Omega,h}.$$

Here, $e_0 \in \Omega_h$. Thus, (3.1) implies (3.2) and (3.3). Therefore, it suffices to show (3.1).

For the nonconforming elements of the first kind, I_h is the (piecewise) linear interpolation operator on e. It follows from the interpolation error estimate and the inverse inequality that

$$\|v - I_h v\|^2_{L^2(\Omega)} = \sum_{e\in\Omega_h}\|v - I_h v\|^2_{0,e} \le \sum_{e\in\Omega_h} ch^4|v|^2_{2,e} \le \sum_{e\in\Omega_h} ch^2|v|^2_{1,e} \le ch^2|v|^2_{1,\Omega,h},$$

hence, (3.1) is established in this case.

For the nonconforming elements of the second kind, construct a function $\tilde v$ on Ω, such that $\tilde v|_e$ is the unique piecewise linear continuous function determined by $v|_{\overline{e}}$ and $\tilde v|_e$ is linear on $\tilde e \subset e$, $\forall\ e \in \Omega_h$. Refer to Fig.3.2 and Fig.3.3. Obviously, $I_h v = I_h\tilde v$, on Ω. The interpolation error estimates and the inverse inequalities yield

$$\|v - \tilde v\|^2_{0,e} = \sum_{\tilde e\subset e}\|v - \tilde v\|^2_{0,\tilde e} \le \sum_{\tilde e\subset e} ch^4|v|^2_{2,\tilde e} = ch^4|v|^2_{2,e} \le ch^2|v|^2_{1,e},$$
$$|v - \tilde v|^2_{1,e} = \sum_{\tilde e\subset e}|v - \tilde v|^2_{1,\tilde e} \le \sum_{\tilde e\subset e} ch^2|v|^2_{2,\tilde e} \le c|v|^2_{1,e},$$
$$\|v - \tilde v\|_{L^2(\Omega)} \le ch|v|_{1,\Omega,h}, \tag{3.4}$$
$$|v - \tilde v|_{1,\Omega,h} \le c|v|_{1,\Omega,h},$$
$$|\tilde v|_{1,\Omega,h} \le c|v|_{1,\Omega,h}. \tag{3.5}$$

If we can show that

$$\|\tilde v - I_h\tilde v\|_{L^2(\Omega)} \le ch|\tilde v|_{1,\Omega,h}, \tag{3.6}$$

then it follows from $I_h v = I_h\tilde v$, on Ω, (3.4) and (3.5) that

$$\|v - I_h v\|_{L^2(\Omega)} \le \|v - \tilde v\|_{L^2(\Omega)} + \|\tilde v - I_h v\|_{L^2(\Omega)} = \|v - \tilde v\|_{L^2(\Omega)} + \|\tilde v - I_h\tilde v\|_{L^2(\Omega)}$$
$$\le ch|v|_{1,\Omega,h} + ch|\tilde v|_{1,\Omega,h} \le ch|v|_{1,\Omega,h},$$

which is (3.1). Hence, what is left is to prove (3.6).

For the quartic rectangular elements, $I_h\tilde v = \tilde v$, on $\tilde e_i$, $i = 5, 6, 7, 8$. Refer to Fig.3.4. Therefore, $\|\tilde v - I_h\tilde v\|^2_{0,e} = \sum_{i=1}^{4}\|\tilde v - I_h\tilde v\|^2_{0,\tilde e_i}$.

On the other hand, on $\overline{e}$, it is easy to see that

$$\tilde v(b_1) = v(b_1) = v(a_1) + v(a_4) - v(a_5) = \tilde v(a_1) + \tilde v(a_4) - \tilde v(a_5).$$

If b_1 is not the corner point of Ω, without loss of generality, we assume that $(I_h\tilde v)(b_1) = \frac{1}{2}(\tilde v(a_1) + \tilde v(a_6))$. Then

$$\|\tilde v - I_h\tilde v\|^2_{0,\tilde e_1} \le ch|\tilde v(b_1) - (I_h\tilde v)(b_1)|$$

$$\le ch|\tilde{v}(a_1)+\tilde{v}(a_4)-\tilde{v}(a_5)-\frac{1}{2}(\tilde{v}(a_1)+\tilde{v}(a_6))|$$
$$\le ch\{|\tilde{v}(a_1)-\tilde{v}(a_5)|+|\tilde{v}(a_4)-\tilde{v}(a_1)|+|\tilde{v}(a_4)-\tilde{v}(a_6)|\}$$
$$\le ch\{|\tilde{v}|_{1,\tilde{e}_5}+|\tilde{v}|_{1,\tilde{e}_9}.\}$$

