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NEW APPROACH TO THE LIMITER FUNCTIONS*

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Abstract

In this paper we discuss three topics on the designing of the limiter functions. (1) To guarantee the TVD property (2) To maintain enough artificial viscosity. (3) A method to form TVB limiter which can ensure second order accuracy even at the extrema of the solution.

Key words: Finite difference methods, TVD schemes, Limiter Function

1. Introduction

Since 1980's, difference schemes with TVD or TVB properties have been used for more and more CFD problems, especially the following system of conservation laws:

$$U_t + F(U)_x = 0. \tag{1.1}$$

The reason is that the TVB property will guarantee the convergence of any subsequence of the difference solution sequence to a week solution of the differential equation. Obviously if the week solution is unique, then the whole sequence will converge to that solution.

One of the frequently used TVD scheme is the second order five-point conservative one:

$$U_i^{n+1} = U_i^n - \lambda(H_{i+1/2} - H_{i-1/2}). \tag{1.2}$$

Here $H_{i+1/2} = H(U_{i-1}^n,\ U_i^n,\ U_{i+1}^n,\ U_{i+2}^n)$, is consistent with F, i.e, $H(U,U,U,U) = F(U)$, and could be written as

$$H_{i+1/2} = F(U_i^n) + Q_{i+1/2} \cdot (F(U_{i+1}^n) - F(U_i^n)), \tag{1.3}$$

here $Q_{i+1/2}$ is usually a nonlinear function of $U_{i-1}^n, \cdots, U_{i+2}^n$, and is called Limiter.

It is this Limiter that has great effect on the scheme. In this paper we will discuss some principles and methods on how to construct that function in order that the scheme has desired properties. For simplicity, we begin with the following scalar linear equation as the model problem:

$$U_t + a \cdot U_x = 0. \tag{1.4}$$

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The corresponding scheme is:

$$U_i^{n+1} = U_i^n - a \cdot \lambda (U_i^n + Q_{i+1/2} \cdot \Delta U_{i+1/2} - U_{i-1}^n - Q_{i-1/2} \cdot \Delta U_{i-1/2}) \tag{1.5}$$

here $\lambda = \frac{\Delta t}{\Delta x}$, $\Delta U_{i+1/2} = U_{i+1}^n - U_i^n$. Without loss of generality, we assume here $a \geq 0$.

Although the above simple model is used for the theoretical analysis, the background problem of this paper is a practical 3-D viscous outer flow one, so some of the numerical examples are about 3-D flow problems.

In section 2, the conditions on Limiter for TVD property are discussed. In section 3, for solving the problems arising in the practical flow calculation, some ideas on maintaining proper artificial viscosity are given. In section 4, a method for constructing a Limiter which will ensure the second order accuracy of the scheme even at the extrema of the solution while keeping the TVB property is presented. The results of numerical experiments are provided in section 5.

2. The Basic Conditions for TVD Limiters

Accordiong to the TVD sufficient condition of Harten in [1], if a scheme can be written as:

$$U_i^{n+1} = U_i^n + C_{i+1/2}^{+} \Delta U_{i+1/2} - C_{i-1/2}^{-} \Delta U_{i-1/2} \tag{2.1}$$

and if

$$C_{i+1/2}^{+}, C_{i+1/2}^{-} \geq 0, \quad C_{i+1/2}^{+} + C_{i+1/2}^{-} \leq 1 \tag{2.2}$$

then the scheme is a TVD one.

The scheme (1.5) can be put into the form (2.1) if we choose:

$$C_{i-1/2}^{-} = a \cdot \lambda (1 + Q_{i+1/2} \frac{\Delta U_{1+1/2}}{\Delta U_{i-1/2}} - Q_{i-1/2}), \quad C_{i+1/2}^{+} = 0. \tag{2.3}$$

Assume that $Q_{i+1/2}$ is a function of the difference ratio $r_{i+1/2} = \frac{\Delta U_{i-1/2}}{\Delta U_{i+1/2}}$, i.e, $Q_{i+1/2} = Q(r_{i+1/2})$, and the function satisfies:

$$\begin{array}{lcl} Q(r) = 0 & & r \leq 0 \\ 1 \geq Q(r) > 0 & \text{if} & r > 0\,. \\ Q(r) = 1/2 & & r = 1 \end{array} \tag{2.4}$$

Furthermore we require the $Q(r)$ is Lipschitz continuous, i.e, there is a $L > 0$ independent of r, such that for any r, r':

$$|Q(r) - Q(r')| \leq L \cdot |r - r'|. \tag{2.5}$$

Thus, there must be $Q(0) = 0$, for any $r > 0$:

$$|Q(r)| = |Q(r) - Q(0)| \leq L \cdot r. \tag{2.6}$$

Therefore, when $a \cdot \lambda \leq \frac{1}{1+L}$, for the coefficient $C_{i-1/2}^{-}$ in (2.3), we have:

$$\text{if} \quad \frac{\Delta U_{i+1/2}}{\Delta U_{i-1/2}} \leq 0: \tag{2.7}$$

$$0 < C^-_{i-1/2} = a \cdot \lambda(1 - Q_{i-1/2}) \le \frac{1}{1+L} < 1 \tag{2.8}$$

$$\text{if} \quad \frac{\Delta U_{i+1/2}}{\Delta U_{i-1/2}} \ge 0 \quad \text{from (2.6):} \tag{2.9}$$

$$0 < C^-_{i-1/2} \le a \cdot \lambda(1 + L \cdot r_{i+1/2} \cdot \frac{1}{r_{i+1/2}}) = a \cdot \lambda(1+L) \le 1. \tag{2.10}$$

