Journal of Computational Mathematics, Vol.16, No.6, 1998, 499–508.

ID-WAVELETS METHOD FOR HAMMERSTEIN INTEGRAL EQUATIONS*1)

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Abstract

The numerical solutions to the nonlinear integral equations of Hammerstein-type

$$y(t) = f(t) + \int_0^1 k(t,s)g(s,y(s))ds, \quad t \in [0,1]$$

are investigated. A degenerate kernel scheme basing on ID-wavelets combined with a new collocation-type method is presented. The Daubechies interval wavelets and their main properties are briefly mentioned. The rate of approximation solution converging to the exact solution is given. Finally we also give two numerical examples.

Key words: Nonlinear integral equation, interval wavelets, degenerate kernel

1. Introduction

In this paper we will consider the numerical solutions of the non-linear integral equations of Hammerstein type:

$$y(t) = f(t) + \int_0^1 k(t,s)g(s,y(s))ds, \quad t \in [0,1] \tag{1}$$

where f, k and g are given function and y is the unknown. There has been much interest in this problem since Hammerstein integral equations, which came from the electromagnetic fluid dynamics, yields strong physical background. Moreover, the Fredholm integral equations of second kind are the special case of the Hammerstein integral equations.

In [6,p.700] the standard collocation method is applied to obtain the approximation solution of Eq.(1). In this approach some iterative method is used for solving the corresponding system of nonlinear equations and definite integrals need to be evaluated at each step of the iteration. In [3] a new collocation-type method for Eq.(1) was introduced in which the collocation method is applied not to the equation in its original form (1), but rather to an equivalent equation for

$$z(t) := g(t, y(t)), \quad t \in [0,1]. \tag{2}$$

* Received April 25, 1995.
1)This work is supported in part by the Foundation of Zhongshan University Advanced Research Centre and Guangdong NFS.

In fact, substituting (2) into (1) we reach at

$$y(t) = f(t) + \int_0^1 k(t,s)z(s)ds, \quad t \in [0,1] \tag{3}$$

and conclude that the new unknown $z(t)$ satisfies the nonlinear integral equation

$$z(t) = g(t, f(t) + \int_0^1 k(t,s)z(s)ds), \quad t \in [0,1]. \tag{4}$$

The advantage of this new method is to avoid evaluating finite integrals at each step of the iteration.

Shen and Xu[5] developed a degenerate kernel scheme for the Hammerstein Equations on the real line by wavelet and the Haar wavelet approximation is used for the linear equations.

By combining the main idea in [3] and [5], we present a new method called Daubechies interval wavelets (ID-wavelets) for obtaining the numerical solutions of Hammerstein type (1). First we use the interval wavelets constructed by I. Daubechies[2] (see next section) to approximate the integral kernel and then obtain the numerical solutions by means of the degenerate kernel scheme and the new collocation-type method. Namely:

1. The kernel $k(t,s)$ is approximated by a degenerate kernel

$$k_j(t,s) = \sum_{m,n=0}^{2^j-1} \alpha_{mn}^j \phi_m^j(t)\phi_n^j(s). \tag{5}$$

where ϕ_k^j are the ID-wavelets described in next section.

2. $z(t)$ is approximated by a linear combination of ID-wavelets:

$$z_j(t) = \sum_{k=0}^{2^j-1} a_k^j \phi_k^j(t) \quad t \in [0,1]. \tag{6}$$

3. Substituting (5) and (6) into (4) and using the orthonormality of $\{\phi_k^j;\ 0 \le k \le 2^j - 1\}$ we have

$$\sum_{k=0}^{2^j-1} a_k^j \phi_k^j(t) = g\Big(t, f(t) + \sum_{m,n=0}^{2^j-1} a_n^j \alpha_{mn}^j \phi_m^j(t)\Big), \tag{7}$$

where the coefficients $a_k^j;\ 0 \le k \le 2^j - 1$ are determined by collocating (7) at the collocation points τ_i^j:

$$\sum_{k=0}^{2^j-1} a_k^j \phi_k^j(\tau_i^j) = g(\tau_i^j, f(\tau_i^j) + \sum_{m,n=0}^{2^j-1} a_n^j \alpha_{mn}^j \phi_m^j(\tau_i^j)), \quad 0 \le i \le 2^j - 1 \tag{8}$$

which is a closed set of 2^j algebraic nonlinear equations for a_k^j.