If b_1 is the corner point of Ω and $(I_h\tilde{v})(b_1)=\tilde{v}(a_1)$, then

$$\|\tilde{v}-I_h\tilde{v}\|_{0,\tilde{e}_1}\le ch|\tilde{v}(b_1)-(I_h\tilde{v})(b_1)|\le ch|\tilde{v}(a_4)-\tilde{v}(a_5)|\le ch|\tilde{v}|_{1,\tilde{e}_5}.$$

By now, we obtain that

$$\|\tilde{v}-I_h\tilde{v}\|_{0,e}\le ch\{|\tilde{v}|_{1,e}+|\tilde{v}|_{1,\ddot{e}}\},$$

where $\ddot{e}$ is the union of the elements adjacent to e. Furthermore, summing up with $e\in\Omega_h$, we get (3.6).

Fig. 3.4 Fig. 3.5

For the Crouzeix–Raviart elements, $\forall\ e\in\Omega_h$, $\tilde{v}-I_h\tilde{v}=0$, on $\tilde{e}_0$. Refer to Fig.3.5. It suffices to consider $\|\tilde{v}-I_h\tilde{v}\|_{0,\tilde{e}_1}$. Here, $\tilde{e}_0$, $\tilde{e}_1\subset e$. In $\overline{e}$, it is easy to see that

$$\tilde{v}(a_1)=(I_h\tilde{v})(a_1),\quad \tilde{v}(a_2)=(I_h\tilde{v})(a_2),$$
$$\tilde{v}(b)=v(b)=v(a_1)+v(a_2)-v(a_0)=\tilde{v}(a_1)+\tilde{v}(a_2)-\tilde{v}(a_0).$$

Without loss of generality, assume that $(I_h\tilde{v})(b)=\frac{1}{2}(\tilde{v}(a_i)+\tilde{v}(a_j))$, where a_i,a_j $(i\le j)$ are the midpoints of any two edges of the elements with b as their endpoint. It follows from the quasi-uniformness of Ω_h that there exists a positive integer J, independent of h, such that $j\le J$. An elementary calculation yields

$$\begin{aligned}\|\tilde{v}-I_h\tilde{v}\|_{0,\tilde{e}_1}&\le ch|\tilde{v}(b)-(I_h\tilde{v})(b)|\\&\le ch\Big|\tilde{v}(a_1)+\tilde{v}(a_2)-\tilde{v}(a_0)-\frac{1}{2}(\tilde{v}(a_i)+\tilde{v}(a_j))\Big|\\&\le ch\{|\tilde{v}(a_2)-\tilde{v}(a_0)|+|\tilde{v}(a_1)-\tilde{v}(a_i)|+|\tilde{v}(a_1)-\tilde{v}(a_j)|\}.\end{aligned}$$

On the other hand, we have $h|\tilde{v}(a_2)-\tilde{v}(a_0)|\le c|\tilde{v}|_{1,\tilde{e}_0}$,

$$h|\tilde{v}(a_1)-\tilde{v}(a_i)|\le h\sum_{k=1}^{i-1}|\tilde{v}(a_k)-\tilde{v}(a_{k+1})|\le c\sum_{k=1}^{i-1}|\tilde{v}|_{1,\tilde{e}_k},$$

$$h|\tilde{v}(a_1)-\tilde{v}(a_j)|\le c\sum_{k=1}^{j-1}|\tilde{v}|_{1,\tilde{e}_k}.$$

Hence

$$\|\tilde{v}-I_h\tilde{v}\|_{0,\tilde{e}_1}\le c\Big\{|\tilde{v}|_{1,\tilde{e}_0}+\sum_{k=1}^{j-1}|\tilde{v}|_{1,\tilde{e}_k}\Big\}.$$

Summing up with $\tilde{e}_1\subset e$, $e\in\Omega_h$ leads to (3.6). By now, the proof of Theorem 3.1 is finished. □

4. The Fundamental Inequalities

In this section, $d=\operatorname{diam}\Omega$ represents the diameter of Ω.

