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SPLITTING A CONCAVE DOMAIN TO CONVEX SUBDOMAINS*

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Abstract

We examine a steady-state heat radiation problem and its finite element approximation in R^d, $d = 2, 3$. A nonlinear Stefan-Boltzmann boundary condition is considered. Another nonlinearity is due to the fact that the temperature is always greater or equal than $0[K]$. We prove two convergence theorems for piecewise linear finite element solutions.

Keywords: Nonlinear elliptic boundary value problems, heat radiation problem, finite elements, variational inequalities.

1. Introduction

It is known from physics that a body loses heat energy from its surface by electromagnetic waves. This phenomenon is called radiation[8,20]. The energetical losses are proportional to the fourth power of the surface temperature (the Kirchhoff law). Thus the radiation cannot be neglected when the surface temperature is high (e.g., in computation of temperature distribution in large dry transformers, electrical engines, $\cdots$). It is represented by the nonlinear boundary condition $\alpha(u - u_0) + n^\top \mathcal{A}$ grad $u + \beta(u^4 - u_0^4) = \tilde{g}$, where $\alpha \geq 0$ is the coefficient of convective heat transfer, u is the temperature of the body, u_0 is the surrounding temperature, n is the outward unit normal to the surface, $\mathcal{A}$ is a symmetric uniformly positive definite matrix of heat conductivities, $\beta = \sigma f_{\text{em}}$, $\sigma = 5.669 \times 10^{-8}[\text{Wm}^{-2}\text{K}^{-4}]$ is the Stefan-Boltzmann constant, $0 \leq f_{\text{em}} \leq 1$ is the relative emissivity function and $\tilde{g}$ is the density of surface heat sources.

Consider the following classical formulation of the radiation problem: Find $u \in C^2(\overline{\Omega})$, $u \geq 0$, such that

$$\begin{aligned} -\text{div}\,(\mathcal{A} \text{ grad } u) &= f \quad \text{in } \Omega, \\ u &= \overline{u} \quad \text{on } \Gamma_1, \\ \alpha u + n^\top \mathcal{A} \text{ grad } u + \beta u^4 &= g \quad \text{on } \Gamma_2, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^d$, $d = \{2, 3\}$, is a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$, Γ_1 and Γ_2 are non-empty disjoint sets, which are relatively open in $\partial\Omega$, and satisfy

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$\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, f is the density of body heat sources, $\overline{u} \geq 0$ is the prescribed temperature and $g = \tilde{g} + \alpha u_0 + \beta u_0^4$.

Similar heat radiation problems were investigated by many authors (see, e.g., [3, 8, 18, 19, 21, 22]). A proof of the existence and uniqueness of the classical solution is given in [2] for a regular boundary.

Throughout the paper we use the standard Sobolev space notation [14, 15, 17]. We will introduce a variational inequality approach to the problem (1.1) and examine its finite element approximation under the maximum angle condition. We also generalize some results of [16, 21] for $d = 2$ to the three-dimensional space.

2. Variational Formulation of a Two-Dimensional Problem

Since the classical solution of the problem (1.1) need not exist, we introduce its variational formulation. To this end we suppose that the entries of $\mathcal{A}$ belong to $L^\infty(\Omega)$, $f \in L^2(\Omega)$, $\overline{u} \in H^1(\Omega)$, $\alpha, \beta \in L^\infty(\Gamma_2)$ and $g \in L^2(\Gamma_2)$. Introduce a space of test functions $V = \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_1\}$ and a set $U = \{v \in H^1(\Omega) | v \geq 0 \text{ in } \Omega,\ v = \overline{u}$ on $\Gamma_1\}$. It is easy to verify that U is convex, closed with respect to the norm $\|.\|_1$ and nonempty as $\overline{u} \in U$. Note that it has no interior points. (To see this for $d = 2$ and $(0,0) \in \Omega$, a simple example can be constructed using the function

$$v_\varepsilon(x_1, x_2) = -\varepsilon\big(-\ln\sqrt{x_1^2 + x_2^2}\big)^{1/4}, \quad (x_1, x_2) \in \Omega, \quad \varepsilon > 0,$$

which has a negative pole and $\|v_\varepsilon\|_1 \to 0$ for $\varepsilon \to 0$, compare [14, p. 10]).

