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LONG TIME ASYMPTOTIC BEHAVIOR OF SOLUTION OF DIFFERENCE SCHEME FOR A SEMILINEAR PARABOLIC EQUATION (I)*1)

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Abstract

In this paper we prove that the solution of implicit difference scheme for a semilinear parabolic equation converges to the solution of difference scheme for the corresponding nonlinear stationary problem as $t \to \infty$. For the discrete solution of nonlinear parabolic problem, we get its long time asymptotic behavior which is similar to that of the continuous solution. For simplicity, we consider one-dimensional problem.

Key words: Asymptotic behavior, implicit difference scheme, semilinear parabolic equation.

1. Introduction

Let $\Omega = (0,l)$, $f(x) \in L^2(\Omega)$, $u_0(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, $\phi(u) = u^3$, we consider the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \phi(u) + f(x) \quad \text{in} \quad \Omega \times R_+ \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

By the usual approach[1–4] we can get the global existence of the solution of (1.1), furthermore, the solution of (1.1) converges to the solution of the following stationary problem (1.2) as $t \to \infty$.

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \phi(u) + f(x) = 0 \qquad \text{in} \quad \Omega \\ u(0,t) = u(l,t) = 0. \end{cases} \tag{1.2}$$

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In this paper we prove that the solution of implicit difference scheme for (1.1) converges to the solution of difference scheme for (1.2) as $t \to \infty$.

2. Finite Difference Scheme

The domain Ω is divided into small segments by points $x_j = jh$ $(j = 0, 1, \cdots, J)$, where $Jh = l$, J is an integer and h is the space stepsize. Let Δt be the time stepsize. For any function $w(x,t)$ we denote the values $w(jh, n\Delta t)$ by w_j^n $(0 \le j \le J$, $n = 0, 1, 2, \cdots)$ and denote the discrete function w_j^n $(0 \le j \le J$, $n = 0, 1, 2, \cdots)$ by w_h^n. We introduce the following notations:

$$\Delta_+ w_j^n = w_{j+1}^n - w_j^n \quad (0 \le j \le J-1, n = 0, 1, 2, \cdots)$$

and

$$\Delta_- w_j^n = w_j^n - w_{j-1}^n \quad (1 \le j \le J, n = 0, 1, 2, \cdots).$$

We denote the discrete function $\frac{\Delta_+ w_j^n}{h}$ $(0 \le j \le J-1$, $n = 0, 1, 2, \cdots)$ by δw_h^n. Similarly, the discrete function $\frac{\Delta_+^2 w_j^n}{h^2}$ $(0 \le j \le J-2$, $n = 0, 1, 2, \cdots)$ is denoted by $\delta^2 w_h^n$.

Denote the scalar product of two discrete functions u_h^n and v_h^m by

$$(u_h^n, v_h^m) = \sum_{j=0}^{J} u_j^n v_j^m h.$$

For $2 \ge k \ge 0$, define discrete norms

$$\|\delta^k w_h^n\|_p = \Big(\sum_{j=0}^{J-k} \Big| \frac{\Delta_+^k w_j^n}{h^k} \Big|^p h \Big)^{\frac{1}{p}}, \qquad +\infty > p > 1$$

and

$$\|\delta^k w_h^n\|_\infty = \max_{j=0,1,\cdots,J-k} \Big| \frac{\Delta_+^k w_j^n}{h^k} \Big|.$$

The difference equation associate with (1.1) is:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\Delta_+ \Delta_- u_j^{n+1}}{h^2} - \phi(u_j^{n+1}) + f_j \tag{2.1}$$

for $j = 1, \cdots, J-1$ and $n = 1, 2, \cdots$, where $f_j = f(x_j)$.

The boundary condition of (2.1) is of the form

$$u_0^n = u_J^n = 0$$

The discrete form corresponding to (1.2) is:

$$\frac{\Delta_+ \Delta_- u_j^*}{h^2} - \phi(u_j^*) + f_j = 0, \quad 0 < j < J \tag{2.2}$$

$$u_0^* = u_J^* = 0$$

Let the discrete function u_h^n and u_h^* be the solution of difference equation (2.1) and (2.2) respectively. For $n = 0, 1, 2, \cdots$, the discrete function $v_h^n = \{v_j^n \mid j = 0, 1, \cdots, J\}$ is defined as $v_j^n = u_j^n - u_j^*$ $(j = 0, 1, \cdots, J)$. Then v_h^n satisfies

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{\Delta_+ \Delta_- v_j^{n+1}}{h^2} - [(u_j^{n+1})^3 - (u_j^*)^3] \tag{2.3}$$

for $j = 1, \cdots, J-1$ and $n = 0, 1, 2, \cdots$ Obviously, $v_0^n = v_J^n = 0, n = 0, 1, 2, \cdots$.