Theorem 4.1. (*Poincaré inequalities in the nonconforming space V_h*)

1) $\|v\|^2_{L^2(\Omega)}\le c\Big(d^2|v|^2_{1,\Omega,h}+d^{-2}|\int_\Omega v|^2\Big)$, $\forall\ v\in V_h$

2) $d^{-2}\|v\|^2_{L^2(\Omega)}\le c|v|^2_{1,\Omega,h}$, $\quad\forall\ v\in V_h^0$

3) *If $v\in V_h$ satisfies $v(x)=0$, $\forall$ interpolation point $x\in\Gamma$, where $\Gamma\subset\partial\Omega$ has at least two interpolation points, then*

$$d^{-2}\|v\|^2_{L^2(\Omega)}\le c|v|^2_{1,\Omega,h}.$$

Proof. $\forall\ v\in V_h$, $I_hv\in S^h(\Omega)\subset H^1(\Omega)$. It follows from the Poincaré inequality in $H^1(\Omega)$ that

$$\begin{aligned}\|I_hv\|^2_{L^2(\Omega)}&\le c\Big(d^2|I_hv|^2_{H^1(\Omega)}+d^{-2}\Big|\int_\Omega I_hv\Big|^2\Big)\\&\le c\Big\{d^2|I_hv|^2_{H^1(\Omega)}+d^{-2}\Big|\int_\Omega v|^2+d^{-2}\Big|\int_\Omega(v-I_hv)\Big|^2\Big\}\\&\le c\Big\{d^2|v|^2_{1,\Omega,h}+d^2|v-I_hv|^2_{1,\Omega,h}+d^{-2}\Big|\int_\Omega v\Big|^2+d^{-2}\Big(\int_\Omega 1^2\Big)\|v-I_hv\|^2_{L^2(\Omega)}\Big\}.\end{aligned}$$

Furthermore, by the triangle inequality and Theorem 3.1 , we obtain

$$\begin{aligned}\|v\|^2_{L^2(\Omega)}&\le 2\{\|v-I_hv\|^2_{L^2(\Omega)}+\|I_hv\|^2_{L^2(\Omega)}\}\\&\le c\Big\{d^2|v|^2_{1,\Omega,h}+d^{-2}\Big|\int_\Omega v|^2+h^2|v|^2_{1,\Omega,h}\Big\}.\end{aligned}$$

Hence, 1) is right.

In the same manner, 2) and 3) can be proved. □

Theorem 4.2. (*Maximum norm estimate in the nonconforming space V_h*)

$$\max_{e\in\Omega_h}\|v\|^2_{L^\infty(e)}\le c\Big\{d^{-2}\|v\|^2_{L^2(\Omega)}+\Big(1+\ln\frac{d}{h}\Big)|v|^2_{1,\Omega,h}\Big\},\quad\forall\ v\in V_h$$

Proof. $\forall\, v \in V_h$, $I_h v \in S^h(\Omega) \subset H^1(\Omega)$. It follows from Lemma 3.3[2] and Theorem 3.1 that

$$\begin{aligned}
\|I_h v\|^2_{L^\infty(\Omega)} &\le c\Big\{d^{-2}\|I_h v\|^2_{L^2(\Omega)} + \Big(\ln\frac{d}{h}\Big)|I_h v|^2_{H^1(\Omega)}\Big\} \\
&\le c\Big\{d^{-2}\|v\|^2_{L^2(\Omega)} + d^{-2}\|v - I_h v\|^2_{L^2(\Omega)} \\
&\quad + \Big(\ln\frac{d}{h}\Big)|v|^2_{1,\Omega,h} + \Big(\ln\frac{d}{h}\Big)|v - I_h v|^2_{1,\Omega,h}\Big\} \\
&\le c\Big\{d^{-2}\|v\|^2_{L^2(\Omega)} + \Big(\ln\frac{d}{h}\Big)|v|^2_{1,\Omega,h}\Big\}, \\
\max_{e\in\Omega_h}\|v\|^2_{L^\infty(e)} &\le 2\Big(\max_{e\in\Omega_h}\|v - I_h v\|^2_{L^\infty(e)} + \|I_h v\|^2_{L^\infty(\Omega)}\Big) \\
&\le c\Big\{d^{-2}\|v\|^2_{L^2(\Omega)} + \Big(1 + \ln\frac{d}{h}\Big)|v|^2_{1,\Omega,h}\Big\}.
\end{aligned}$$