Define a symmetric bilinear continuous form

$$a(v, w) = \int_\Omega (\text{grad } v)^\top \mathcal{A} \text{ grad } w dx + \int_{\Gamma_2} \alpha v w ds, \quad v, w \in H^1(\Omega),$$

and a linear continuous form

$$F(v) = \int_\Omega f v dx + \int_{\Gamma_2} g v ds, \quad v \in H^1(\Omega).$$

Using positive definiteness of $\mathcal{A}$, the Friedrichs inequality and the fact that $\Gamma_1 \neq \emptyset$, we obtain the V-ellipticity of $a(.,.)$,

$$a(v, v) \geq C\|v\|_1^2 \quad \forall v \in V. \tag{2.1}$$

In this chapter, we will examine the case $d = 2$. Let $v \in U$ be arbitrary and suppose that a solution $u \in U$ of (1.1) exists. Multiplying (1.1) by the function $v - u \in V$ and then integrating over Ω, we get by Green's theorem the following variational equality

$$a(u, v - u) + \int_{\Gamma_2} \beta u^4 (v - u) ds = F(v - u) \quad \forall v \in U.$$

From here we obviously get the variational inequality

$$a(u, v - u) + \int_{\Gamma_2} \beta u^4 (v - u) ds \geq F(v - u) \quad \forall v \in U. \tag{2.2}$$

A function $u \in U$ satisfying (2.2) is called a weak solution (or variational inequality solution) of the classical problem (1.1). It is clear that the classical solution is also the weak solution. Using the well-known results from potential theory, we show later (see Remark 2.5) that there exists precisely one weak solution $u \in U$ of (2.2).

Define a functional of potential energy

$$J(v) = \frac{1}{2}a(v,v) + \frac{1}{5}\int_{\Gamma_2} \beta v^5 ds - F(v), \quad v \in H^1(\Omega). \tag{2.3}$$

According to the trace theorem [17, p. 84, 86], for $d = 2$ and $q \in [1, \infty)$ there exists a unique linear continuous mapping $Z : H^1(\Omega) \to L^q(\partial\Omega)$ such that for any $v \in C^\infty(\overline{\Omega})$ we have $Zv = v$ on $\partial\Omega$, i.e.,

$$\|v\|_{0,q,\partial\Omega} \le C_q \|v\|_{1,2,\Omega} \quad \forall v \in H^1(\Omega). \tag{2.4}$$

Therefore, the boundary integral in (2.3) is finite and J is thus correctly defined.

By (2.3) we can easily compute the directional derivative of J at the point $v \in H^1(\Omega)$ in the direction $w \in H^1(\Omega)$,

$$\begin{aligned} J'(v;w) &= \lim_{t\to 0} \frac{1}{t}(J(v+tw) - J(v)) = \lim_{t \to 0}\frac{1}{t}\Big(ta(v;w) + \frac{1}{2}t^2 a(w;w) \\ &\quad + \int_{\Gamma_2} \beta\Big(tv^4 w + 2t^2 v^3 w^2 + 2t^3 v^2 w^3 + t^4 v w^4 + \frac{1}{5}t^5 w^5\Big) ds - tF(w)\Big) \\ &= a(v;w) + \int_{\Gamma_2} \beta v^4 w ds - F(w). \end{aligned} \tag{2.5}$$

Analogously we find an expression for the second Gâteaux derivative of J,

$$J''(v;w,w) = a(w,w) + 4\int_{\Gamma_2} \beta v^3 w^2 ds, \quad v, w \in H^1(\Omega). \tag{2.6}$$