3. Preliminary Results

Lemma 1. *For any discrete function $u_h = \{u_j \mid j = 0, 1, \cdots, J\}$ satisfying the homogeneous discrete boundary condition $u_0 = u_J = 0$, we have*

$$\|u_h\|_2 \le k_1 \|\delta u_h\|_2,$$
$$\|\delta u_h\|_2 \le k_1 \|\delta^2 u_h\|_2,$$

where k_1 is a constant independent of u_h and h.

Proof. The first inequality is from [5], since

$$\sum_{j=0}^{J-1} (\Delta_+ u_j)^2 = -\sum_{j=1}^{J-1} u_j \Delta_+ \Delta_- u_j,$$

we can get the second inequality. ■

By [5], we have the following Lemma 2:

Lemma 2. *For any discrete function $u_h = \{u_j \mid j = 0, 1, \cdots, J\}$, there is*

$$\|\delta^k u_h\|_\infty \le k_2 \|u_h\|_2^{1-\frac{2k+1}{2n}} (\|\delta^n u_h\|_2 + \|u_h\|_2)^{\frac{2k+1}{2n}},$$

where $0 \le k < n$ and k_2 is a constant independent of u_h and h.

Lemma 3. *Let the discrete function $u_h^* = \{u_j^* \mid j = 0, 1, \cdots J\}$ be the solution of the difference equation* (2.2). *There are*

$$\|\delta^2 u_h^*\|_2 \le k_3,$$
$$\|\delta u_h^*\|_\infty \le k_4, \quad \|u_h^*\|_\infty \le k_5,$$

where k_3, k_4, k_5 are constants independent of h.

Proof. It follows from (2.2) that

$$\sum_{j=1}^{J-1} \Big(\frac{\Delta_+ \Delta_- u_j^*}{h^2}\Big)^2 h - \sum_{j=1}^{J-1} \frac{\Delta_+ \Delta_- u_j^*}{h^2} (u_j^*)^3 h + \sum_{j=1}^{J-1} f_j \frac{\Delta_+ \Delta_- u_j^*}{h^2} h = 0,$$

since

$$\sum_{j=1}^{J-1} (u_j^*)^3 \frac{\Delta_+ \Delta_- u_j^*}{h^2} h = -\sum_{j=0}^{J-1} [(u_{j+1}^*)^3 - (u_j^*)^3] \frac{u_{j+1}^* - u_j^*}{h^2} h$$

$$= -\sum_{j=0}^{J-1}(u^*_{j+1}-u^*_j)^2\frac{(u^*_{j+1})^2+u^*_{j+1}u^*_j+(u^*_j)^2}{h^2}h \le 0,$$

we get

$$\sum_{j=1}^{J-1}\frac{\Delta_+\Delta_- u_j^{*\,2}}{h^2}h \le \sum_{j=1}^{J-1} f_j^2 h \tag{3.1}$$

(3.1) together with the previous Lemmas imply the conclusion. ■

Lemma 4. *Let the discrete function* u_h^n *and* u_h^* *be the solution of difference equation (2.1) and (2.2) respectively. There exist positive constants* k_6 *and* α *independent of* $h, n, \Delta t$ *such that*

$$\|u_h^n - u_h^*\|_2^2 \le k_6 e^{-\alpha n\Delta t}.$$

Proof. By (2.3), we have

$$\sum_{j=1}^{J-1}(v_j^{n+1}-v_j^n)v_j^{n+1}h = \sum_{j=1}^{J-1}\frac{\Delta_+\Delta_- v_j^{n+1}}{h^2}v_j^{n+1}h\Delta t - \sum_{j=1}^{J-1}[(u_j^{n+1})^3-(u_j^*)^3]v_j^{n+1}h\Delta t,$$