By now, the theorem is proved. $\square$

Theorem 4.3. (Extension theorem) *Let $\Gamma \subset \partial\Omega$ be an open boundary. $\{\xi_j\}_1^m$ $(m \ge 2)$ denotes the set of the interpolation points on Γ (ordered in some way). Give $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T \in \Re^m$. If $w_h \in V_h$ satisfies*

$$\begin{cases} A(w_h, v) = 0, & \forall\, v \in V_h^0 \\ w_h(\xi_j) = \lambda_j, & j = 1, 2, \cdots, m \\ w_h(x) = 0, & \forall \textit{ interpolation point } x \in \partial\Omega\backslash\Gamma \end{cases} \tag{4.1}$$

then

$$c\|\lambda_h\|^2_{H_{00}^{\frac{1}{2}}(\Gamma)} \le A(w_h, w_h) \le C\|\lambda_h\|^2_{H_{00}^{\frac{1}{2}}(\Gamma)}, \tag{4.2}$$

where λ_h denotes the piecewise linear continuous function on Γ which satisfies

$$\lambda_h(\xi_j) = \lambda_j,\ j = 1, 2, \cdots, m,\ \lambda_h(\nu) = 0,\ \forall \textit{ endpoint } \nu \textit{ of } \Gamma.$$

Proof. Construct the harmonic function $w \in H^1(\Omega)$, which satisfies

$$\begin{cases} a(w, v) = 0, & \forall\, v \in H_0^1(\Omega) \\ w = \lambda_h, & \text{on } \Gamma \\ w = 0, & \text{on } \partial\Omega\backslash\Gamma \end{cases} \tag{4.3}$$

It follows from the priori estimate of the elliptic problems[7,15,16] that even if Ω is concave, there exists $\varepsilon \ge \dfrac{1}{2}$, such that $w \in H^{\varepsilon+1}(\Omega)$. In addition, we note that w_h is the nonconforming finite element approximation of w.

Theorem 2.9 indicates that

$$\begin{aligned}
A(w_h, w_h) &\le 2\{A(w, w) + A(w - w_h, w - w_h)\} \le c\{a(w, w) + |w - w_h|^2_{1,\Omega,h}\} \\
&\le c\{\|w\|^2_{H^1(\Omega)} + h^{2\varepsilon}\|w\|^2_{H^{\varepsilon+1}(\Omega)}\}.
\end{aligned}$$

Furthermore, with the priori estimate of the elliptic problems and the fractional order inverse inequality implied by the interpolation theorem of Sobolev spaces, we have

$$\begin{aligned} A(w_h, w_h) &\le c\{\|w\|^2_{H^{\frac{1}{2}}(\partial\Omega)} + h^{2\varepsilon}\|w\|^2_{H^{\varepsilon+\frac{1}{2}}(\partial\Omega)}\} \\ &\le c\|w\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \le c\|w\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)} = c\|\lambda_h\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)}. \end{aligned} \tag{4.4}$$

In the above last inequality, we have applied the fact that $\|w\|_{H^{\frac{1}{2}}(\partial\Omega)}$ is equivalent to $\|w\|_{H^{\frac{1}{2}}_{00}(\Gamma)}$ for $w = 0$ on $\partial\Omega\backslash\Gamma$ (cf.[2]).