Lemma 2.1. *The functional J is continuous on $H^1(\Omega)$ with respect to the $\|.\|_1$-norm and is strictly convex over the set U.*

Proof. To check the continuity of J it suffices to prove that the Gâteaux differential of J is the Fréchet differential, i.e., we show that for any $v \in H^1(\Omega)$

$$\lim_{\substack{\|w\|_1 \to 0 \\ w \in H^1(\Omega)}} \frac{|J(v+w) - J(v) - J'(v;w)|}{\|w\|_1} = 0. \tag{2.7}$$

By (2.3), (2.5), the continuity of $a(.,.)$, the Cauchy-Schwarz inequality, the imbedding $L^5(\partial\Omega) \subset L^2(\partial\Omega)$ and (2.4), we obtain

$$\begin{aligned} |J(v+w) - J(v) - J'(v;w)| &\le \frac{1}{2}a(w,w) + \int_{\Gamma_2} \beta\Big|2v^3w^2 + 2v^2w^3 + vw^4 + \frac{1}{5}w^5\Big| ds \\ &\le C_1 \|w\|^2_{1,2,\Omega} + C_2(\|v\|^3_{0,5,\partial\Omega}\|w\|^2_{0,5,\partial\Omega} + \|v\|^2_{0,5,\partial\Omega}\|w\|^3_{0,5,\partial\Omega} \\ &\quad + \|v\|_{0,5,\partial\Omega}\|w\|^4_{0,5,\partial\Omega} + \|w\|^5_{0,5,\partial\Omega}) \le C(v)\|w\|^2_{1,2,\Omega} \end{aligned} \tag{2.8}$$

whenever $\|w\|_{1,2,\Omega} \le 1$. Thus (2.7) is valid and J is continuous.

From (2.6) and (2.1) we see that there exists a constant $C > 0$ such that

$$J''(v; w, w) \geq C\|w\|_1^2 \quad \forall v \in U \quad \forall w \in V. \tag{2.9}$$

This means that the functional J is strictly convex over the set U which completes the proof.

Remark 2.2. The functional J is not convex over the whole space $H^1(\Omega)$ due to the term $\frac{1}{5}\int_{\Gamma_2} \beta v^5 ds$. That is why we have restricted ourselves to sets of non-negative functions. This is in accordance with physics because the temperature cannot decrease below $0[K]$. Therefore, we have to deal with two nonlinearities. The first one is due to the Stefan-Boltzmann boundary condition and the second one is due to the constrain $v \geq 0$ contained in the definition of the convex set U.

Theorem 2.3. *There exists a unique solution of the variational problem: Find $u \in U$ such that*

$$J(u) = \inf_{v \in U} J(u). \tag{2.10}$$

Proof. By (2.3), (2.1) and the continuity of F, there exist positive constants C_1, C_2 such that

$$J(v) = \frac{1}{2}a(v, v) + \frac{1}{5}\int_{\Gamma_2} \beta v^5 ds - F(v) \geq C_1\|v\|_1^2 - C_2\|v\|_1 \quad \forall v \in U,$$

since the integral over Γ_2 is non-negative. Hence, the functional J is coercive on U, i.e.,

$$J(v) \to \infty \quad \text{as } \|v\|_1 \to \infty, \quad v \in U. \tag{2.11}$$

Let $\tilde{v} \in U$ be arbitrary. Then by (2.11) there exists $r > 0$ such that for any $v \in U$ for which $\|v\|_1 > r$, we have $J(\tilde{v}) < J(v)$. Thus the minimization problem (2.10) over the unbounded set U is equivalent to the minimization of J over the bounded closed set $\tilde{U} = \{v \in U \quad | \; \|v\|_1 \leq r\}$. The rest of the proof follows from Lemma 2.1, the fact that $H^1(\Omega)$ is a reflexive Banach space and the next theorem (see [6, p. 199]).