this implies

$$\sum_{j=1}^{J-1}(v_j^{n+1})^2h - \sum_{j=1}^{J-1}(v_j^n)^2h + \sum_{j=1}^{J-1}(v_j^{n+1}-v_j^n)^2h + 2\sum_{j=0}^{J-1}\Big(\frac{\Delta_+ v_j^{n+1}}{h}\Big)^2 h\Delta t + 2\sum_{j=1}^{J-1}(v_j^{n+1})^2[(u_j^{n+1})^2+u_j^{n+1}u_j^*+(u_j^*)^2]h\Delta t = 0,$$

the last term in the above equality is positive, then

$$\sum_{j=1}^{J-1}(v_j^{n+1})^2h - \sum_{j=1}^{J-1}(v_j^n)^2h + 2\sum_{j=0}^{J-1}\Big(\frac{\Delta_+ v_j^{n+1}}{h}\Big)^2 h\Delta t \le 0,$$

by Lemma 1, there is a constant $\alpha > 0$ such that

$$\sum_{j=1}^{J-1}(v_j^{n+1})^2h - \sum_{j=1}^{J-1}(v_j^n)^2h + \alpha\sum_{j=1}^{J-1}(v_j^{n+1})^2h\Delta t \le 0.$$

Therefore,

$$\|v_h^{n+1}\|_2^2 \le 2e^{-\alpha\Delta t}\|v_h^n\|_2^2,$$

the proof of the lemma is completed. ■

Lemma 5. *Let the discrete function* u_h^n *and* u_h^* *be the solution of difference equation (2.1) and (2.2) respectively, there exists constant* $k_7 > 0$ *independent of* $h, n, \Delta t$ *such that*

$$\|u_h^n - u_h^*\|_6 \le k_7.$$

Proof. It follows from (2.3) that

$$\begin{aligned}
&\sum_{j=1}^{J-1}(v_j^{n+1})^6 h - \sum_{j=1}^{J-1}(v_j^{n+1})^5 v_j^n h = \sum_{j=1}^{J-1}(v_j^{n+1})^5 h(v_j^{n+1} - v_j^n) \\
&= \sum_{j=1}^{J-1}(v_j^{n+1})^5 h\Delta t\Big(\frac{\Delta_+\Delta_- v_j^{n+1}}{h^2} - [(u_j^{n+1})^3 - (u_j^*)^3]\Big) \\
&= -\sum_{j=0}^{J-1}[(v_{j+1}^{n+1})^5 - (v_j^{n+1})^5]\frac{\Delta_+ v_j^{n+1}}{h^2} h\Delta t \\
&- \sum_{j=1}^{J-1}(v_j^{n+1})^6[(u_j^{n+1})^2 + u_j^{n+1}u_j^* + (u_j^*)^2]h\Delta t \\
&= -\sum_{j=0}^{J-1}\Big(\frac{\Delta_+ v_j^{n+1}}{h}\Big)^2 G(v_j^{n+1}, v_{j+1}^{n+1})h\Delta t \\
&- \sum_{j=1}^{J-1}(v_j^{n+1})^6[(u_j^{n+1})^2 + u_j^{n+1}u_j^* + (u_j^*)^2]h\Delta t \\
&\le -\sum_{j=0}^{J-1}\Big(\frac{\Delta_+ v_j^{n+1}}{h}\Big)^2 G(v_j^{n+1}, v_{j+1}^{n+1})h\Delta t,
\end{aligned}$$

where

$$G(x,y) = x^4 + x^3y + x^2y^2 + xy^3 + y^4 \ge 0, \forall x, y \in R,$$

hence

$$\sum_{j=1}^{J-1}(v_j^{n+1})^6 h \le \sum_{j=1}^{J-1}(v_j^{n+1})^5 v_j^n h. \tag{3.2}$$