On the other hand, it follows from the construction of I_h that

$$I_h w_h \in H^1(\Omega),\ I_h w_h = 0,\ \text{on } \partial\Omega\backslash\Gamma, \quad I_h w_h = \lambda_h,\ \text{on } \Gamma.$$

With Theorem 3.1, the Poincaré inequality and the trace theorem, we obtain

$$\begin{aligned} A(w_h, w_h) &\ge c|w_h|^2_{1,\Omega,h} \ge c|I_h w_h|^2_{H^1(\Omega)} \ge c\|I_h w_h\|^2_{H^1(\Omega)} \\ &\ge c\|I_h w_h\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \ge c\|I_h w_h\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)} = c\|\lambda_h\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)}, \end{aligned}$$

Therefore, (4.2) follows from (4.4). □

Lemma 4.4. *Suppose that $d \gg h$ and there exists a positive constant β, such that $\sup\limits_{B_r\subset\Omega} r \ge \beta d$. Let ν_1 be a corner point of Ω. v is the piecewise linear continuous function on $\partial\Omega$ which satisfies*

$$v(x) = 0,\ \forall \text{ interpolation point } x \in \partial\Omega,\ v(\nu) = 0,\ \forall \text{ corner point } \nu \text{ of } \Omega,\ \nu \ne \nu_1.$$

Then, there exists a positive constant c, independent of d, such that

$$\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \le c|v(\nu_1)|. \tag{4.5}$$

Proof. Without loss of generality, we assume that Ω is a triangle. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be the three edges of Ω with ν_1, ν_2, ν_3 as their opposite corner points. The definition of the Sobolev space $H^{\frac{1}{2}}(\partial\Omega)$ is (cf.[15])

$$\begin{aligned} \|v\|^2_{H^{\frac{1}{2}}(\partial\Omega)} &\triangleq \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega} v^2 ds + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^2} ds(x)ds(y) \\ &= \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega} v^2 ds + \sum_{i,j=1}^{3}\int_{\Gamma_i}\int_{\Gamma_j} \frac{|v(x)-v(y)|^2}{|x-y|^2} ds(x)ds(y). \end{aligned}$$

Since $d \gg h$, $\sup\limits_{B_r\subset\Omega} r \ge \beta d$, a simple calculus calculation yields

$$\frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega} v^2 ds \le c|v(\nu_1)|^2,$$

$$\int_{\Gamma_1}\int_{\Gamma_1} = 0, \ \int_{\Gamma_1}\int_{\Gamma_2} = \int_{\Gamma_2}\int_{\Gamma_1} \le c|v(\nu_1)|^2,$$
$$\int_{\Gamma_2}\int_{\Gamma_2} \le 2|v(\nu_1)|^2, \quad \int_{\Gamma_2}\int_{\Gamma_3} = \int_{\Gamma_3}\int_{\Gamma_2} < 2|v(\nu_1)|^2,$$
$$\int_{\Gamma_3}\int_{\Gamma_3} \le 2|v(\nu_1)|^2, \ \int_{\Gamma_3}\int_{\Gamma_1} = \int_{\Gamma_1}\int_{\Gamma_3} \le c|v(\nu_1)|^2.$$

Here, we omit the integrands for conciseness and c is independent of d, h.

By now, (4.5) is established. □

Lemma 4.5. *Let Γ_h be the quasi-uniform mesh of the interval $\Gamma = [0, L]$. Suppose that $v(x)$ is the piecewise linear continuous function on Γ_h and $v(0) = 0$. Then, there exists a constant c, independent of L, such that*

$$\int_\Gamma \frac{(v(x))^2}{x} dx \le \Big(1 + \ln \frac{L}{h}\Big)\|v\|^2_{L^\infty(\Gamma)}. \tag{4.6}$$

The proof of Lemma 4.5 is trivial, so we omit it here.

Theorem 4.6. (Extension theorem) *Suppose that $d \gg h$ and there exists a positive constant β, such that $\sup\limits_{B_r \subset \Omega} r \ge \beta d$. Let $\{\nu_k\}_1^J$ be the set of the corner points of Ω. $\Gamma_{ij} \subset \partial\Omega$ denotes the edge of Ω with ν_i, ν_j as its endpoints. Suppose that V_h is the nonconforming finite element space of the second kind. Then*

$$\|v^{ij}\|^2_{H^{\frac12}_{00}(\Gamma_{ij})} \le c\Big(1 + \ln\frac{d}{h}\Big)\Big\{\Big(1 + \ln\frac{d}{h}\Big)|v|^2_{1,\Omega,h} + (1 + d^{-2})\|v\|^2_{L^2(\Omega)}\Big\}, \ \forall\, v \in V_h$$

where v^{ij} is the piecewise linear continuous function on Γ_{ij} which satisfies

$$v^{ij}(x) = v(x), \ \forall \text{ interpolation point } x \in \Gamma_{ij}, \ v^{ij}(\nu_i) = v^{ij}(\nu_j) = 0,$$

and the positive constant c is independent of d.