In consequence, we get the following conclusion:

Theorem 2.1. *If the Limiter function $Q(r)$ satisfies condition* (2.4) *and is Liptchitz continuous, then when $a \cdot \lambda \le \frac{1}{1+L}$, here L is the Liptchitz constant, the scheme* (1.5) *is TVD.*

The next question is about the accuracy of scheme (1.5), if it satisfies the conditions in theorem (2.1). Here the accuracy is in the sense of truncation error.

From Taylor expansion:

$$\begin{aligned}
\Delta U_{i+1/2} &= U_{i+1} - U_i = h \cdot U_i' + \frac{h^2}{2} U_i'' + \frac{h^3}{6} U_i^{(3)} + O(h^4) \\
\Delta U_{i-1/2} &= U_i - U_{i-1} = h \cdot U_i' - \frac{h^2}{2} U_i'' + \frac{h^3}{6} U_i^{(3)} + O(h^4) \\
\Delta U_{i-3/2} &= U_{i-1} - U_{i-2} = h \cdot U_i' - \frac{3h^2}{2} U_i'' + \frac{7h^3}{6} U_i^{(3)} + O(h^4)
\end{aligned} \tag{2.11}$$

Substituting (2.11) into (1.5) yields:

$$\begin{aligned}
U_i^{n+1} =& U_i^n - a \cdot \lambda(\Delta U_{i-1/2} + Q_{i+1/2} \cdot \Delta U_{i+1/2} - Q_{i-1/2} \cdot \Delta U_{i-1/2}) \\
=& U_i^n - a \cdot \lambda \Big[h \cdot (1 + Q_{i+1/2} - Q_{i-1/2}) \cdot U_i' + \frac{h^2}{2}(-1 + Q_{i+1/2} + Q_{i-1/2}) \cdot U_i'' \\
&+ \frac{h^3}{6}(1 + Q_{i+1/2} - Q_{i-1/2}) \cdot U_i^{(3)} + O(h^4) \Big].
\end{aligned} \tag{2.12}$$

Eq.(2.12) means that to have second order accuracy, we only need:

$$Q_{i+1/2} - Q_{i-1/2} = O(h^2) \tag{2.13}$$

$$Q_{i+1/2} + Q_{i-1/2} - 1 = O(h). \tag{2.14}$$

From the Liptchitz continuity of Q, we have:

$$\begin{aligned}
|Q_{i+1/2} - Q_{i-1/2}| =& \Big| Q\Big(\frac{\Delta U_{i-1/2}}{\Delta U_{i+1/2}}\Big) - Q\Big(\frac{\Delta U_{i-3/2}}{\Delta U_{i-1/2}}\Big) \Big| \le L \cdot \Big| \frac{\Delta U_{i-1/2}}{\Delta U_{i+1/2}} - \frac{\Delta U_{i-3/2}}{\Delta U_{i-1/2}} \Big| \\
=& L \cdot \Big| \frac{\Delta U^2_{i-1/2} - \Delta U_{i-3/2} \cdot \Delta U_{i+1/2}}{\Delta U_{i+1/2} \cdot \Delta U_{i-1/2}} \Big| = O(h^2).
\end{aligned} \tag{2.15}$$

In the derivation of the last equality, the expansion (2.11) and the following results are used:

$$\Delta U_{i+1/2}, \quad \Delta U_{i-1/2} = O(h). \tag{2.16}$$

Thus the (2.13) is proved. For the proof of (2.14), note that $Q(1) = 1/2$, so we have:

$$\begin{aligned} |Q_{i+1/2} - 1/2| =& \Big| Q\Big(\frac{\Delta U_{i-1/2}}{\Delta U_{i+1/2}}\Big) - Q\Big(\frac{\Delta U_{i+1/2}}{\Delta U_{i+1/2}}\Big)\Big| \\ \le & L \cdot \Big|\frac{\Delta U_{i-1/2} - \Delta U_{i+1/2}}{\Delta U_{i+1/2}}\Big| = O(h). \end{aligned} \tag{2.17}$$

Similarly $|Q_{i-1/2} - 1/2| = O(h)$ can be proved. Notice that all the above expansions are on the basis of (2.16), i.e. we require here $U_i^{'} \neq 0$ or x_i is not an extramum of U. Otherwise the scheme will degenerate into first order one.

Up to now, we have proved the following conclusion:

Theorem 2.2. *If $Q(r)$ satisfies the conditions in theorem (2.1), then except in the vicinity of the extrama of the solution U, scheme (1.5) has second-order accuracy.*

It is not difficult to know that many widely used Limiters satisfy the above conditions. For example, the following:

Minimod: $Q(r) = 1/2 \cdot \text{minimod}(1, r)$

here

$$\text{minimod}(1, r) = \begin{cases} \min \quad (1, r) & \text{if } r > 0 \\ 0 & \phantom{\text{if }} r \le 0 \end{cases}$$

Monotonic: $Q(r) = 1/2 \cdot \frac{r+|r|}{1+r}$.