The desired approximation to the solution y of (1) is obtained, within our present method, by substituting the approximation z_j into the right-hand side of (3). That is, the approximation to y is y_j, where

$$y_j(t) := f(t) + \int_0^1 k(t,s) z_j(s) ds. \tag{9}$$

The principal task in this paper is to show that the approximation z_j converges under suitable conditions to an exact solution of (4), and to analyze the rate of this convergence. This is carried out in Section 3. In Section 2 wavelets and ID-wavelets are introduced. The final section examines two numerical examples in some detail.

2. Wavelets and ID-Wavelets

In this section we shall state some concepts and results about wavelets and ID-wavelets.

Let $L^2[0,1]$ denote the Hilbert space of square integrable real-valued functions on [0,1] with the norm

$$\|x\| = \Big(\int_0^1 |x(t)|^2 dt\Big)^{\frac{1}{2}} := (\langle x, x\rangle)^{\frac{1}{2}}, \quad x \in L^2[0,1].$$

Following the idea of Daubechies[1] and Mallat [4], wavelet is such a special function $\psi(x)$ that its shape looks like a little wave and the set $\{\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k),$ $j, k \in Z\}$ constitutes a Riesz basis of Hilbert space $L^2(R)$. More concretely, functions $\phi(x), \psi(x) \in L^2(R) \cap L^1(R)$ are called scaling function and wavelet respectively if the following holds:

(a) $V_j \subset V_{j+1}, \quad j \in Z;$

(b) $\overline{\bigcup_{j\in Z} V_j} = L^2(R), \quad \bigcap_{j\in Z} V_j = \{0\};$

(c) $W_j \oplus V_j = V_{j+1}, \quad W_j \perp V_j;$

(d) $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1};$

(e) $\{\phi(x-k), k \in Z\}$ forms a Riesz basis of V_0,

where

$$V_j = \text{Span}\ \{\phi_{j,k}(x) := 2^{j/2}\phi(2^j x - k), \quad k \in Z\}, \tag{10}$$

$$W_j = \text{Span}\ \{\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \quad k \in Z\}. \tag{11}$$

Following the idea of Daubechies, the compactly supported orthonormal scaling functions and wavelets are constructed from the following two scale equations:

$$\phi(x) = \sum_{k\in Z} h_k \phi(2x - k) \tag{12}$$

$$\psi(x) = \sum_{k\in Z} g_k \phi(2x - k) \tag{13}$$

where $\{h_k\}, \{g_k\} \in l^2(Z)$ satisfy:

$$\begin{array}{lll} B1 & h_k = 0 & k < 1-N \text{ or } k > N \\ B2 & \sum_{k=1-N}^{N} h_k = 2 & \\ B3 & \sum_{k=1-N}^{N} (-1)^k k^j h_k = 0, & j = 0, \cdots, N-1 \\ B4 & \sum_{k=1-N}^{N} \overline{h}_k \cdot h_{k-2n} = 0, & (n \neq 0) \\ B5 & g_k = (-1)^k \cdot \overline{h}_{1-k} & \end{array}$$

(12) and (13) will generate the Mallat algorithm, i.e., the decomposition and the reconstruction formulae of signal (see [4]).

The wavelets defined in R^1 mentioned above and the corresponding Mallat algorithms are very useful in theory and applications. However, sometimes their applications are limited since the above wavelets constitute a basis for space $L^2(R)$ rather than space $L^2[a,b]$ while most practical problems are posed on finite interval $[a,b]$.