Theorem 2.4. *Let $\tilde{U}$ be a non-empty convex closed and bounded subset of a reflexive Banach space. Let J be a continuous and convex functional defined on $\tilde{U}$. Then*

(a) $\inf_{v \in \tilde{U}} J(v) > -\infty$;

(b) *there exists at least one $u \in \tilde{U}$ such that $J(u) = \inf_{v \in \tilde{U}} J(v)$;*

(c) *if, moreover, J is strictly convex on $\tilde{U}$ then there exists precisely one u with the property* (b).

Remark 2.5. The unique solution of (2.10) is called a variational solution. It is known (see [1, p. 118]) that the minimization of a convex Gâteaux-differentiable functional J over a convex subset $U \subset H^1(\Omega)$ is equivalent to the following variational inequality problem: Find $u \in U$ such that

$$J'(u; v - u) \geq 0 \quad \forall v \in U. \tag{2.12}$$

By (2.5), the inequality (2.12) takes just the form (2.2). Hence, there is also precisely one solution of the variational inequality problem (2.2).

3. Convergence of Finite Element Approximations

Finite element solution of a heat radiation problem with the vanishing right-hand side is investigated in [9, 10] provided all angles of all triangles are less than $\frac{\pi}{2}$. Here we will employ the maximum angle condition which is much more weaker.

Throughout this section we suppose that Ω is polygonal. We shall employ standard linear triangular elements [15]. Triangulations $\mathcal{T}_h$ are supposed to satisfy standard consistency condition, i.e., those points, where one type of the boundary condition changes into another, belong to the set of vertices of all $K \in \mathcal{T}_h$. Let $\mathcal{F}$ be a family of triangulations of $\overline{\Omega}$. For a given triangulation $\mathcal{T}_h \in \mathcal{F}$ set $W_h = \{v_h \in H^1(\Omega) \mid v_h|_K \in P_1(K)\ \forall K \in \mathcal{T}_h\}$, where $P_1(K)$ is the space of linear polynomials over K. Recall (see [14, p. 27]) that functions from W_h are continuous. Assume, for simplicity that

$$\exists h_0 > 0 \quad \forall h \in (0, h_0) : \ \overline{u} \in W_h. \tag{3.1}$$

Then the following set will be non-empty $U_h = U \cap W_h$.

Analogously to Theorem 2.3 we can prove that for a sufficiently small $h > 0$ there exists a unique solution of the discrete problem: Find $u_h \in U_h$ such that

$$J(u_h) = \inf_{v_h \in U_h} J(v_h), \tag{3.2}$$

where J is defined by (2.3).

We prove now the convergence $u_h \to u$ for $h \to 0$ without any smoothness assumptions upon u. Recall that a family $\mathcal{T}$ of triangulations of $\overline{\Omega}$ is said to satisfy the maximum angle condition if there exists a constant $\gamma_0 < \pi$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $K \in \mathcal{T}_h$ we have $\gamma_K \le \gamma_0$, where γ_K is the maximum angle of K.

For such a family we have by [12, p. 225] that $\|w - \pi_h w\|_1 \le Ch|w|_2\ \forall w \in H^2(\Omega)$, where π_h is the standard linear interpolation operator and $\pi_h w \in W_h$.

Theorem 3.1. *Let there exist a constant $C_u > 0$ such that $u \ge C_u$ and let $\mathcal{F}$ be a family of triangulations of $\overline{\Omega}$ satisfying the maximum angle condition. Then $\|u - u_h\|_1 \to 0$ as $h \to 0$, where u and u_h are, respectively, the unique solutions of* (2.10) *and* (3.2).

