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DISCRETIZATION OF JUMP STOCHASTIC DIFFERENTIAL EQUATIONS IN TERMS OF MULTIPLE STOCHASTIC INTEGRALS*

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Abstract

In the Stratonovich-Taylor and Stratonovich-Taylor-Hall discretization schemes for stochastic differential equations (SDEs), there appear two types of multiple stochastic integrals respectively. The present work is to approximate these multiple stochastic integrals by converting them into systems of simple SDEs and solving the systems by lower order numerical schemes. The reliability of this approach is clarified in theory and demonstrated in numerical examples. In consequence, the results are applied to the strong discretization of both continuous and jump SDEs.

Key words: Brownian motion, Poisson process, stochastic differential equation, multiple stochastic integral, strong discretization.

1. Introduction

For the strong discretization of SDEs, any numerical method which only depends on the values of Brownian paths or Poisson paths at the partition nodes cannot achieve an order higher than 0.5 in general[2,4,8]. Therefore the evaluation of multiple stochastic integrals on the intervals between nodes is a major obstacle that must be overcome. Some attempts have been made previously in different approaches to approximate multiple stochastic integrals. [2] suggests an approximation in terms of Fourier Gaussian coefficients of the Brownian bridge process. As the layer of integration increases, the treatment becomes complicated and the computation is laborious to generate a lot of independent Gaussian random variables. For 2-dimensional Brownian motions, Gaines and Lyons applied in [1] the Marsaglia rectangle-wedge-tail method to generate stochastic area Ito integrals. However this method is not easy to be extended to general cases.

[6] indicates to model multiple Ito integrals by the rectangular rule, the trapezoidal rule as well as the Fourier method with the discussion of how small the time step should be taken to ensure the necessary accuracy. Our approach is in some sense the systematization and development of Milstein's work. In section 2, we propose to treat multiple stochastic integrals as systems of SDEs which can be solved by STH

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scheme of lower order. The results will be applied to the strong discretization of the Ginzburg-Landau system driven by both Brownian motion and Poisson process.

2. The Approximation of Multiple Stochastic Integrals

Let M be the set of the empty index and all multiple indices $\alpha = (\alpha_1, \cdots, \alpha_l)$ such that $\alpha_i \in \{-1, 0, \cdots, m\}$ for $i = 1, \cdots, l$. For $\alpha = (\alpha_1, \cdots, \alpha_l)$, define $|\alpha| = l$, $(\alpha) = \#\{\alpha_i : \ \alpha_i = 0\}$, $\|\alpha\| = |\alpha| + (\alpha)$. For a square integrable $\mathcal{F}_t$-predictable process f_t, define the multiple Stratonovich integrals $J_{\alpha,\rho,\tau}$ recursively by

$$J_\alpha[f]_{\rho,\tau} = \begin{cases} \int_\rho^{\tau+} J_{\alpha-}[f]_{\rho,s} dN_s^{-\alpha_l} & \text{if } \alpha_l = -1 \\ \int_\rho^{\tau} J_{\alpha-}[f]_{\rho,s} ds & \text{if } \alpha_l = 0 \\ \int_\rho^{\tau} J_{\alpha-}[f]_{\rho,s} dW_s^{\alpha_l} & \text{if } \alpha_l > 0 \end{cases} \tag{2.1}$$

and agree that $J_\alpha[f]_{\rho,\tau} = f_\tau$ when α is the empty index ϕ.

Let $B_r(L_{-2}, \ L_0, \cdots, L_m)$ be the set of all formal brackets of the indeterminates $L_{-2}, L_0, \cdots, L_m$. The meaning of L_j will be clarified in section 3. For $B \in B_r(L_{-2}, L_0, \cdots, L_m)$, the degree $|B|$ is defined recursively by $|B| = |B_1| + B_2|$ and $|L_i| = 1$, $i = -2, 0, \cdots, m$. Let $\mathcal{B} \subset B_r(L_{-2}, L_0, \cdots, L_m)$ be a Philip Hall basis of $\mathcal{L}(L_{-2}, L_0, \cdots, L_m)$ with a total order $\preceq$ such that L_{-2} is the first element with respect to $\preceq$. For any $B = (ad(B_1))^j(B_2) \in \mathcal{B}$ with $B_1 \neq B_2$, we define, as in [3], the stochastic integral