In order to solve this problem, Daubechies devoleped the ID-wavelets in 1994, which form an orthonormal basis of Hilbert space $L^2[0,1]$. The main results are listed as following (See [2] for details):

Theorem 2.1. *Suppose $\{\psi_k(x); -N \leq k \leq N-1\}$ and $\psi(x)$ are ID-wavelets (corresponding scaling functions: $\{\phi_k(x); -N \leq k \leq N-1\}$ and $\phi(x)$), choose any J such that $2^J \geq 2N$ and let*

$$\phi_k^j(x) = \begin{cases} 2^{j/2}\phi_k(2^j x), & 0 \leq k \leq N-1; \\ 2^{j/2}\phi(2^j x - k), & N \leq k \leq 2^j - N - 1; \\ 2^{j/2}\phi_{k-2^j}(2^j(x-1)), & 2^j - N \leq k \leq 2^j - 1. \end{cases}$$

$$\psi_k^j(x) = \begin{cases} 2^{j/2}\psi_k(2^j x), & 0 \leq k \leq N-1; \\ 2^{j/2}\psi(2^j x - k), & N \leq k \leq 2^j - N - 1; \\ 2^{j/2}\psi_{k-2^j}(2^j(x-1)), & 2^j - N \leq k \leq 2^j - 1. \end{cases}$$

Then the collection

$$\bigcup_{j \geq J} \{\psi_k^j; 0 \leq k \leq 2^j - 1\} \bigcup \{\phi_k^J; 0 \leq k \leq 2^J - 1\}$$

is an orthonormal basis for $L^2[0,1]$. If r is the Holder index of ϕ, ψ (i.e. $\phi, \psi \in C^r$), then this collection is also an unconditional basis for $C^s([0,1])$ for $s < r$; a bounded function f is in $C^s([0,1])$ if and only if

$$|\langle f, \psi_k^j \rangle| \leq C_f 2^{-j(s+1/2)}, \tag{14}$$

where C_f is independent of j or k.

ID-Wavelets possess some good properties as follows (which enable them being applied successfully to many fields):

$$
\begin{array}{lll}
C1 & \text{supp } \phi_k(x),\ \text{supp } \psi_k(x) = [0, N+k], & k = 0, \cdots, N-1, \\
 & \text{supp } \phi_k(x),\ \text{supp } \psi_k(x) = [1-N+k, 0], & k = -N, \cdots, -1, \\
 & \text{supp } \phi(x),\ \text{supp } \psi(x) = [1-N, N], & \\
C2 & \phi(x), \phi_k(x) \in C^{\mu N}, \quad k = -N, \cdots, N-1, & \mu \approx 0.2075(\text{ as } N \to \infty), \\
 & \psi(x), \psi_k(x) \in C^{\mu N}, \quad k = -N, \cdots, N-1, & \\
C3 & \int_0^1 x^l \psi(x) dx = 0, \quad l = 0, \cdots, N-1, & \\
 & \int_0^1 x^l \psi_k(x) dx = 0, \quad l = 0, \cdots, N-1, & k = -N, \cdots, N-1.
\end{array}
$$

Proposition 2.2. *Setting* $P_j f(x) = \sum_{k=0}^{2^j-1} \langle f,\ \phi_k^j \rangle \phi_k^j(x)$, *one can have the following inequalities:*

(1) $\|P_j\| \leq 1$ *for all* $j > \ln(2N)/(\ln 2)$,

(2) $\lim_{j \to \infty} \|f - P_j f\| = 0$, *for all* $f \in L^2[0,1]$.