By Holder's inequality, (3.2) yields that

$$\sum_{j=1}^{J-1}(v_j^{n+1})^6 h \le \sum_{j=1}^{J-1}(v_j^n)^6 h,$$

this complete the proof. ■

A simple computation shows that

Lemma 6. *Suppose the sequence* $\{a_n\}$ *satisfies*

$$a_{n+1} \le e^{-c_1\Delta t}a_n + c_2 e^{-c_3(n+1)\Delta t}\Delta t,$$

where $a_n \ge 0$, $\forall n \in N$, $c_i > 0$, $i = 1, 2, 3$, *then there exist* $c_4 > 0$, $\sigma > 0$ *such that*

$$a_n \le c_4 e^{-\sigma n\Delta t}.$$

4. Asymptotic Behavior of Implicit Difference Solution

In this section, we intend to study the asymptotic behavior of solutions of (2.1). It follows from (2.3) that

$$\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + \|\delta(v_h^{n+1} - v_h^n)\|_2^2 + 2\Delta t \sum_{j=1}^{J-1} \Big(\frac{\Delta_+\Delta_- v_j^{n+1}}{h^2}\Big)^2 h$$
$$= 2\sum_{j=1}^{J-1}((u_j^{n+1})^3 - (u_j^*)^3)\frac{\Delta_+\Delta_- v_j^{n+1}}{h^2} h\Delta t.$$

From Lemma 1 it follows that there exists $\theta > 0$ such that

$$\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + \|\delta(v_h^{n+1} - v_h^n)\|_2^2 + \Delta t\|\delta^2 v_h^{n+1}\|_2^2 + \theta\Delta t\|\delta v_h^{n+1}\|_2^2$$
$$\le 2\sum_{j=1}^{J-1}((u_j^{n+1})^3 - (u_j^*)^3)\frac{\Delta_+\Delta_- v_j^{n+1}}{h^2} h\Delta t. \tag{4.1}$$

A simple computation shows that

$$\sum_{j=1}^{J-1}(u_j^{n+1})^3 \Delta_+\Delta_- v_j^{n+1} = -\sum_{j=0}^{J-1}((u_{j+1}^{n+1})^3 - (u_j^{n+1})^3)(v_{j+1}^{n+1} - v_j^{n+1})$$
$$= -\sum_{j=0}^{J-1}(u_{j+1}^{n+1} - u_j^{n+1})[(u_{j+1}^{n+1})^2 + u_{j+1}^{n+1}u_j^{n+1} + (u_j^{n+1})^2](v_{j+1}^{n+1} - v_j^{n+1})$$
$$= -\sum_{j=0}^{J-1}(\Delta_+ v_j^{n+1})^2[(u_{j+1}^{n+1})^2 + u_{j+1}^{n+1}u_j^{n+1} + (u_j^{n+1})^2]$$
$$-\sum_{j=0}^{J-1}\Delta_+ u_j^*[(u_{j+1}^{n+1})^2 + u_{j+1}^{n+1}u_j^{n+1} + (u_j^{n+1})^2]\Delta_+ v_j^{n+1}.$$

Similarly,

$$\sum_{j=1}^{J-1}(u_j^*)^3\Delta_+\Delta_- v_j^{n+1} = -\sum_{j=0}^{J-1}\Delta_+ u_j^*[(u_{j+1}^*)^2 + u_{j+1}^*u_j^* + (u_j^*)^2]\Delta_+ v_j^{n+1}.$$

Hence from (4.1) it follows that

$$\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + \|\delta(v_h^{n+1} - v_h^n)\|_2^2 + \Delta t\|\delta^2 v_h^{n+1}\|_2^2 + \theta\Delta t\|\delta v_h^{n+1}\|_2^2$$
$$\le -2\sum_{j=0}^{J-1}\Big(\frac{\Delta_+ v_j^{n+1}}{h}\Big)^2[(u_{j+1}^{n+1})^2 + u_{j+1}^{n+1}u_j^{n+1} + (u_j^{n+1})^2]h\Delta t$$
$$-2\sum_{j=0}^{J-1}\frac{\Delta_+ u_j^*}{h}\frac{\Delta_+ v_j^{n+1}}{h}[(u_{j+1}^{n+1})^2 + u_{j+1}^{n+1}u_j^{n+1} + (u_j^{n+1})^2]h\Delta t$$
$$+2\sum_{j=0}^{J-1}\frac{\Delta_+ u_j^*}{h}\frac{\Delta_+ v_j^{n+1}}{h}[(u_{j+1}^*)^2 + u_{j+1}^*u_j^* + (u_j^*)^2]h\Delta t$$
$$= -2\sum_{j=0}^{J-1}\Big(\frac{\Delta_+ v_j^{n+1}}{h}\Big)^2[(u_{j+1}^{n+1})^2 + u_{j+1}^{n+1}u_j^{n+1} + (u_j^{n+1})^2]h\Delta t$$