MUSCL: $Q(r) = 1/2 \cdot \max[0, \min(2, 2r, \frac{1+r}{2})]$.

Superbee: $Q(r) = 1/2 \cdot \max[0, \min(2r, 1), \min(r, 2)]$

The conditions in the above theorem are relatively easy to meet, so besides the above Limiters, one can form some new Limiters which will also guarantee the TVD condition and second order accuracy. Here we want to emphasize that if $Q(r)$ satisfies the conditions in the above theorems, then the function $QM(r)$ defined by $Q(r)$:

$$QM(r) = Q(\min(r, 1/r)) \tag{2.18}$$

still satisfies those conditions. The proof is as follows:

First it is easy to see the $QM(r)$ still satisfies condition (2.4), so we only need to prove:

Lemma 2.3. *The $QM(r)$ is Liptchitz continuous with the same Liptchitz constant L as that of $Q(r)$.*

Proof. First, for any $r_1, r_2 > 0$:

(1) when $r_1 \le 1$, $r_2 \le 1$

$$QM(r_1) = Q(r_1) \qquad QM(r_2) = Q(r_2). \tag{2.19}$$

So the lemma is obvious in this case.

(2) when $r_1 \ge 1$, $r_2 \ge 1$, from the L-continuity of $Q(r)$ and $|r_1 \cdot r_2| \ge 1$:

$$|QM(r_2) - QM(r_1)| = \Big|Q\Big(\frac{1}{r_2}\Big) - Q\Big(\frac{1}{r_1}\Big)\Big| \le L \cdot \Big|\frac{1}{r_2} - \frac{1}{r_1}\Big|$$

$$=L \cdot \Big|\frac{r_1 - r_2}{r_1 \cdot r_2}\Big| \le L \cdot |r_2 - r_1|. \tag{2.20}$$

(3) when $r_1 \le 1$, $r_2 \ge 1$ It can be similarly proved when $r_1 \ge 1$, $r_2 \le 1$)

$$\begin{aligned}|QM(r_2) - QM(r_1)| =&\Big|Q\Big(\frac{1}{r_2}\Big) - Q(r_1)\Big| \\ \le& L \cdot \Big|\frac{1}{r_2} - r_1\Big| = L \cdot \Big|\frac{1 - r_1 \cdot r_2}{r_2}\Big|.\end{aligned} \tag{2.21}$$

Because $|r_2 - r_1| = |\frac{r_2^2 - r_1 \cdot r_2}{r_2}|$, so we only need to prove $|\frac{1-r_1 \cdot r_2}{r_2}| \le |\frac{r_2^2 - r_1 \cdot r_2}{r_2}|$.

When $r_1 \cdot r_2 \le 1$, from $r_2^2 \ge 1$ the above is obvious.

When $r_1 \cdot r_2 > 1$, then $r_1 \cdot r_2 - 1 \ge 0$, from $r_1 \le r_2$ comes $r_2^2 - r_1 \cdot r_2 \ge 0$. therefor:

$$\begin{aligned}|r_2^2 - r_1 \cdot r_2| - |r_1 \cdot r_2 - 1| =& r_2^2 - r_1 \cdot r_2 - r_1 \cdot r_2 + 1 \\ =& r_2^2 - 2 \cdot r_1 r_2 + r_1^2 + 1 - r_1^2 \\ =& (r_2 - r_1)^2 + (1 - r_1^2) \ge 0.\end{aligned} \tag{2.22}$$

Combine the above inequalities, we have:

$$|QM(r_2) - QM(r_1)| \le L \cdot \Big|\frac{1 - r_1 \cdot r_2}{r_2}\Big| \le L \cdot \Big|\frac{r_2^2 - r_1 \cdot r_2}{r_2}\Big| = L \cdot |r_2 - r_1\Big|. \tag{2.23}$$

So in the case of $r_1, r_2 \ge 0$, the lemma is proved. In the cases of one or both rs less than 0, the corresponding QM will become 0, then the prove is very simple, it is ommited here. ■

From the above theorems and lemma, we know that if the Limiter $QM(r)$ is used in (1.5), the scheme will still be a second order TVD scheme.

Notice that this Limiter (which we call the Limiter of QM type in the following discussion) is not the same as the symmetric Limiters of H.C. Yee in [11].

3. The Relation Between the Limiter and the Artificial Viscosity

Numerical viscosity included in almost all of the schemes used in CFD. The central difference schemes usually have an explicit artificial viscosity term, while the upwind biased flux spliting schemes include an implicit one sometimes called the scheme viscosity. To include artificial viscosity is not only for the purpose of shock capturing but also in many cases for the stably converging of the numerical solution. Especially when the meshes used in the numerical calculation are not fine enough to make the physical viscosity play the key role in stabilizing the solution.