Proof. (1). Defining $Q_j f(x) = \sum_{k=0}^{2^j-1} \langle f, \psi_k^j \rangle \psi_k^j(x)$, from $(b), (c)$ and Theorem 2.1 we have

$$f(t) = P_j f(t) + \sum_{j' \geq j} Q_{j'} f(t)$$

Hence

$$
\begin{aligned}
\|f(t)\|^2 = \langle f, f \rangle &= \Big\langle P_j f + \sum_{j' \geq j} Q_{j'} f, P_j f + \sum_{j' \geq j} Q_{j'} f \Big\rangle \\
&= \langle P_j f, P_j f \rangle + \Big\langle \sum_{j' \geq j} Q_{j'} f, \sum_{j' \geq j} Q_{j'} f \Big\rangle \geq \|P_j f\|^2
\end{aligned}
$$

i.e. $\|P_j\| \leq 1$.

(2) follows from Theorem 2.1 and conditions $(a), (b)$.

For 2-dimension case, by means of the tensor product one can easily obtain a basis for $L^2([0,1]^2)$.

3. Theoretical Results

First we make some assumptions on the functions f, k, and g in (1):

A1. $f \in L^2[0,1]$,

A2. the kernel $k \in L^2[0,1]^2$, and satisfies

$$\lim_{t \to t'} \int_0^1 |k(t,s) - k(t',s)|^2 ds = 0, \quad t' \in [0,1],$$

A3. $g(t,y)$ is continuous on $[0,1] \times R$ and Lipschitz continuous with respect to y, i.e. $|g(t,y_1) - g(t,y_2)| \leq C_1 |y_1 - y_2|$ for some constant $C_1 > 0$, $t \in [0,1]$ and all $y_1, y_2 \in B(y^*, \delta)$, where $B(y^*, \delta) = \{y \in L^2[0,1] : \|y - y^*\| \leq \delta\}$, $\delta > 0$.

A4. the partial derivative $g^{(0,1)}(t,y) := (\partial/\partial y)g(t,y)$ is continuous in t on [0,1] and is Lipschitz continuous in y around y^*.

Define operators as follows:

$$\begin{aligned}
&G(x)(t) := g(t,x(t)),\\
&(Kx)(t) := \int_0^1 k(t,s)x(s)ds,\\
&(K_jx)(t) := \int_0^1 k_j(t,s)x(s)ds, \qquad x \in L^2[0,1]\\
&T(x)(t) := f(t) + (Kx)(t),\\
&T_j(x)(t) := f(t) + (K_jx)(t).
\end{aligned}$$

Then the integral equations (1), (4) and (9) can be rewritten as

$$y = TG(y), \quad y \in L^2[0,1] \tag{15}$$

$$z = GT(z), \quad z \in L^2[0,1] \tag{16}$$

and

$$z_j = P_jGT_j(z_j), \quad z_j \in \text{ Span } \{\phi_k^j\}_{k=0}^{2^j-1} \tag{17}$$

respectively.

An operator A defined in an open set $\Omega \subset E$, where E is a Banach space, is called Frechet differentiable at the point $x_0 \in \Omega$ if there exists a linear continuous operator B (usually denoted by $A'(x_0)$) in E such that $A(x_0+h) - A(x_0) = Bh + \omega(x_0,h)$ where $\|\omega(x_0;h)\|/\|h\| \to 0$ as $\|h\| \to 0$. The bounded linear operator $B = A'(x_0)$ is called the Frechet derivative of the operator A at the point x_0. For example:

$$(GT)'(x_0)x(t) = g^{(0,1)}\Big(t, f(t) + \int_0^1 k(t,s)x_0(s)ds\Big) \cdot \int_0^1 k(t,s)x(s)ds.$$

For the further discussion, the following two conditions are assumed to hold:

A5. 1 is a regular value of $(GT)'(z^*)$, i.e. the inverse $(I - (GT)'(z^*))^{-1}$ exists and is a bounded linear operator.

A6. $z^* \in L^2[0,1]$ is a local solution of Eq.(16).