Proof. Let v^B be the piecewise linear continuous function on $\partial\Omega$, such that $v^B = v^{ij}$, on Γ_{ij}, $\forall\, \Gamma_{ij} \subset \partial\Omega$. Let v_k, $k = 1, 2, \cdots, J$ be the piecewise linear continuous function on $\partial\Omega$, such that

$$v_k(\nu_k) = \Big(I_h v\Big)(\nu_k), \ v_k(\nu_j) = 0, \ j = 1, 2, \cdots, k-1, k+1, \cdots, J,$$
$$v_k(x) = 0, \ \forall \text{ interpolatoin point } x \in \partial\Omega.$$

It is easy to see that $I_h v = v^B + \sum\limits_{k=1}^{J} v_k$, on $\partial\Omega$. It follows from the definition of the Sobolev space $H^{\frac12}_{00}(\Gamma_{ij})$ that[2,15]

$$\|v^{ij}\|^2_{H^{\frac12}_{00}(\Gamma_{ij})} \le c\Big\{\|v^B\|^2_{H^{\frac12}(\partial\Omega)} + \int_{\Gamma_{ij}} \Big(\frac{|v^{ij}(x)|^2}{|x - \nu_i|} + \frac{|v^{ij}(x)|^2}{|x - \nu_j|}\Big) ds(x)\Big\}. \tag{4.7}$$

With the trace theorem, Lemma 4.4, Lemma 3.3[2] and Theorem 3.1, we have

$$\begin{aligned}
\|v^B\|^2_{H^{\frac{1}{2}}(\partial\Omega)} &= \|I_h v - \sum_{k=1}^{J} v_k\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \le c\Big\{\|I_h v\|^2_{H^{\frac{1}{2}}(\partial\Omega)} + \sum_{k=1}^{J} \|v_k\|^2_{H^{\frac{1}{2}}(\partial\Omega)}\Big\} \\
&\le c\Big\{\|I_h v\|^2_{H^1(\Omega)} + \sum_{k=1}^{J} |\Big(I_h v\Big)(\nu_k)|^2\Big\} \le c\{\|I_h v\|^2_{H^1(\Omega)} + \|I_h v\|^2_{L^\infty(\Omega)}\Big\} \\
&\le \Big\{\Big(1+\ln\frac{d}{h}\Big)|I_h v|^2_{H^1(\Omega)} + (1+d^{-2})\|I_h v\|^2_{L^2(\Omega)}\Big\}, \\
&\le \Big\{\Big(1+\ln\frac{d}{h}\Big)|v|^2_{1,\Omega,h} + (1+d^{-2})\|v\|^2_{L^2(\Omega)}\Big\}. \qquad (4.8)
\end{aligned}$$

It follows from Lemma 4.5, Lemma 3.3[2] and Theorem 3.1 that

$$\begin{aligned}
\int_{\Gamma_{ij}} \Big(\frac{|v^{ij}(x)|^2}{|x-\nu_i|} + \frac{|v^{ij}(x)|^2}{|x-\nu_j|}\Big) ds(x) &\le c\Big(1+\ln\frac{d}{h}\Big)\|v^{ij}\|^2_{L^\infty(\Gamma_{ij})} \\
&\le c\Big(1+\ln\frac{d}{h}\Big)\|I_h v\|^2_{L^\infty(\Omega)} \\
&\le c\Big(1+\ln\frac{d}{h}\Big)\Big\{\Big(\ln\frac{d}{h}\Big)|I_h v|^2_{H^1(\Omega)} + d^{-2}\|I_h v\|^2_{L^2(\Omega)}\Big\} \\
&\le c\Big(1+\ln\frac{d}{h}\Big)\Big\{\Big(\ln\frac{d}{h}\Big)|v|^2_{1,\Omega,h} + d^{-2}\|v\|^2_{L^2(\Omega)}\Big\}. \qquad (4.9)
\end{aligned}$$

(4.7), (4.8) and (4.9) indicate that Theorem 4.6 holds. □

Remark 4.1. Although we deal with the Dirichlet form $a(u,v) = \int_\Omega \nabla u \nabla v$ in this paper, all the above conclusions are true for the general form

$$a(u,v) = \int_\Omega \Big[\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0(x)uv\Big],$$

where $a_{ij}(x), a_0(x)$ are bounded, piecewise smooth on Ω, $a_0(x) \ge 0$ and (a_{ij}) is a symmetric, uniformly positive definite matrix on Ω.