As mentioned in section 1, the background problem of this paper is a viscous 3-D static outer flow problem, the Re number is in the order of 10^6 and the angle of attack is fairly large. The distribution of pressure is given in Fig.3. When we used flux vector spliting scheme plus the Monotonic or Minimod Limiter to solve the above problem, the numerical solution is 'flowing' up and down, i.e. changing with the advance of the time steps (the dashed line in Fig.3) and not converging to a fixed place. Other researchers carrying out the calculation for the same problem also found similar phenomena.

In fact, for the scheme (1.5), the sign, size and property of numerical viscosity are all related to the Limiter. Consider the semidiscrete scheme corresponding to (1.5):

$$U_t = -\frac{a}{h} \cdot (\Delta U_{i-1/2} + Q_{i+1/2} \cdot \Delta U_{i+1/2} - Q_{i-1/2} \cdot \Delta U_{i-1/2}). \tag{3.1}$$

From (2.11), we have the expansion similar to (2.12):

$$\begin{aligned} U_t =& - a \cdot U_i' + \Big(a \cdot \Big[(1 - Q_{i+1/2} - Q_{i-1/2}) - (Q_{i+1/2} - Q_{i-1/2}) \cdot \frac{U_i'}{\frac{h}{2}U_i''} \\ & - (1 + Q_{i+1/2} - Q_{i-1/2}) \cdot \frac{U_i^{(3)}}{U_i''} \cdot \frac{h}{3}\Big] \cdot \frac{h}{2}U_i''\Big) + O(h^3). \end{aligned} \tag{3.2}$$

If Limiter Q satisfies (2.13) and (2.14), the three terms in the square bracket are all $O(h)$, and the sum of them is in fact the coefficient of the numerical viscosity. To ensure the stability, the coefficient must be positive. (notice here $a > 0$)

Now let us analyse this coefficient. From the expansion (2.11), we obtain:

$$\begin{aligned} U_i' &= \frac{\Delta U_{i+1/2} + \Delta U_{i-1/2}}{2h} + O(h^2) \\ U_i'' &= \frac{\Delta U_{i+1/2} - \Delta U_{i-1/2}}{h^2} + O(h^2) \end{aligned} \tag{3.3}$$

and:

$$U_i^{(3)} = \frac{\Delta U_{i+1/2} - 2\Delta U_{i-1/2} + \Delta U_{i-3/2}}{h^3} + O(h). \tag{3.4}$$

From $|Q_{i+1/2} - Q_{i-1/2}| = O(h^2)$, omit the higher order terms:

$$\begin{aligned} &(1 - Q_{i+1/2} - Q_{i-1/2}) - (Q_{i+1/2} - Q_{i-1/2}) \cdot \frac{U_i'}{\frac{h}{2}U_i''} - (1 + Q_{i+1/2} - Q_{i-1/2}) \cdot \frac{U_i^{(3)}}{U_i''} \cdot \frac{h}{3} \\ &= 2 \cdot \frac{[(1/2 - Q_{i+1/2}) \cdot \Delta U_{i+1/2} - (1/2 - Q_{i-1/2}) \cdot \Delta U_{i-1/2}]}{\Delta U_{i+1/2} - \Delta U_{i-1/2}} - \frac{1}{3} \cdot \frac{\Delta^3 U_{i-3/2}}{\Delta^2 U_{i-1/2}}. \end{aligned} \tag{3.5}$$

Here $\Delta^3 U_{i-3/2} = \Delta U_{i+1/2} - 2\Delta U_{i-1/2} + \Delta U_{i-3/2}$, $\Delta^2 U_{i-1/2} = \Delta U_{i+1/2} - \Delta U_{i-1/2}$.

The omitted terms are higher order ones and will not affect the sign of the main term, so the right hand side of the above equation should be positive.

Now let us see in what cases the above condition can be violated.

(1) When:

$$0 < \Delta U_{i+1/2} < \Delta U_{i-1/2} < \Delta U_{i-3/2} \tag{3.6}$$

it is obvious that in this case, $r_{i+1/2} > 1 \quad r_{i-1/2} > 1$.

If the Limiter $Q(r)$ is linear on the interval $(1, \max(r_{i+1/2}, r_{i-1/2}))$ with the β as the slop, we have:

$$(1/2 - Q_{i+1/2}) = Q\Big(\frac{\Delta U_{i+1/2}}{\Delta U_{i+1/2}}\Big) - Q\Big(\frac{\Delta U_{i-1/2}}{\Delta U_{i+1/2}}\Big) = \beta \cdot \frac{(\Delta U_{i+1/2} - \Delta U_{i-1/2})}{\Delta U_{i+1/2}} \tag{3.7}$$

$$(1/2 - Q_{i-1/2}) = \beta \cdot \frac{(\Delta U_{i-1/2} - \Delta U_{i-3/2})}{\Delta U_{i-1/2}}. \tag{3.8}$$

Substituting the above two equations into (3.5), the coefficient of numerical viscosity is just:

$$(2\beta - \tfrac{1}{3}) \cdot \frac{\Delta^3 U_{i-3/2}}{\Delta^2 U_{i-1/2}}. \tag{3.9}$$

If $\beta > \frac{1}{6}$ and:

$$\frac{\Delta^3 U_{i-3/2}}{\Delta^2 U_{i-1/2}} < 0 \tag{3.10}$$

the value of (3.9) is negative.