In the discussion below, we always assume Eq.(16) having a unique local solution $z^* \in L^2[0,1]$, since before trying to find the numerical solution we may conclude that Eq.(16) is solvable by means of its physical or mechanic background. Moreover, this conclusion can be obtained under some more strong assumptions in mathematics. For example, if $g(t,y)$ is Lipschitz continuous about y on R with Lipschitz constant L and $L\|k\|_{L^2[0,1]} < 1$, then in virtue of the theorem of contraction mapping there exists a unique solution to Eq.(16).

A5–A6 imply that z^* is also an isolated solution of Eq.(16). In fact, if $z^{**} \neq z^*$ is another solution of Eq.(16) and $z^{**} \in B(z^*,\rho) = \{z \in L^2[0,1] : \|z - z^*\| < \rho\}$ with $\rho < 1/(C_1\|[I - (GT)'(z^*)]^{-1}\| \cdot \|k\|_{L^2[0,1]})$, we have $\|z^* - z^{**}\| = \|GT(z^*) - GT(z^{**})\| < \|z^* - z^{**}\|$. (See the proof of theorem (3.2) for the detail) This contradiction concludes that $z^* = z^{**}$,i.e. z^* is an isolated solution of Eq.(16) in $B(z^*,\rho)$.

Lemma 3.1. *Suppose A*1 *to A*6 *hold, the project operators* P_j *and* Q_j *are defined as in Proposition* 2.2. *Then for sufficiently large* $j, I - (P_jGT_j)'(z^*)$ *is invertible and* $[I - (P_jGT_j)'(z^*)]^{-1}$ *is uniformly bounded.*

Proof. Since 1 is a regular value of $(GT)'(z^*)$, the inverse $(I - (GT)'(z^*))^{-1}$ exists and thus

$$\begin{aligned}&[I - (P_jGT_j)'(z^*)][I - (GT)'(z^*)]^{-1} \\ =&I + [(GT)'(z^*) - (P_jGT_j)'(z^*)][I - (GT)'(z^*)]^{-1}\end{aligned} \tag{18}$$

Since $\|K - K_j\| \to 0$ and $\|I - P_j\| \to 0$ (as $j \to \infty$), it follows that for a sufficiently large j

$$\begin{aligned}&\|[(GT)'(z^*) - (P_jGT_j)'(z^*)][I - (GT)'(z^*)]^{-1}\| \\ =&\|[(GT)'(z^*) - (P_jGT)'(z^*) + (P_jGT)'(z^*) - (P_jGT_j)'(z^*)][I - (GT)'(z^*)]^{-1}\| \\ \le&[\|I - P_j\| \cdot \|(GT)'(z^*)\| + \|G'(Tz^*) \cdot Tz^* - G'(T_jz^*)T_jz^*\|] \\ &\cdot \|[I - (GT)'(z^*)]^{-1}\| \quad \text{(by Propositon 2.2)} \\ \le&C_2\|I - P_j\| + C_3\|G'(Tz^*) \cdot Tz^* - G'(Tz^*) \cdot T_jz^* + G'(Tz^*) \cdot T_jz^* - G'(T_jz^*) \cdot T_jz^*\| \\ \le&C_4\|I - P_j\| + C_5\|K - K_j\| \quad \text{(by A4)} \\ \le&1/2.\end{aligned}$$

Hence the operator $I + [(GT)'(z^*) - (P_jGT_j)'(z^*)][I - (GT)'(z^*)]^{-1}$ is invertible for a sufficiently large j and there exists a constant C_6 such that

$$\|[I + [(GT)'(z^*) - (P_jGT_j)'(z^*)][I - (GT)'(z^*)]^{-1}]^{-1}\| \le C_6$$

for all such j. This implies from (18) that $I-(P_jGT_j)'(z^*)$ is invertible for a sufficiently large j and thus

$$\begin{aligned}\|[I - (P_jGT_j)'(z^*)]^{-1}\| =&\|[I - (GT)'(z^*)]^{-1} \cdot \{I + [(GT)'(z^*) - (P_jGT_j)'(z^*)] \\ &\cdot [I - (GT)'(z^*)]^{-1}\}^{-1}\| \le C_6\|[I - (GT)'(z^*)]^{-1}\|.\end{aligned}$$

The proof is completed.