$$
\begin{aligned}
&-2\sum_{j=0}^{J-1}\frac{\Delta_+u_j^*}{h}\frac{\Delta_+v_j^{n+1}}{h}A_jh\Delta t\\
\le&-2\sum_{j=0}^{J-1}\frac{\Delta_+u_j^*}{h}\frac{\Delta_+v_j^{n+1}}{h}A_jh\Delta t, \qquad (4.2)
\end{aligned}
$$

where

$$
\begin{aligned}
A_j=&(v_{j+1}^{n+1})^2+v_{j+1}^{n+1}v_j^{n+1}+(v_j^{n+1})^2+2v_{j+1}^{n+1}u_{j+1}^*+2v_j^{n+1}u_j^*\\
&+v_j^{n+1}u_{j+1}^*+v_{j+1}^{n+1}u_j^*. \qquad (4.3)
\end{aligned}
$$

(4.2) implies that there exist $\rho>0$, $\mu>0$ independent of $h,n,\Delta t$ such that

$$
\begin{aligned}
&\|\delta v_h^{n+1}\|_2^2-\|\delta v_h^n\|_2^2+\|\delta(v_h^{n+1}-v_h^n)\|_2^2+\Delta t\|\delta^2v_h^{n+1}\|_2^2+\rho\Delta t\|\delta v_h^{n+1}\|_2^2\\
\le&\mu\sum_{j=0}^{J-1}\Big(\frac{\Delta_+u_j^*}{h}\Big)^2A_j^2h\Delta t. \qquad (4.4)
\end{aligned}
$$

By Lemma 3, it follows from (4.4) that

$$
\begin{aligned}
&\|\delta v_h^{n+1}\|_2^2-\|\delta v_h^n\|_2^2+\|\delta(v_h^{n+1}-v_h^n)\|_2^2+\Delta t\|\delta^2v_h^{n+1}\|_2^2+\rho\Delta t\|\delta v_h^{n+1}\|_2^2\\
\le&\tau\Big(\sum_{j=0}^{J-1}(v_j^{n+1})^4h\Delta t+\sum_{j=0}^{J-1}(v_j^{n+1})^2h\Delta t\Big)\\
\le&\tau(\|v_h^{n+1}\|_6^3\|v_h^{n+1}\|_2+\|v_h^{n+1}\|_2^2)\Delta t. \qquad (4.5)
\end{aligned}
$$

From Lemma 4 and Lemma 5, there are constants $M>0,\alpha>0$ inpendent of $h,n,\Delta t$ such that

$$
\begin{aligned}
&\|\delta v_h^{n+1}\|_2^2-\|\delta v_h^n\|_2^2+\|\delta(v_h^{n+1}-v_h^n)\|_2^2+\Delta t\|\delta^2v_h^{n+1}\|_2^2+\rho\Delta t\|\delta v_h^{n+1}\|_2^2\\
\le&M\exp\Big\{-\frac{\alpha}{2}(n+1)\Delta t\Big\}\Delta t. \qquad (4.6)
\end{aligned}
$$

Therefore by Lemma 6, we have

Theorem 1. *Let the discrete function u_h^n and u_h^* be the solution of difference equation* (2.1) *and* (2.2) *respectively. There exist constants $M_1>0$, $\beta>0$ independent of $h,n,\Delta t$ such that*

$$\|\delta(u_h^n-u_h^*)\|_2^2\le M_1e^{-\beta n\Delta t}.$$

By (4.6), it suffices to show that from Theorem 1:

Theorem 2. *Let the discrete function u_h^n and u_h^* be the solution of difference equation* (2.1) *and* (2.2) *respectively. For any positive integer s, there exist constants $M_2>0,\lambda>0$ independent of $h,n,\Delta t$ such that*

$$\sum_{i=0}^{s}\|\delta^2(u_h^{n+i}-u_h^*)\|_2^2\Delta t\le M_2e^{-\lambda n\Delta t}.$$

Remark. Let u^* be the solution of (1.2), $\phi_h = \{\phi_j \mid j = 0, 1, \cdots, J\}$ be the discrete function satisfies $\phi_j = u^*(x_j), j = 0.1, \cdots, J$. By the well-known energy method, there is $C > 0$ such that

$$\|\delta(u_h^* - \phi_h)\|_2 \leq Ch^2.$$

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