If $\beta < \frac{1}{6}$ and:

$$\frac{\Delta^3 U_{i-3/2}}{\Delta^2 U_{i-1/2}} > 0 \tag{3.11}$$

the value of (3.9) is also negative.

In fact, for the MUSCL Limiter, the (3.7) becomes true if $r_{i+1/2}, r_{i-1/2} \le 3.0$ with $\beta = \frac{1}{4}$.

For the Superbee Limiter, the (3.7) becomes true if $r_{i+1/2}, r_{i-1/2} \le 2.0$ with $\beta = \frac{1}{2}$.

For the Minimod Limiter, the (3.7) is held for any $r_{i+1/2}, r_{i-1/2} \ge 1.0$ with $\beta = 0$.

Although the Monotonic Limiter is not a linear function, when $r_{i+1/2}, r_{i-1/2} \ge 1.0$ it varies in the similar way.

Therefore we can say that all the above Limiters can't ensure the coefficient being positive under the conditions of both (3.10) and (3.11).

The cases of (3.6), (3.10), (3.11) are just what happened in our calculations. The pressure curve at the right neighbour of point x_0 in Fig.3 show that here $U' > 0$, $U'' < 0$, so comes (3.6). The absolute value of U'' first increase, then decrease, i.e. the value of $U^{(3)}$ first is negative then positive, thus make the cases (3.11) and (3.10) alternately happen in that narrow area. So if the Limiter is not properly designed, the 'fluctuating' phenomenon of the numerical solution will occur.

To make the values of (3.9) be always positive, the Limiter of QM type in the last section was tried, but no satisfactory result has been obtained. Some analysis show that the reason is the convexity on [0,1] of the basic Limiter $Q(r)$, i.e. for any $s, t \in [0,1]$, $s < t$, there always be:

$$Q(t) \ge \frac{1/2 - Q(s)}{1 - s} \cdot (t - s) + Q(s). \tag{3.12}$$

This is the common feature of the Limiters from Minimod to Superbee, the equality is hold only for the Minimod Limiter.

Further analysis indicate that if Q is convex, the coefficient of artificial voscosity is definitely negative under (3.6), (3.11), but if Q is concave, the result may be different. Let's see the following example:

Denote $\alpha = \min\left(|1-\frac{1}{r_{i+1/2}}|,\ |1-\frac{1}{r_{i-1/2}}|\right)$ it is easy to know $\alpha = O(h)$, so we construct a Limiter as:

$$Q(s) = \frac{1}{2}\cdot e^{\frac{s-1}{\alpha^2}} = \frac{1}{2}\cdot\frac{1}{e^{\frac{1-s}{\alpha^2}}} \tag{3.13}$$

it is obvious: $Q(1) = 1/2$, and $Q(0) = 1/2\cdot e^{\frac{-1}{\alpha^2}}$ is a vary small quantity. To make $Q(0) = 0$ so that the conditions in theorem (2.1) are satisfied, a small smoothness could be made in the neighborhood of 0, but this will not affect the main property of that Limiter. For simplicity, we omitted it here.

Thus when $r_{i+1/2}, r_{i-1/2} > 1$, the corresponding QM Limiter is

$$QM_{i+1/2} = 1/2\cdot e^{\frac{1/r_{i+1/2}-1}{\alpha^2}},\quad QM_{i-1/2} = 1/2\cdot e^{\frac{1/r_{i-1/2}-1}{\alpha^2}}. \tag{3.14}$$

Substituting this into (3.5), now the coefficient is:

$$\begin{aligned}
&1-2QM_{i+1/2}+2\cdot\frac{QM_{i-1/2}-QM_{i+1/2}}{\frac{1}{r_{i+1/2}}-1}-\frac{1}{3}\cdot\frac{\Delta^3U_{i-3/2}}{\Delta^2U_{i-1/2}}\\
=&1-e^{\frac{-(1-1/r_{i+1/2})}{\alpha^2}}-\Big(\frac{e^{-\frac{(1-1/r_{i-1/2})}{\alpha^2}}}{1-1/r_{i+1/2}}-\frac{e^{-\frac{(1-1/r_{i+1/2})}{\alpha^2}}}{1-1/r_{i+1/2}}\Big)\\
&-\frac{1}{3}\cdot\frac{\Delta^3U_{i-3/2}}{\Delta^2U_{i-1/2}}
\end{aligned} \tag{3.15}$$

By the definition of α:

$$\frac{e^{-\frac{(1-1/r_{i-1/2})}{\alpha^2}}}{1-1/r_{i+1/2}} < \frac{1}{\alpha}\cdot e^{-\frac{1}{\alpha}},\quad \frac{e^{-\frac{(1-1/r_{i+1/2})}{\alpha^2}}}{1-1/r_{i+1/2}} < \frac{1}{\alpha}\cdot e^{-\frac{1}{\alpha}} \tag{3.16}$$

From the properties of exponential function, for any $k > 0$:

$$\frac{1}{\alpha^{k+1}}\cdot e^{-\frac{1}{\alpha}} \underset{\alpha\to 0}{\to} 0. \tag{3.17}$$

Because $\alpha = O(h)$, the second and third terms in (3.15) are less than $O(h^3)$, so they are higher order terms and will not affect the sign of the quantity. Omitting those terms, the quantity in (3.15) become:

$$1-\frac{1}{3}\cdot\frac{\Delta^3U_{i-3/2}}{\Delta^2U_{i-1/2}}. \tag{3.18}$$

The second term is $O(h)$, so when h is small enough, the coefficient is positive.