We now state the following result on the rate of z_j converging to an exact solution of (16).

Theorem 3.2. *Under the same conditions as in Lemma* 3.1*, there exists a neighbourhood of* z^* *in which for sufficiently large* j *equation (17) has a unique solution* z_j *and the following estimate holds:*

$$\|z^* - z_j\| \le C_7\|z^* - P_jz^*\| + C_8\|(K - K_j)z^*\| \tag{19}$$

Proof. First of all, we define operators U_j by $U_jz = z^*+[I-(P_jGT_j)'(z^*)]^{-1}[P_jGT_jz-GTz^* - (P_jGT_j)'(z^*)(z - z^*)]$. Then we consider the neighbourhood $B(z^*, \delta) = \{z \in L^2[0,1] : \|z - z^*\| < \delta\}$ of z^* with $\delta < 1/(C_1C_6\|[I - (GT)'(z^*)]^{-1}\| \cdot \|k\|_{L^2[0,1]})$. For $u_1, u_2 \in B(z^*, \delta)$ we have

$$\|U_ju_1 - U_ju_2\| =\|[I - (P_jGT_j)'(z^*)]^{-1}[P_jGT_ju_1 - P_jGT_ju_2 - (P_jGT_j)'(z^*)(u_1 - u_2)]\|$$

$$
\begin{aligned}
&=\|[I-(P_jGT_j)'(z^*)]^{-1}[(P_jGT_j)'(\xi)(u_1-u_2)-(P_jGT_j)'(z^*)(u_1-u_2)]\| \\
&\quad(\xi=u_1+\theta(u_2-u_1),\ 0<\theta<1) \\
&\le C_6\|[I-(GT)'(z^*)]^{-1}\|\cdot C_1\|k\|_{L^2[0,1]}\|\xi-z^*\|\cdot\|u_1-u_2\| \\
&\le \delta C_6\|[I-(GT)'(z^*)]^{-1}\|C_1\|k\|_{L^2[0,1]}\|u_1-u_2\|<\|u_1-u_2\|
\end{aligned}
$$

This means that U_j is contraction mapping over the ball $B(z^*,\delta)$ for a sufficiently large j. By the theorem of contraction mapping, we conclude that equation $u=U_ju$ has unique solution z_j in the ball $B(z^*,\delta)$ for a sufficiently large j. i.e.

$$z_j=z^*+[I-(P_jGT_j)'(z^*)]^{-1}[P_jGT_jz_j-GTz^*-(P_jGT_j)'(z^*)(z_j-z^*)]$$

that is, $z_j=P_jGT_j(z_j)$. This shows that Eq.(17) has local unique solution.

We now establish the estimate (19). Since $z^*=GT(z^*)$ and $z_j=P_jGT_j(z_j)$, it follows that

$$
\begin{aligned}
z^*-z_j&=z^*-P_jz^*+P_jGT(z^*)-P_jGT_j(z^*)+P_jGT_j(z^*)-P_jGT_j(z_j)\\
&=[I-(P_jGT_j)'(z^*)]^{-1}[z^*-P_jz^*+P_jGT(z^*)-P_jGT_j(z^*)+\omega_j]
\end{aligned}
$$

where $\omega_j=P_jGT_jz^*-P_jGT_jz_j-(P_jGT_j)'(z^*)(z^*-z_j)$.