$$C_{B,\rho,\tau} = \int_\rho^\tau c_{B,\rho,t} \tag{2.2}$$

where $c_{B,\rho,t}$ is defined recursively by

$$c_{B,\rho,t} = \frac{1}{j!} C^j_{B_1,\rho,t} c_{B_2,\rho,t} \tag{2.3}$$

with

$$c_{L_j,\rho,t} = \begin{cases} dt, & j = 0 \\ \circ dW_t, & j \in \{1, \cdots, m\} \end{cases} \tag{2.4}$$

and

$$c_{L_{-2},\rho,t} = dN_t. \tag{2.5}$$

Define

$$V_t^j = \begin{cases} \frac{1}{(\tau-\rho)^{1/2}} N_{(1-t)\rho+t\tau}, & j = -1, \\ \frac{(1-t)\rho + t\tau}{\tau - \rho}, & j = 0, \\ \frac{1}{(\tau-\rho)^{1/2}} W^j_{(1-t)\rho+t\tau}, & j = 0, \cdots, m, \end{cases} \tag{2.6}$$

Then, with respect to $\mathcal{F}_t^V = \mathcal{F}_{\rho+t(\tau-\rho)}$, V_t^j, $j = 0, \cdots m$ are still independent Brownian motions and V_t^{-1} is a jump process such that $V_t^{-1} - \lambda(\tau-\rho)^{1/2}t$ is a martingale. Let

$$J_\alpha^V[g]_{0,1} = \int_0^1 \int_0^{s_l} \cdots \int_0^{s_2} g_{s_1} \circ dV_{s_1}^{\alpha_1} \cdots \circ dV_{s_{l-1}}^{\alpha_{l-1}} \circ dV_{s_l}^{\alpha_l} \tag{2.7}$$

It is straightforward that

$$J_\alpha[g]_{\rho,\tau} = (\tau-\rho)^{\|\alpha\|/2} J_\alpha^V[\tilde{g}]_{0,1} = (\tau-\rho)^{\|\alpha\|/2} X_1^l \tag{2.8}$$

where

$$\tilde{g}_s = g_{(1-s)\rho+s\tau}, \quad s \in [0,1]$$

and $(X_t^1, \cdots, X_t^l)$ is the solution of the following linear stochastic system

$$\begin{cases} dX_t^1 = \tilde{g} \circ dV_t^{\alpha_1}, \\ dX_t^2 = X_t^1 \circ dV_t^{\alpha_2}, \\ \quad \cdots \\ dX_t^l = X^{l-1} \circ dV_t^{\alpha_l}, \end{cases} \quad t \in [0,1] \tag{2.9}$$

with initial conditions

$$X_0^i = 0, \quad i = 1, \cdots, l.$$

Given a partition $0 = t_0 < t_1 < \cdots < t_K = 1$, The STH scheme of order 0.5 for (2.9) is

$$\begin{aligned} Y_{k+1}^1 &= Y_k^1 + \tilde{g}_{t_k} \triangle V_k^{\alpha_1}, \\ Y_{k+1}^2 &= Y_k^2 + Y_k^1 \triangle V_k^{\alpha_2} + \frac{1}{2}(1 - \delta_{0\alpha_2} - \delta_{-1\alpha_2})\delta_{\alpha_1\alpha_2}\tilde{g}_{t_k}\triangle t_k, \\ Y_{k+1}^i &= Y_k^i + Y_k^{i-1} \triangle V_k^{\alpha_i} + \frac{1}{2}(1 - \delta_{0\alpha_i} - \delta_{-1\alpha_i})\delta_{\alpha_{i-1}\alpha_i} Y_k^{i-2}\triangle t_k, \ i = 3, \cdots, l. \end{aligned} \tag{2.10}$$

For higher accuracy, we may use the following STH scheme of order 1.0.

$$\begin{aligned} Y_{k+1}^1 =& Y_k^1 + \tilde{g}_{t_k} \triangle V_k^{\alpha_1}, \\ Y_{k+1}^2 =& Y_k^2 + Y_k^1 \triangle V_k^{\alpha_2} + (1-\delta_{\alpha_1 0})(1-\delta_{\alpha_2 0})\tilde{g}_{t_k} D_{(\alpha_1,\alpha_2),t_k,t_{k+1}} - \frac{1}{2}\delta_{-1\alpha_1}\delta_{-1\alpha_2}\tilde{g}_{t_k}\triangle N_n, \\ Y_{k+1}^i =& Y_k^i + Y_k^{i-1} \triangle V_k^{\alpha_i} + (1-\delta_{\alpha_{i-1}0})(1-\delta_{\alpha_i 0}) Y^{i-2} D_{(\alpha_{i-1},\alpha_i),t_k,t_{k+1}} \\ & - \frac{1}{2}\delta_{-1\alpha_{i-1}}\delta_{-1\alpha_i} Y_n^{i-2} \triangle N_n, \quad i = 3, \cdots, l \end{aligned} \tag{2.11}$$