The above example indicate that at least the coefficient will be positive when Q is concave enough. But the Limiter in the above example is concave too much and will cause too big artificial viscosity.

The Limiter $Q(s)$ could be designed as in the following shape:

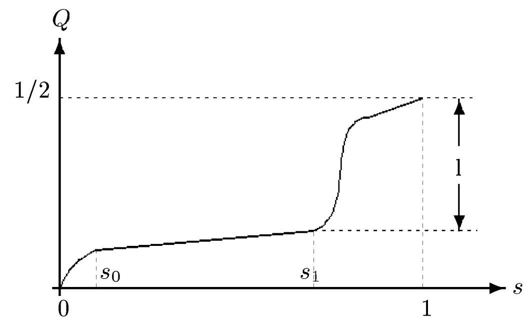


Fig.1

Here Q is linear on $[s_0, s_1]$ with the slop $\beta > 0$, when $s_0 \leq \frac{1}{r_{i+1/2}}, \frac{1}{r_{i-1/2}} \leq s_1$, the coefficient in (3.5) is:

$$\begin{aligned} &1 - 2QM_{i+1/2} + 2 \cdot \frac{QM_{i-1/2} - QM_{i+1/2}}{\frac{1}{r_{i+1/2}} - 1} - \frac{1}{3} \cdot \frac{\Delta^3 U_{i-3/2}}{\Delta^2 U_{i-1/2}} \text{ if } \frac{1}{r_{i+1/2}} < \frac{1}{r_{i-1/2}} \\ &1 - 2QM_{i-1/2} + 2 \cdot \frac{QM_{i-1/2} - QM_{i+1/2}}{1 - r_{i+1/2}} - \frac{1}{3} \cdot \frac{\Delta^3 U_{i-3/2}}{\Delta^2 U_{i-1/2}} \text{ if } \frac{1}{r_{i+1/2}} > \frac{1}{r_{i-1/2}}, \end{aligned} \tag{3.19}$$

By the linearity of Q:

$$\begin{aligned} 2 \cdot (QM_{i-1/2} - QM_{i+1/2}) =& 2 \cdot \Big(Q\Big(\frac{1}{r_{i-1/2}}\Big) - Q\Big(\frac{1}{r_{i+1/2}}\Big)\Big) \\ =& 2\beta \cdot \Big(\frac{1}{r_{i-1/2}} - \frac{1}{r_{i+1/2}}\Big). \end{aligned} \tag{3.20}$$

so for the first case of (3.19), the third term is:

$$2 \cdot \frac{QM_{i-1/2} - QM_{i+1/2}}{\frac{1}{r_{i+1/2}} - 1} = -2\beta \cdot \frac{\big(\frac{1}{r_{i-1/2}} - \frac{1}{r_{i+1/2}}\big)}{1 - \frac{1}{r_{i+1/2}}} > -2\beta, \tag{3.21}$$

for the second case, this term is positive.

By Fig.1 and $s_0 \leq \frac{1}{r_{i+1/2}}, \frac{1}{r_{i-1/2}} \leq s_1$, it is obvious that:

$$1 - 2QM_{i+1/2}, \quad 1 - 2QM_{i-1/2} \geq 2 \cdot \boldsymbol{l} \tag{3.22}$$

combine the above analysis, we know that for both cases in (3.19), the quantity there will not be less than:

$$2 \cdot \boldsymbol{l} - 2\beta - \frac{1}{3} \cdot \frac{\Delta^3 U_{i-3/2}}{\Delta^2 U_{i-1/2}}, \tag{3.23}$$

That means if we choose $\boldsymbol{l} - \beta > \frac{1}{6} \cdot \frac{\Delta^3 U_{i-3/2}}{\Delta^2 U_{i-1/2}}$ the coefficient of artificial viscosity (if h is small enough) will be positive.

To determine the parameters such as $\boldsymbol{l}, \beta, s_0, s_1$ properly, we need some preknowledge about the feature and property of the solution U, for example, the range of $\frac{1}{r_{i+1/2}}$,

$\frac{1}{r_{i-1/2}}$, the scale of $\frac{\Delta^3 U_{i-3/2}}{\Delta^2 U_{i-1/2}}$ etc. But at least the above analysis indicate that the QM type plus concave shape is a right direction to design a Limiter such that it can maintain adequate artificial viscosity.

With the parameters determined, to implement the program is not difficult. The Q can be formed on [0,1] as a piecewise polynomial and what we have to do is to calculate the polynomial coefficients for each piece, afterwards the $s = r$, or $s = \frac{1}{r}$ can be used as the variable to get the value $Q(s)$.

4. A Uniform Second Order TVB Limiter

For TVD schemes there is a common defect that they will degenerate to first order accuracy near the extrema of U. The reason is (2.16) will no longer be correct at the extrema and become the following form:

$$\Delta U_{i+1/2}, \quad \Delta U_{i-1/2} = O(h^2) \tag{4.1}$$

In this section we simply use the above result to modify the Limiter in the vicinity of the extrema to maintain the second order truncation error while only lead to $O(h^2)$ Total Variation increase there. Because there is only finite number of extrema, the TV increase during the whole procedure of time evolution is bounded by a constant independent of step size Δt, and Δx, i.e. the scheme is a TVB one.