By Lemma 3.1, one can obtain immediately

$$
\begin{aligned}
\|z^*-z_j\|&\le C_9\|z^*-P_jz^*+P_jGT(z^*)-P_jGT_j(z^*)+\omega_j\|\\
&\le C_9[\|z^*-P_jz^*\|+\|P_jGT(z^*)-P_jGT_j(z^*)\|+\|\omega_j\|]\\
&\le C_9\|z^*-P_jz^*\|+C_{10}\|T(z^*)-T_j(z^*)\|+C_9\|\omega_j\|\\
&\qquad\text{(by Proposition 2.2 and A3)}\\
&=C_9\|z^*-P_jz^*\|+C_{11}\|(K-K_j)z^*\|+C_9\|\omega_j\|.
\end{aligned}
$$

To complete the proof, we need to show that $\|\omega_j\|=o(\|z^*-z_j\|)$. In fact,

$$
\begin{aligned}
\|\omega_j\|&=\|P_jGT_jz^*-P_jGT_jz_j-(P_jGT_j)'(z^*)(z^*-z_j)\|\\
&\le\|GT_jz^*-GT_jz_j-(GT_j)'(z^*)(z^*-z_j)\|\quad\text{(by Proposition 2.2)}\\
&=\|[(GT_j)'(z^*+\alpha\triangle z)-(GT_j)'(z^*)]\triangle z\|\quad(\text{where }0<\alpha<1,\triangle z:=z^*-z_j)\\
&\le C_{12}\|T_jz^*-T_jz_j\|\cdot\|\triangle z\|\quad\text{(by A4)}=C_{12}\|K_jz^*-K_jz_j\|\cdot\|\triangle z\|\\
&=C_{12}\|K_j\triangle z\|\cdot\|\triangle z\|\le C_{12}\|K_j\|_{L^2([0,1]^2)}\cdot\|\triangle z\|^2=o(\|z^*-z_j\|)\ (j\to\infty).
\end{aligned}
$$

Corollary 3.3. *If* $z^*\in H^s[0,1]$ $(s<r, r$ *is defined in Theorem* 2.1$)$, *then with* L^2- *norm* $\|z^*-z_j\|\le C_{13}2^{-js}\cdot\|z^*\|$, *where* $C_{13}>0$ *is independent of* j.

Proof. From the definitions of P_j and Q_j, we have

$$
\begin{aligned}
\|z^*-P_jz^*\|^2&=\Big\|\sum_{j'\ge j}Q_{j'}z^*\Big\|^2=\Big\|\sum_{j'\ge j}\sum_{k=0}^{2^{j'}-1}\langle z^*,\psi_k^{j'}\rangle\psi_k^{j'}\Big\|^2=\sum_{j'\ge j}\sum_{k=0}^{2^{j'}-1}|\langle z^*,\psi_k^{j'}\rangle|^2\\
&\le\sum_{j'\ge j}\sum_{k=0}^{2^{j'}-1}(C2^{-j'(s+1/2)})^2\quad\text{(by Theorem 2.1)}
\end{aligned}
$$

$$= \sum_{j' \geq j} 2^{j'} \cdot C_{14} \cdot 2^{-2j's-j'} = \sum_{j' \geq j} C_{14} \cdot 2^{-2j's} < C_{15} \cdot 2^{-2js}.$$

Hence, $\|z^* - P_j z^*\| \leq C_{16} \cdot 2^{-js}$. Estimate $\|(K - K_j)z^*\| \leq C'_{16} \cdot 2^{-js}$ can arrive in the same way. Thus, Theorem 3.2 results in Corollary 3.3.

According to the definitions of y^* and y_j, the following proposition holds immediately

Proposition 3.4. $\|y^* - y_j\| \leq \|K\| \cdot \|z^* - z_j\|$.

This proposition shows that the rate of convergence of y_j to y^* is no worse than that of z_j to z^*. Numerical experiment also suggests that, under suitable conditions, y_j may converge to y^* faster than z_j to z^*.

Remark. It is easy to show that all results mentioned above still hold in the following norm: $\|f\|_C = \max_{0 \leq x \leq 1} |f(x)|$, $\forall f \in C[0,1]$, because we have the following uniform estimates (which are easy to prove): (1) $\|P_j\|_C \leq$Const. for all $j > \ln(2N)/(\ln 2)$; (2) $\lim_{j \to \infty} \|f - P_j f\|_C = 0$, for all $f \in C[0,1]$.