Proof. Let $\varepsilon > 0$ be given. According to [4, p. 618], there exists $w \in C^\infty(\overline{\Omega}) \cap V$ such that $\|u_0 - w\|_1 < \frac{\varepsilon}{2}$, where $u_0 = u - \overline{u} \in V$. Moreover, we see that $\pi_h w \in V \cap W_h$ and, by the maximum angle condition $\|w - \pi_h w\|_1 < \frac{\varepsilon}{2}$, when h is small enough. Setting $v_h = \pi_h w + \overline{u}$, we find that

$$\|u - v_h\|_1 = \|u_0 - \pi_h w\|_1 \le \|u_0 - w\|_1 + \|w - \pi_h w\|_1 < \varepsilon$$

for sufficiently small $h > 0$. Hence,

$$\|u - v_h\|_1 \to 0 \quad \text{as} \quad h \to 0. \tag{3.3}$$

The function w has been obtained by a regularization technique of [17, p. 58]. Therefore, for a sufficiently small ε we get that $w + \overline{u} \ge 0$, since $u = u_0 + \overline{u} \ge C_u > 0$.

As π_h is the linear interpolation operator, we get that $\pi_h(w+\overline{u}) \geq 0$. Moreover, by (3.1), $v_h = \pi_h w + \overline{u} \geq 0$ and thus $v_h \in U_h$, since $\pi_h w = 0$ on Γ_1.

Obviously,

$$J(u) \leq J(u_h) \leq J(v_h). \tag{3.4}$$

By Lemma 2.1 the functional J is continuous and, therefore, the relations (3.3) and (3.4) yield

$$\lim_{h\to 0} J(u_h) = J(u). \tag{3.5}$$

Introduce the Taylor expansion of J at the point u (see [1, p. 52]),

$$J(u_h) = J(u) + J'(u; u_h - u) + \frac{1}{2} J''(u + \theta_h(u_h - u); u_h - u, u_h - u),$$

where $\theta_h \in (0,1)$. This and (2.9) imply that

$$J(u_h) - J(u) \geq J'(u; u_h - u) + C\|u - u_h\|_1^2 \geq C\|u - u_h\|_1^2, \tag{3.6}$$

where $C > 0$ is independent of h. From here, (3.5) and (2.12) we observe that $\|u - u_h\|_1 \to 0$ as $h \to 0$.

If the variational solution u is sufficiently smooth we can derive even a linear rate of convergence in the $\|.\|_1$-norm.

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *be fulfilled and let* $u \in H^2(\Omega)$. *Then* $\|u - u_h\|_1 \leq Ch \quad$ *as* $h \to 0$.

Proof. Let $w \in V \cap C^\infty(\overline{\Omega})$ be an arbitrary function such that $\|w\|_{C(\overline{\Omega})} < C_u$. Then clearly $u + w, u - w \in U$. Setting $v = u \pm w$ in (2.12), we find that $J'(u; w) \geq 0$ and $J'(u; -w) \geq 0$. Now from the linearity of $J'(u; \cdot)$ and the density of $V \cap C^\infty(\overline{\Omega})$ in V (see [4, p. 618]), we obtain the well-known necessary Euler extremum condition:

$$J'(u; v) = 0 \quad \forall v \in V. \tag{3.7}$$

Since $u \in H^2(\Omega)$, the function u is continuous by the Sobolev imbedding theorem and thus $\pi_h u \in W_h$ is well-defined. We see that $\pi_h u \in U_h$ and clearly,

$$J(u_h) \leq J(\pi_h u), \tag{3.8}$$

due to (3.2). Using now (3.7), (3.6), (3.8), again (3.7), (2.8) and the maximum angle condition in that order, we arrive at

$$\begin{aligned} C\|u - u_h\|_1^2 &= C\|u - u_h\|_1^2 + J'(u; u_h - u) \leq J(u_h) - J(u) \leq J(\pi_h u) - J(u) \\ &= J(\pi_h u) - J(u) - J'(u; \pi_h u - u) \leq C(u)\|u - \pi_h u\|_1^2 \leq C_u h^2, \end{aligned}$$

where $C, C_u > 0$ are independent of h and C_u depends on u.