In the following, the (1.4) is still used as the model equation to discuss the modification of the Limiter. From (2.12) it is obvious that to make (2.1) second order is equivalent to make:

$$h \cdot U_i'(Q_{i+1/2} - Q_{i-1/2}) + \frac{h^2}{2} \cdot U_i''(Q_{i+1/2} + Q_{i-1/2} - 1) = O(h^3) \tag{4.2}$$

For the scheme at point x_i, we first assume that x_{i-k} is an extremum point of U. (The extremum point is easy to be detected in practical calculation by checking whether $\frac{\Delta U_{i-k+1/2}}{\Delta U_{i-k-1/2}} < 0$.) so we have $U_{i-k}' = 0$. From Taylor expansion:

$$\begin{aligned} U_i' =& U_{i-k}' + k \cdot h U_{i-k}'' + \frac{(kh)^2}{2} U_{i-k}^{(3)} + O(h^3) \\ =& kh \cdot (U_i'' - kh \cdot U_i^{(3)} + O(h^2)) + \frac{(kh)^2}{2} U_{i-k}^{(3)} + O(h^3) \\ =& kh \cdot U_i'' - (kh)^2 \cdot U_i^{(3)} + \frac{(kh)^2}{2} U_{i-k}^{(3)} + O(h^3) \\ =& kh \cdot U_i'' + O((kh)^2). \end{aligned} \tag{4.3}$$

Substituting the above into the left side of (4.2), we have:

$$\begin{aligned} & h \cdot U_i'(Q_{i+1/2} - Q_{i-1/2}) + \frac{h^2}{2} \cdot U_i''(Q_{i+1/2} + Q_{i-1/2} - 1) \\ & \qquad = [(1+2k) \cdot Q_{i+1/2} + (1-2k) \cdot Q_{i-1/2} - 1] \frac{h^2}{2} \cdot U_i'' + R_i \end{aligned} \tag{4.4}$$

here

$$R_i = (Q_{i+1/2} - Q_{i-1/2}) \cdot O(h \cdot (kh)^2). \tag{4.5}$$

Now let the value of Limiter at $i - 1/2$, i.e. $Q_{i-1/2} = Q(r_{i-1/2})$ be the TVD Limiter satisfying the conditions in theorem (2.1), but the value at $i + 1/2$ be:

$$Q_{i+1/2} = \frac{(1 + (2k-1) \cdot Q_{i-1/2})}{1+2k} = \frac{1 - 2Q_{i-1/2}}{1+2k} + Q_{i-1/2}. \tag{4.6}$$

It is easy to verify that if $R_i = O(h^3)$, with the above definition the (4.2) will be satisfied.

Of course we can choose $Q_{i+1/2}$ as the normal TVD Limiter but let:

$$Q_{i-1/2} = \frac{(1 - (2k+1) \cdot Q_{i+1/2})}{1-2k} = \frac{1 - 2Q_{i+1/2}}{1-2k} + Q_{i+1/2}, \tag{4.7}$$

this definition can also make (4.2) be satisfied when $R_i = O(h^3)$.

When x_{i+k} is the extremum point, similar modification can be made either as:

$Q_{i+1/2}$ is the normal TVD Limiter with:

$$Q_{i-1/2} = \frac{(1 + (2k-1) \cdot Q_{i+1/2})}{1+2k} = \frac{1 - 2Q_{i+1/2}}{1+2k} + Q_{i+1/2} \tag{4.8}$$

or as:

$Q_{i-1/2}$ is the normal TVD Limiter with:

$$Q_{i+1/2} = \frac{(1 - (2k+1) \cdot Q_{i1/2})}{1-2k} = \frac{1 - 2Q_{i-1/2}}{1-2k} + Q_{i-1/2}. \tag{4.9}$$

To ensure the scheme is a TVB one under some conditions, the Limiters should be modified on the following principle:

Principle 4.1. *Between the two Limiters: $Q_{i-1/2}$, $Q_{i+1/2}$, the one corresponding to the bigger ΔU must be the TVD Limiter with the other modified.*

For example, if $|\Delta U_{i-1/2}| < |\Delta U_{i+1/2}|$, the $Q_{i+1/2}$ should be the original TVD Limiter and the $Q_{i-1/2}$ could be modified using (4.7), or (4.8).

Now we begin to prove that with the above modified Limiter, the scheme (1.5) is a uniform second order TVB scheme. In order to avoid confusion, in the following the modified Limiter defined in (4.6) to (4.9) is denoted by QB, while the TVD Limiter in section 2 is Q.

In this section, we assume the solution U of equation (1.4) has a compact support on the x axis, i.e. there are $\alpha, \beta \in \boldsymbol{R}$ such that U vanishes outside (α, β). We also assume the time upper bound T is a finite number, i.e. we only need to get the solution $U(t, x)$ of (1.4) with $t < T$. Many practical CFD problems satisfy the above assumptions.