Remark 4.2. Besides [4, 5, 14], the nonconforming finite element space of the first kind may include other elements which are continuous at the vertices of the elements of the mesh, if (2.6) is true.

Remark 4.3. Besides [6,13], the nonconforming finite element space of the second kind may include other elements which are continuous at the midpoints, even the Gaussian quadrature points, of the edges of the elements of the mesh, if (2.5) and (3.6) are established.

References

[1] R.A. Adams, Sobolev Spaces. Academic Press, New York, 1975.

[2] J.H. Bramble, J.E. Pasciak, A.H. Schatz, The construction of preconditioners for elliptic problems by substructuring, *I. Math. Comp.,* **47** (1986), 103–134.

[3] J.H. Bramble, J.E. Pasciak, A.H. Schatz, An iterative method for elliptic problems on regions partitioned into substructures, *Math. Comp.,* **46** (1986), 361–369.
[4] G.F. Carey, An analysis of finite element equations and mesh subdivision, *Comput. Methods Appl. Mech. Engrg.,* **9** (1976), 165–179.
[5] P.G. Ciarlet,The Finite Element Method for Elliptic Problems, North–Holland, Amsterdam, 1978.
[6] M. Crouzeix, P.A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, *RAIRO Numer. Anal.,* **7**:R3 (1973), 33–76.
[7] P. Grisvard, Elliptic Problems in Nonsmooth Domain, Pitman, Boston, 1985.
[8] J. Gu,Domain Decomposition Methods with Nonconforming Finite Elements, Ph. D. thesis. (in Chinese), Depart. of Applied Math., Tsinghua University, China, 1994.
[9] J. Gu, X. Hu, On domain decomposition methods in two–subdomain nonoverlap cases, *Chinese J. Num. Math. & Appl.,* **17**:1 (1995), 78–94.
[10] J. Gu, X. Hu, On an essential estimate in the analysis of domain decomposition methods, *J. Comput. Math.,* **12**:2 (1994), 132–137.
[11] J. Gu, X. Hu, Trace averaging domain decomposition method with nonconforming finite elements, *J. Comput. Math.,* **14**:1 (1996), 40–53.
[12] J. Gu, X. Hu, The construction of preconditioners for elliptic problems discretized by nonconforming finite elements via substructuring (II) (in Chinese), *Mathematica Numerica Sinica,* **18**:4 (1996), 337–348.
[13] H. Han, Nonconforming elements in the mixed finite element method, *J. Comput. Math.,* **2**:3 (1984), 223–233.
[14] J. Jiang, X. Cheng, A nonconforming element like Wilson's for second order problems, (in Chinese), *Mathematica Numerica Sinica,* **14**:3 (1992), 274–278.
[15] J.L. Lions, E. Magenes, Non-homogeneous Boundary Value Problems and Applications, I. Grund. B. 181. Berlin, Springer, 1972.
[16] J. Nečas, Les Méthodes Directes en Théorie des Équations Elliptiques, Academia, Prague, 1967.
[17] Z. Shi, The F–E–M–Test for convergence of nonconforming finite elements, *Math. Comp.,* **49** (1987), 391–405.
[18] G. Strang, G.J. Fix, An Analysis of the Finite Element Method, Prentice–Hall, Englewood Cliffs, New Jersey, 1973.
[19] F. Stummel,Basic compactness properties of nonconforming and hybrid finite element spaces, *RAIRO Numerical Analysis,* **4**:1 (1980), 81–115.
[20] O.B. Widlund, An extension theorem for finite element spaces with three applications. In Numerical Techniques in Continuum Mechanics, Vol.16. Braaunschweig Wiesbaden, 1987.