where

$$D_{(j_1,j_2),t_k,t_{k+1}} = \begin{cases} \displaystyle\int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \circ dV_s^{j_1} \circ dV_t^{j_2}, & j_1 < j_2 \\ 2(\triangle V_k^{j_1})^2 & j_1 = j_2 \\ \displaystyle\triangle V_k^{j_1} \triangle V_k^{j_2} - \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \circ dV_s^{j_1} \circ dV_t^{j_2}, & j_1 > j_2, \end{cases}$$

with $D_{(j_1,j_2)}$, $j_1 < j_2$, approximated by the STH scheme of order 0.5. In particular, if $g_t \equiv 1$, we get the approximation of $J_{\alpha,\rho,\tau}$. Denote by C_B^V the multiple stochastic integral with respect to V as C_B with respect to W, so that

$$C_{B,\rho,\tau} = (\tau-\rho)^{\|B\|/2} C^V_{B,0,1}. \tag{2.12}$$

Preposition 2.1. F*or any $B \in \mathcal{B}$, there exists a system of stochastic differential equations*

$$\begin{aligned} &dX_t^i = \mathcal{M}_i(X_t^1,\cdots,X_t^{i-1}) \circ dV_t^{\gamma_i},\ \gamma_i \in \{0,\cdots,m\} \\ &X_0^i = 0, \quad i = 1,\cdots,d(B), \end{aligned} \tag{2.13}$$

for some $d(B) \le |B|$, so that $C^V_{B,0,1} = X_1^{d(B)}$. Here $\mathcal{M}_1 \equiv 1$ and $\mathcal{M}_i(x^1,\cdots,x^{i-1})$ is a monomial of $x^1,\cdots,x^{i-1}$ when $i \ge 2$.

Proof. For $|B| = 1$, or $B = L_j$, the result holds with $\gamma_1 = j$. Assume the lemma holds for $1 \le |B| \le l$. Then for $|B| = l+1$, B can be written as $(ad(B_1))^j(B_2)$, $1 \le |B_1| \le |B_2| \le l$. By induction hypothesis, we have the systems

$$dX_t^i = \mathcal{M}_i(X_t^1,\cdots,X_t^{i-1}) \circ dV_t^{\gamma_i}, \quad i = 1,\cdots,d(B_1)$$

and

$$dX_t^{d(B_1)+i} = \mathcal{M}_{d(B_1)+i}(X_t^{d(B_1)+1},\cdots,X_t^{d(B_1)+i-1}) \circ dV_t^{\gamma_{d(B_1)+i}}, \quad i = 1,\cdots,d(B_2),$$

with $X_0^i = 0$, $i = 1,\cdots,d(B) = d(B_1) + d(B_2)$ such that $C^V_{B_1,0,1} = X_1^{d(B_1)}$ and $C^V_{B_2,0,1} = X_1^{d(B)}$. Thus we can establish the system

$$\begin{cases} dX_t^i = \mathcal{M}_i(X_t^1,\cdots,X_t^{i-1}) \circ dV_t^{\gamma_i}, \quad i = 1,\cdots,d(B)-1, \\ dX_t^{d(B)} = (X^{d(B_1)})^j \mathcal{M}_{d(B)} \circ dV_t^{\gamma_{d(B)}}/j! \end{cases}$$

with initial conditions $X_0^i = 0$, $i = 1,\cdots,d(B)$, so that $C^V_{B,0,1} = X_1^{d(B)}$ and the result follows by induction. □

By the convergence of ST schemes,

$$E\{|\hat{C}_{B,t_n,t_{n+1}} - C_{B,t_n,t_{n+1}}|^{2q}|\mathcal{F}_\rho\}^{1/q} \le C(r',q)h^{2r'}\triangle^{\|B\|} \tag{2.14}$$

In order to achieve a overall convergence of order r when $\hat{C}_{B,t_n,t_{n+1}}$ are used to solve SDEs (see Section 3), it is necessary that $\triangle^{\|B\|}h^{2r'} = O(\triangle^{2r+1})$ so that

$$h = O(\triangle^{(2r+1-\|B\|)/2r'}). \tag{2.15}$$

Similar estimation of step size can be made for the approximation of J_α.