4. Numerical Examples

Example 1. The integral equation reformulation of the nonlinear two-point boundary value problem $y''(t) - \exp(y(t)) = 0$, $t \in (0,1)$, $y(0) = y(1) = 0$ is evidently of some interest in magnetohydrodynamics. This problem has the unique solution $y^*(t) = -\ln(2) + 2\ln(c/\cos(c(t-1/2)/2))$, (where c is the only solution of $c/\cos(c/4) = \sqrt{2}$) and may be reformulated as the integral equation $y(t) = \int_0^1 k(t,s)\exp(y(s))ds$, $t \in [0,1]$ where the kernel

$$k(t,s) = \begin{cases} -s(1-t), & s \leq t \\ -t(1-s), & s > t \end{cases}$$

is the Green's function for the homogeneous problem $y''(t) = 0$, $t \in [0,1]$, $y(0) = y(1) = 0$.

Equation above was solved by our new method with the collocation points chosen to be $\tau_{ji} = i/2^j$, $i = 0, \cdots, 2^j - 1$, and the basis functions ϕ_k^j $(k = 0, \cdots, 2^j - 1)$ are taken as ID-wavelets for $N = 2$ (Usually called D_4).

The programs are coded in TRUE BASIC, and the tests are done on common PC Computer.

The maximum errors listed in Table 1 and Table 2 are estimated by taking the largest of the computed errors at $t = (i-1)/256$, $i = 0, 1, \cdots, 256$.

The experiment results are listed as follows:

Table 1

N=2				
2^j	$\|z^* - z_j\|$	ξ_1	$\|y^* - y_j\|$	ξ_2
4	7.28005e–3	1.96	1.65104e–5	1.99
8	1.87459e–3	1.99	4.16571e–6	2.01
16	4.71934e–4	2.05	1.03589e–6	2.06
32	1.13784e–4	2.29	2.48087e–7	2.30
64	2.32252e–5		5.04622e–8	

Table 2

N=2				
2^j	$\|z^* - z_j\|$	ξ_1	$\|y^* - y_j\|$	ξ_2
4	4.79168e-2	2.03	8.23608e-6	2.24
8	1.17453e-2	2.01	1.74268e-6	2.24
16	2.92196e-3	2.01	3.68902e-7	2.25
32	7.25444e-4	2.01	7.75521e-8	2.25
64	1.80108e-4		1.63033e-8	

where ξ_1 and ξ_2 are calculated by

$$\xi_1 \approx \ln(\|z^* - z_j\|/\|z^* - z_{2j-1}\|)/\ln(2)$$

and

$$\xi_2 \approx \ln(\|y^* - y_j\|/\|y^* - y_{2j-1}\|)/\ln(2)$$

respectively.

It is observed from Table 1 that whatever are errors $\|y^* - y_j\|$ or decay exponential ξ_2 obtained in this paper are better than those of [3].

Example 2. The approximation procedure of Example 1 is also used to solve Liouville equation:

$$\begin{cases} -y'' + y = (\pi^2 + 1)\sin \pi x \\ y(0) = y(1) = 0 \end{cases}$$

The procedure for finding approximate solutions is the same as that of Example 1.

This example shows that the errors $\|y^* - y_j\|$ are smaller than those given in [7] but decay exponentials are not as large as those of [7], which is due to that what we employed is D_4 wavelets rather than D_8 wavelets used in [7].

The boundary conditions in above two examples yield periodicity, but our method is available to non-periodic case. As we know, the triangle function method is a traditional approach for solving the problems with periodic boundary conditions, so it is a interesting problem to compare the wavelet method with the triangle function method. However, if ID-wavelet method is employed, we can show in the similar way as [8] that the coefficients matrice of operator in this basis are sparse and the complexity of computation will be reduced.

Acknowledgement. The authors are grateful to referees for their opinions and suggestions which are very helpful and valuable in the revising this paper.

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