4. Three-dimensional Radiation Heat Transfer Problem

In [7], the Monte Carlo method is used to solve three-dimensional heat radiation problem. Finite element approximations of an axially symmetric heat radiation problem are studied in [3]. In this chapter, we need not assume that Ω is axially symmetric. Then, however, we encounter some troubles with the definition domain of the functional J. Namely, the traces of functions from $H^1(\Omega)$ need not be in $L^5(\partial\Omega)$, in general (see the following example).

Example 4.1. Let us use standard spherical coordinates (r, φ, θ) to describe the set $\Omega = \{(x_1, x_2, x_3) \in R^3 | x_1^2 + x_2^2 + x_3^2 < 1,\, x_i > 0,\, i = 1, 2, 3\}$. Consider the function r^α, where α is a real parameter. Then for $\alpha > -\dfrac{1}{2}$ we have

$$\|r^\alpha\|_1^2 = \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (r^{2\alpha} + \alpha^2 r^{2\alpha-2}) r^2 \sin\theta \; d\theta \, d\varphi \, dr = \frac{\pi}{2}\Big(\frac{1}{2\alpha+3} + \frac{\alpha^2}{2\alpha+1}\Big) \in (0, \infty).$$

The triple integral is not finite whenever $\alpha \in \Big(-\infty, -\dfrac{1}{2}\Big]$. Now we show that the trace of r^α is in $L^5(\partial\Omega)$ only if $\alpha > -\dfrac{2}{5}$.

It suffices to investigate traces on one of the three sectors of a circle which are contained in $\partial\Omega$. Denote by S that sector for which $\theta = \dfrac{\pi}{2}$, i.e.,

$$S = \{(x_1, x_2, 0) \in R^3 | x_1^2 + x_2^2 < 1,\, x_1 > 0,\, x_2 > 0\}.$$

Then

$$\|r^\alpha\|_{0,5,S}^5 = \int_0^1 \int_0^{\frac{\pi}{2}} r^{5\alpha} r \; d\varphi dr = \frac{\pi}{2(5\alpha+2)} \in (0, \infty)$$

if and only if $\alpha > -\dfrac{2}{5}$. Hence, if $\alpha \in \Big(-\dfrac{1}{2}, -\dfrac{2}{5}\Big]$ then $r^\alpha \in H^1(\Omega)$, but its trace is not in $L^5(\partial\Omega)$.

For $d = 3$ we can guarantee that the traces of functions from $H^1(\Omega)$ are only in $L^4(\partial\Omega)$, i.e., there exists a constant $C > 0$ such that (see [17, p. 84])

$$\|v\|_{0,4,\partial\Omega} \le C\|v\|_{1,2,\Omega} \quad \forall v \in H^1(\Omega). \tag{4.1}$$

If we would restrict the definition domain of the functional (2.3) to the space $H^2(\Omega)$ (or $W_3^1(\Omega)$) then J would be not coercive on nonnegative functions from these spaces with respect to the norm $\|.\|_2$ (or $\|.\|_{1,3}$). Therefore, we shall consider the functional J over the linear space $Z = \{v \in H^1(\Omega) \mid \; v|_{\Gamma_2} \in L^5(\Gamma_2)\}$, equipped with the norm $\|v\|_Z = (\|v\|_{1,2,\Omega}^2 + \|v\|_{0,5,\Gamma_2}^2)^{\frac{1}{2}}$.

Lemma 4.2. *The space Z is a Banach space.*

Proof. Let $\{v_k\} \subset Z$ be a sequence such that $\lim\limits_{k,m\to\infty} \|v_k - v_m\|_Z = 0$. Since $H^1(\Omega)$ and $L^5(\Gamma_2)$ are Banach spaces, there exist $v \in H^1(\Omega)$ and $w \in L^5(\Gamma_2)$ such that $v_k \to v$ in $H^1(\Omega)$ and $v_k|_{\Gamma_2} \to w$ in $L^5(\Gamma_2)$ for $k \to \infty$. By (4.1) we find that $v_k|_{\Gamma_2} \to v|_{\Gamma_2}$ in $L^4(\Gamma_2)$ and thus $v|_{\Gamma_2} = w$. Hence, the Cauchy sequence $\{v_k\}$ converges in Z and the space Z is therefore complete.