The following table lists the absolute errors between some exact values with respect to two fix Brownian paths and a Poisson path on $[0, 1/4]$ as well as the differences between the approximations of J_α and those of C_B. The effect tends to improve as h reduced.

Table 2.1.

$-\log_2(h)$	E_1	E_2	E_3
6	0.048693	0.120910	0.017604
7	0.040896	0.113549	0.018261
8	0.037810	0.078257	0.008925
9	0.011098	0.077780	0.008871
10	0.009927	0.074667	0.008516
11	0.001983	0.053942	0.006152
12	0.009144	0.008626	0.000984

where

$$E_1 = |\triangle W^1_{1/4}\triangle W^2_{1/4} - J_{(1,2),0,1/4} - J_{(2,1),0,1/4}|$$
$$E_2 = |\triangle W^1_{1/4}\triangle N_{1/4} - J_{(1,-1),0,1/4} - J_{(-1,1),0,1/4}|$$
$$E_3 = |C_{[L_1,[L_{-1},L_2]],0,1/4} - J_{(1,-1,2),0,1/4} - J_{(-1,1,2),0,1/4}|$$

3. Discretization of SDEs

The following is the SDE for stochastic diffusion.

$$X_t = X_0 + \int_{t_0}^t b^0(s,X_s)ds + \int_{t_0}^t b(s,X_s)\circ dW_s \tag{3.1}$$

with initial value X_0 where $W_t = (W^1_t, \cdots, W^m_t)$ is a standard m-dimensional $\mathcal{F}_t$ adapted Brownian motion, b^0 is a non-random d-dimensional vector function and b is a non-random $d\times m$ matrix function with sufficient smoothness. More generally, we will consider the stochastic jump-diffusion equation

$$X_t = X_{t_0} + \int_{t_0}^t b^0(s,X_s)ds + \int_{t_0}^t b(s,X_s)dW_s + \int_{t_0}^{t+} c(s,X_s)dN_s \tag{3.2}$$

where c is a d-dimensional vector-valued function and N_t is a Poisson process of intensity λ.

Define some operator corresponding to (3.2) as follows.

$$L_{-1}f(t,x) = f(t,x+c(t,x)) - f(t,x), \quad L_{-2} = \log(1+L_{-1})$$
$$L_0 = \frac{\partial}{\partial t} + \sum_{k=1}^d b^{k,0}\frac{\partial}{\partial x^k},$$
$$L_j = \sum_{k=1}^d b^{k,j}\frac{\partial}{\partial x^k}, \quad j = 1,\cdots,m \tag{3.3}$$

Agree that $L_\phi = \mathrm{id}$ and $L_\alpha = L_{\alpha_1}L_{\alpha_2}\cdots L_{\alpha_l}$. For a finite hierarchical set $A\in M_m$, the Stratonovich-Taylor (ST) scheme of integral order r is

$$Y^k_{n+1} = \sum_{\alpha\in A_r} J_{\alpha,t_n,t_{n+1}}L_\alpha\Phi^k(t_n,Y_n), \tag{3.4}$$

see [2] and [4] for details.

Let $\pi_A : \hat{\mathcal{A}}(L_{-1}, \cdots, L_m) \to \mathcal{A}(L_{-1}, \cdots, L_m)$ be the truncation mapping such that

$$\pi_A\Big(\sum_{\alpha \in M_m} a_\alpha L_\alpha \Big) = \sum_{\alpha \in A} a_\alpha L_\alpha \tag{3.5}$$

and define

$$P_{\rho,\tau} = \overleftarrow{\prod_{B \in \mathcal{B}}} \exp(C_{B,\rho,\tau} B). \tag{3.6}$$

See [7] for the diffusion case and [5] for jump case. Then (3.4) is equivalent to the Stratonovich-Taylor-Hall (STH) scheme

$$Y^k_{n+1} = \pi_A(P_{t_n,t_{n+1}})\Phi^k(t_n, Y_n). \tag{3.7}$$

The multiple stochastic integrals J_α appear in (3.4) while C_B in (3.7). In implementation, their approximation can be done off-line since the multiple stochastic integrals is independent of the equations. We save the values of J_α and C_B with respect to a large number of independent Brownian paths and Poisson paths in an accessible library The future use of them is as automatic as the use of random numbers. The values of the multiple stochastic integrals can be imported from this library when the ST or STH schemes are employed.

4. Numerical Examples

First we consider the continuous stochastic Ginzburg-Landau equation

$$dX_t = (-X_t^3 + X_t/2)dt + X_t \circ dW_t \tag{4.1}$$

with the exact solution

$$X_t = \frac{X_0 \exp(t/2 + W_t)}{\sqrt{1 + 2X_0^2 \int_0^t \exp(s + 2W_s)ds}}. \tag{4.2}$$

Let

$$b^0(x) = -x^3 + x/2, \qquad b^1(x) = x, \qquad a(x) = -x^3 + x.$$

$$L_0 = \frac{\partial}{\partial t} + (-x^3 + x/2)\frac{\partial}{\partial x}, \quad L_1 = x\frac{\partial}{\partial x}, \quad L_0^I = \frac{\partial}{\partial t} + (x^2/2)\frac{\partial^2}{\partial x^2} + (-x^3 + x)\frac{\partial}{\partial x}.$$

The STH scheme of order 0.5, 1.0, 1.5 and 2.0 for (4.1) are respectively

$$Y_{n+1} = Y_n + b^1(Y_n)\triangle W_n^1 + a(Y_n)\triangle t_n, \tag{4.3}$$

$$Y_{n+1} = Y_n + \sum_{j=0}^{1} b^j(Y_n)\triangle W_n^j + L_1 b^1(Y_n)(\triangle W_n^1)^2/2, \tag{4.4}$$

$$Y_{n+1} = Y_n + \sum_{j=0}^{1} b^j(Y_n)\triangle W_n^j + L_1 b^1(Y_n)(\triangle W_n^1)^2/2 + L_1 b^0(Y_n)\triangle t_n \triangle W_n^1$$

$$+ L_1^2 b^1(Y_n)(\triangle W_n^1)^3/6 + [L_0, L_1]\Phi(Y_n)C_{[L_0,L_1],t_n,t_{n+1}} + L_0^I a(Y_n)\triangle {t_n}^2/2 \tag{4.5}$$

and

$$\begin{aligned} Y_{n+1} =& Y_n + \sum_{j=0}^{1} b^j(Y_n)\triangle W_n^j + \frac{1}{2}\sum_{j=0}^{1} L_j b^j(Y_n)(\triangle W_n^j)^2 + L_1 b^0(Y_n)\triangle t_n \triangle W_n^1 \\ &+ L_1^2 b^1(Y_n)(\triangle W_n^1)^3/6 + [L_0, L_1]\Phi(Y_n)C_{[L_0,L_1],t_n,t_{n+1}} \\ &+ [L_0, L_1]b^1(Y_n)\triangle W_n^1 C_{[L_0,L_1],t_n,t_{n+1}} + L_1^2 b^0(Y_n)\triangle t_n(\triangle W_n^1)^2/2 \\ &+ L_1^3 b^1(Y_n)(\triangle W_n^1)^4/24 + [L_1,[L_0, L_1]]\Phi(Y_n)C_{[L_1,[L_0,L_1]],t_n,t_{n+1}} \end{aligned} \tag{4.6}$$

where Φ is the identity on $I\!R$, $\triangle W_n^0 = \triangle t_n = t_{n+1} - t_n$ and $\triangle W_n^1 = \triangle W_n = W_{t_{n+1}} - W_{t_n}$.

In this case, the STH1.0 (Milstein) scheme, believed to perform better than the STH0.5 (Euler) scheme, is still simple and easy to implement since it does not involve any multiple stochastic integrals. In Fig. 4.1, we plot the approximations using various STH schemes of order 1.0, 1.5, 2.0 with a step size $\triangle = 2^{-2}$ and the exact solution X of (4.2) along a Brownian path.

Fig.4.1

The approximation using the STH1.0 scheme does not converge to the true solution after time $t = 3$ while that using the STH1.5 scheme diverges after $t = 6$ and the result using the high order STH2.0 scheme perform well for this coarse step size.