Lemma 4.3. *The space Z is reflexive.*

Proof. We shall proceed similarly to [3, p. 1079]. By [6, p. 187] the Cartesian product of two reflexive spaces is also reflexive. Hence, the space $T \equiv H^1(\Omega) \times$

$L^5(\Gamma_2) = \{(v,w) \,|\, v \in H^1(\Omega),\ w \in L^5(\Gamma_2)\}$, equipped with the norm $\|(v,w)\|_T = (\|v\|^2_{1,2,\Omega} + \|w\|^2_{0,5,\Gamma_2})^{\frac{1}{2}}$, is a reflexive normed space. The mapping

$$v \in Z \mapsto (v, v|_{\Gamma_2}) \in T \tag{4.2}$$

is obviously an isomorphism between Z and some closed subspace T_1 of T. According to [6, p. 187], T_1 is reflexive since T is reflexive. Therefore, due to the isomorphism (4.2), the space Z is also reflexive.

Lemma 4.4. *The functional J given by* (2.3) *is continuous on Z.*

Proof. For $v, w \in Z$ we have

$$\begin{aligned}|J(v+w) - J(v)| \le & \Big|\frac{1}{2}a(v+w, v+w) - \frac{1}{2}a(v,v)\Big| + \Big|\frac{1}{5}\int_{\Gamma_2} \beta((v+w)^5 - v^5)ds\Big| \\ & + |F(v+w) - F(v)| \le |a(v,w)| + \frac{1}{2}|a(w,w)| + |F(w)| \\ & + \int_{\Gamma_2} \Big|\beta\Big(v^4 w + 2v^3w^2 + 2v^2w^3 + vw^4 + \frac{1}{5}w^5\Big)\Big|ds \\ \le & C_1(\|v\|_1\|w\|_1 + \|w\|_1^2 + \|w\|_1) + C_2(\|v\|^4_{0,5,\Gamma_2}\|w\|_{0,5,\Gamma_2} \\ & + \|v\|^3_{0,5,\Gamma_2}\|w\|^2_{0,5,\Gamma_2} + \|v\|^2_{0,5,\Gamma_2}\|w\|^3_{0,5,\Gamma_2} \\ & + \|v\|_{0,5,\Gamma_2}\|w\|^4_{0,5,\Gamma_2} + \|w\|^5_{0,5,\Gamma_2}),\end{aligned}$$

where C_1, C_2 are positive constants, since $\|w\|_1 \le \|w\|_Z$ and $\|w\|_{0,5,\Gamma_2} \le \|w\|_Z$, we find that

$$\lim_{\substack{w \in Z \\ \|w\|_Z \to 0}} |J(v+w) - J(v)| = 0.$$

Now for $\overline{u} \in Z$ we define

$$U = \{v \in Z \,|\, v \ge 0 \text{ in } \Omega,\ v = \overline{u} \text{ on } \Gamma_1\}. \tag{4.3}$$

Lemma 4.5. *The functional J is strictly convex over the set U given by* (4.3)*.*

Proof. For any $v, w, z \in U$ we have by (2.6) and (2.1) that

$$J''(v; z - wz - w) = a(z-w,\, z-w) + 4\int_{\Gamma_2} \beta v^3 (z-w)^2 ds \ge C\|z-w\|_1^2, \tag{4.4}$$

where C is a positive constant. Since $\|z-w\|_Z = 0$ if and only if $\|z-w\|_1 = 0$, we see that J is strictly convex on U.

Lemma 4.6. *Let there exist a positive constant C_β such that*

$$\beta \ge C_\beta \text{ on } \Gamma_2. \tag{4.5}$$

Then the functional J is coercive on U.