In Table 4.2, we list the absolute errors of the approximate solution Y_N, $N = 8.0/\triangle$ at $T = 8.0$ along a Brownian path using various ST and STH schemes with step size $\triangle$ ranging from 2^{-1} to 2^{-9}. The step size should be refined for lower order schemes to achieve the same accuracy as higher order schemes. For instance, to obtain an accuracy of one decimal place, the step size for the schemes STH0.5 and STH1.0 are at most 2^{-6} and 2^{-5} respectively while that for the schemes STH1.5 (ST1.5) and STH2.0 (ST2.0) are at most 2^{-3} (2^{-3}) and 2^{-1} (2^{-2}).

(4.1) is only driven by the Brownian motion. Now we are going to see how the ST and STH schemes will perform when a jump term is introduced to the equation.

Table 4.2.

	$\|X_{8.0} - Y_N\|$					
$-\log_2(\triangle)$	STH0.5	STH1.0	ST1.5	STH1.5	ST2.0	STH2.0
1	0.534475	0.461381	0.149671	0.133301	0.074258	0.048041
2	0.396289	0.473161	0.095910	0.113372	0.034015	0.033146
3	0.047480	0.177173	0.010179	0.006955	0.003423	0.007804
4	0.046505	0.101961	0.001989	0.001700	0.000136	0.001625
5	0.111503	0.060454	0.000555	0.002836	0.000076	0.000692
6	0.059801	0.026938	0.000538	0.000563	0.000421	0.000052
7	0.085788	0.013831	0.000163	0.000371	0.000148	0.000048
8	0.054299	0.005240	0.000190	0.000055	0.000060	0.000047
9	0.028922	0.002380	0.000127	0.000034	0.000009	0.000051

Consider the equation

$$dX_t = (-X_t^3 + X_t/2)dt + X_t \circ dW_t + X_t dN_t. \tag{4.7}$$

It has the exact solution

$$X_t = \frac{X_0 \exp\{t/2 + W_t + (\ln 2)N_t\}}{\sqrt{1 + 2X_0^2 \int_0^t \exp(2 + 2W_s + 2(\ln 2)N_s)ds}}. \tag{4.8}$$

The following schemes of order 0.5, 1.0 and 1.5 respectively are used to approximate the exact solution.

$$\text{ST0.5: } Y_{n+1} = Y_n + (-Y_n^3 + Y_n)\triangle t_n + Y_n \triangle W_n + Y_n \triangle N_n, \tag{4.9}$$

$$\begin{aligned}\text{ST1.0: } Y_{n+1} =& Y_n + (-Y_n^3 + Y_n/2)\triangle t_n + Y_n \triangle W_n + Y_n \triangle N_n \\ &+ Y_n(J_{(-1,-1,),t_n,t_{n+1}} + J_{(-1,1),t_n,t_{n+1}} + J_{(1,-1),t_n,t_{n+1}} + J_{(1,1),t_n,t_{n+1}}),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\text{STH1.0: } Y_{n+1} =& Y_n + (-Y_n^3 + Y_n/2)\triangle t_n + Y_n \triangle W_n + Y_n \triangle N_n \\ &+ Y_n((\triangle N_n)^2/2 - \triangle N_n/2 + \triangle W_n \triangle N_n + (\triangle W_n)^2/2),\end{aligned} \tag{4.11}$$

$$\begin{aligned}\text{ST1.5: } Y_{n+1} =& Y_n + (-Y_n^3 + Y_n/2)\triangle t_n + Y_n \triangle W_n + Y_n \triangle N_n \\ &+ Y_n(J_{(-1,-1,),t_n,t_{n+1}} + J_{(-1,1),t_n,t_{n+1}} + J_{(1,-1),t_n,t_{n+1}} + J_{(1,1),t_n,t_{n+1}}) \\ &(-Y_n^3 + Y_n/2)J_{(0,-1),t_n,t_{n+1}} + (-7Y_n^3 + Y_n/2)J_{(-1,0),t_n,t_{n+1}} \\ &(-Y_n^3 + Y_n/2)J_{(0,1),t_n,t_{n+1}} + (-3Y_n^3 + Y_n/2)J_{(1,0),t_n,t_{n+1}} \\ &Y_n(J_{(-1,-1,-1),t_n,t_{n+1}} + J_{(-1,-1,1),t_n,t_{n+1}} + J_{(-1,1,-1),t_n,t_{n+1}}\end{aligned}$$