Proof. For any $v \in U$ we find that

$$J(v) = \frac{1}{2}a(v,v) + \frac{1}{5}\int_{\Gamma_2} \beta v^5 ds - F(v) \ge C_1\|v\|_1^2 + \frac{1}{5}C_\beta\|v\|^5_{0,5,\Gamma_2} - C_2\|v\|_1.$$

Obviously, if $\|v\|_Z \to \infty$ then $\|v\|_1 \to \infty$ or $\|v\|_{0,5,\Gamma_2} \to \infty$, and thus $\lim\limits_{\substack{v \in Z \\ \|v\|_Z \to \infty}} J(v) = \infty$.

Theorem 4.7. *Let $\overline{u} \in Z$ and let* (4.5) *hold. Then there exists a unique function u which minimizes the functional J over the set U. The function u is a variational solution of the problem* (1.1).

Proof. Due to the coercivity of J (see Lemma 4.6), the minimization of J over the unbounded set U can be transformed to the minimization of J over the bounded subset $\tilde{U} = \{v \in U \quad | \quad \|v\|_Z \le r\}$ like in Theorem 2.3. The existence of a unique minimizer u now follows from Theorem 2.4 and Lemmas 4.2–4.5.

By (2.12), we again derive the associated variational inequality of the form (2.4).

5. Remarks on Approximation on Polyhedra

Let Ω be a bounded polyhedral domain with a Lipschitz-continuous boundary (an example of a polyhedron whose boundary is not Lipschitz-continuous is given in [11, p. 48]). Let the one-dimensional boundary of the two-dimensional set $\Gamma_1 \subset \partial\Omega$ consist of a finite number of straight line segments. Denote by $\mathcal{T}_h$ a decomposition of $\overline{\Omega}$ into closed tetrahedra K in the usual sense, i.e., the union of all $K \in \mathcal{T}_h$ is $\overline{\Omega}$, the interiors of all tetrahedra are mutually disjoint, and any face of any tetrahedron $K \in \mathcal{T}_h$ is either a subset of $\overline{\Gamma}_1$ or $\overline{\Gamma}_2$ or a face of another $K' \in \mathcal{T}_h$. A constructive proof of the existence of such a decomposition is given in [11, p. 61]. The space W_h is defined in the same way as for $d = 2$. Let us assume that $\overline{u} \in W_h$ for all h sufficiently small (cf. (3.1)). Then from Theorem 2.4 we again see that for any $\mathcal{T}_h$ (h small) there exists one and only one solution of the minimization problem: Find $u_h \in U_h$ such that $J(u_h) = \min\limits_{v_h \in U_h} J(v_h)$, where $U_h = U \cap W_h$.

A family of decompositions $\mathcal{F} = \{\mathcal{T}_h\}$ into tetrahedra is said to satisfy the maximum angle condition if there exists a constant $\gamma_0 < \pi$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $K \in \mathcal{T}_h$ we have $\gamma_K \le \gamma_0$ and $\varphi_K \le \gamma_0$, where γ_K is the maximum angle of all triangular faces of the tetrahedron K, and φ_K is the maximum angle between faces of K.

Recall that (see [13, p. 516]) if $\mathcal{F}$ satisfies the maximum angle condition then

$$\|v - \pi_h v\|_{1,\infty} \le Ch|v|_{2,\infty} \tag{5.1}$$

for sufficiently smooth v. Here π_h is the standard linear interpolation operator. From property (5.1) we can prove the convergence of u_h to u in the $\|.\|_Z$-norm as in Chapter 3 using the inequality $\|v\|_Z \le C\|v\|_{1,\infty}$ for any v from the Sobolev space $W^1_\infty(\Omega)$.

All results of this paper can be modified to the case $\Gamma_1 = \emptyset$ if $\alpha > 0$ or $\beta > 0$ on some set of a positive measure. Various approaches to the solution of the radiation heat transfer problem can be further find, e.g., in [8, 18, 21, 22]. A condition, which is similar to the Stefan-Boltzmann boundary condition arises in mathematical modelling of electrolysis processes, see [5].

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