$$
\begin{aligned}
&+ J_{(1,-1,-1),t_n,t_{n+1}} + J_{(-1,1,1),t_n,t_{n+1}} + J_{(1,-1,1),t_n,t_{n+1}} \\
&+ J_{(1,1,-1),t_n,t_{n+1}} + J_{(1,1,1),t_n,t_{n+1}}) \\
&+ (3Y_n^5 - 7Y_n^3 + Y_n)(\triangle t_n)^2/2, \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
\text{STH1.5: } Y_{n+1} =& Y_n + (-Y_n^3 + Y_n/2)\triangle t_n + Y_n \triangle W_n + Y_n \triangle N_n \\
&+ Y_n((\triangle N_n)^2/2 - \triangle N_n/2 + \triangle W_n \triangle N_n + (\triangle W_n)^2/2) \\
&+ (-Y_n^3 + Y_n/2)\triangle t_n \triangle N_n - 6Y_n^3 C_{[L_{-1},L_0]} \\
&+ (-3Y_n^3 + Y_n/2)\triangle t_n \triangle W_n + 2Y_n^3 C_{[L_0,L_1]} \\
&+ Y_n((\triangle N_n)^3/6 - (\triangle N_n)^2/2 + \triangle N_n/3 + \triangle W_n(\triangle N_n)^2/2 \\
&+ \triangle N_n(\triangle W_n)^2/2 - \triangle W_n \triangle N_n/2 + (\triangle W_n)^3/6) \\
&+ (3Y_n^5 - 7Y_n^3 + Y_n)(\triangle t_n)^2/2. \qquad (4.13)
\end{aligned}
$$

Table 4.3 lists the absolute errors of the different approximate solutions Y_N, $N = 5.0/\triangle$, at $T = 5.0$.

Table 4.3

	$\|X_{5.0} - Y_N\|$				
$-\log_2(\triangle)$	ST0.5	ST1.0	STH1.0	ST1.5	STH1.5
5	146.014883	8.876596	5.667589	13.096284	9.143036
6	81.617830	4.028475	3.508073	1.772738	1.584532
7	30.780046	4.346578	7.165639	0.308736	0.244532
8	28.200415	1.999825	2.559961	0.128743	0.174067
9	1.313635	1.200021	1.149696	0.009785	0.000406
10	8.578004	0.663945	0.597334	0.008762	0.007101
11	3.304512	0.508462	0.420424	0.005866	0.004410
12	1.910364	0.139633	0.176351	0.000443	0.000465
13	0.309483	0.087264	0.098924	0.000221	0.000332
14	0.344547	0.458526	0.032951	0.000197	0.000164
15	1.075811	0.038975	0.022530	0.000033	0.000026
16	0.846444	0.011873	0.013506	0.000019	0.000011

The rate of convergence increases very slowly for the ST0.5 scheme and it is impossible for it to reach high accuracy by simply refining the partition because the round off errors will accumulate when the time step is too small, see the first column. On contrast, the schemes of order 1.0 and 1.5 can achieve higher accuracy. And the latter converges more rapidly. These results are almost equally matched to those in the non jump case, refer to table 4.2 for comparison.

References

[1] J.G. Gaines, T.J. Lyons, Random generation of stochastic area integrals, *SIAM J. Appl. Math.,* **54** (1994), 1132–1146.

[2] P.E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer-Verlag, Berlin, 1992.

[3] C.W. Li, X.Q. Liu, Algebraic structure of multiple stochastic integrals with respect to Brownian motions and Poisson processes, *Stochastic and Stochastics Reports,* **60** (1997), 23–40.

[4] X.Q. Liu, C.W. Li, Discretization of stochastic differential equations by the product expansion for the Chen series, *Stochastics and Stochastics Reports,* **61** (1997), 107–120.

[5] X.Q. Liu, C.W. Li, Product expansion for stochastic jump diffusions and its application to numerical approximation (preprint).

[6] G.N. Milstein, Numerical Integration of Stochastic Differential Equations, Ural State University Press, Sverdlovsk, 1988, revised translation, Kluwer Academic Publishers, Dordrecht, 1995.

[7] J. Sussmann, Product expansions of exponential Lie series and the discretization of stochastic differential equations, in *Stochastic Differential System, Stochastic Control Theory and Application,* W. Fleming and P.L. Lions (eds.), Springer, (1988), 563-582.

[8] D. Talay, Simulation of stochastic differential systems, in *Probabilistic Methods in Applied Physics,* P. Krée and W. Wedig (eds), *Lecture Notes in Physics,* **451,** Springer-Verlag, Berlin, 54-96.