Under the above assumptions, it is obvious that the number of mesh points at x direction is $N_1 = \frac{B_1}{h}$, here $B_1 = \beta - \alpha$. The number of time steps is denoted as $N_2 = \frac{T}{\Delta t}$. If the CFL number $\lambda = \frac{\Delta t}{h}$ is bounded, i.e. $0 \leq \lambda_1 \leq \lambda \leq \lambda_2$, then we have:

Theorem 4.1. *If the numerical solution of* (1.4) *has only finite number of isolated discontinuous points, then the scheme* (1.5) *with the modified Limiter* QB *is a TVB scheme.*

Here a point x_i is a discontinuous point that means $\Delta U_i = O(1)$, and it is a isolated one that means there should be $\Delta U_{i-1}, \Delta U_{i+1} = O(h)$

Proof. In this proof, U_i^n, $i = 0, 1, \cdots, N_1$, $n = 0, 1, \cdots, N_2$, denote the numerical solution of (1.4).

The scheme (1.5) with QB as its Limiter is:

$$\begin{aligned} U_i^{n+1} =& U_i^n - a \cdot \lambda(U_i^n + QB_{i+1/2} \cdot \Delta U_{i+1/2} - U_{i-1}^n - QB_{i-1/2} \cdot \Delta U_{i-1/2}) \\ =& UD_i^{n+1} + a\lambda(Q_{i+1/2} - QB_{i+1/2}) \\ & \cdot \Delta U_{i+1/2} - a\lambda(Q_{i-1/2} - QB_{i-1/2}) \cdot \Delta U_{i-1/2} \end{aligned} \tag{4.10}$$

here Q is the TVD Limiter from which the QB is formed in (4.6) to (4.9), and $UD_i^{n+1} = U_i^n - a \cdot \lambda(U_i^n + Q_{i+1/2} \cdot \Delta U_{i+1/2} - U_{i-1}^n - Q_{i-1/2} \cdot \Delta U_{i-1/2})$ is just the value of U_i on n+1 time level when the original TVD Limiter is used.

From the TVD property: $TV(UD^{n+1}) \leq TV(UD^n)$:

$$\begin{aligned} TV(U^{n+1}) =& \sum_{i=1}^{N_1} |U_i^{n+1} - U_{i-1}^{n+1}| \\ & \leq TV(U^n) + 2a\lambda \cdot \sum_{i=1}^{N_1} |Q_{i+1/2} - QB_{i+1/2}| \cdot |\Delta U_{i+1/2}^n| \\ & \quad + 2a\lambda \cdot \sum_{i=1}^{N_1} |Q_{i-1/2} - QB_{i-1/2}| \cdot |\Delta U_{i-1/2}^n|. \end{aligned} \tag{4.11}$$

The last two term in the above inequality can not be combined because the formula used to calculate $QB_{i+1/2}$ at point x_i maybe different from the formula for $QB_{(i+1)-1/2}$ at point x_{i+1}. (For simplicity, we did not introduce different notations for them.) Notice that between the two Limiter values $QB_{i+1/2}$, $QB_{i-1/2}$ associated with point x_i, there is only one which is different from Q's value.

To estimate the scale of last two terms, note that no matter which formula of (4.6) to (4.9) is used, we always have:

$$|Q_{i+1/2} - QB_{i+1/2}| \leq |2Q_{i-1/2} - 1| + |Q_{i+1/2} - Q_{i-1/2}| \tag{4.12}$$

and the above inequality is correct for any k, (In fact, when $k = 0$, there is a better estimation.) so from the condition (2.4), there will be:

$$|Q_{i+1/2} - QB_{i+1/2}| \leq 2. \tag{4.13}$$

First, assume the numerical solution U^n is smooth enough in the interval $[x_{i-2}, x_{i+2}]$ which is the 'dependent interval' of the value U_i^{n+1}, thus all the estimations in section 2 are correct here.

From expansion (2.11), no matter whether or not $U_i^{'} = 0$, there always be:

$$\Delta U_{i+1/2} = \Delta U_{i-1/2} + O(h^2), \tag{4.14}$$

so if $\Delta U_{i-1/2} = O(h)$, there must be $\Delta U_{i+1/2} = O(h)$ too, thus the (2.13) and $|Q_{i-1/2} - 1/2| = O(h)$ are correct, substituting them into (4.12) lead to:

$$|Q_{i+1/2} - QB_{i+1/2}| = O(h) \tag{4.15}$$

Now we have the following conclusion:

$$\text{From (4.13):} \quad \begin{aligned} &\text{if:} \quad |\Delta U_{i+1/2}| = O(h) \\ &\qquad |Q_{i+1/2} - QB_{i+1/2}| \cdot |\Delta U_{i+1/2}| = O(h^2) \\ &\text{if:} \quad |\Delta U_{i+1/2}| = O(h^2) \quad \text{also:} \\ &\qquad |Q_{i+1/2} - QB_{i+1/2}| \cdot |\Delta U_{i+1/2}| = O(h^2) \end{aligned} \tag{4.16}$$

Similarly we can prove that:

$$|Q_{i-1/2} - QB_{i-1/2}| \cdot |\Delta U_{i-1/2}| = O(h^2). \tag{4.17}$$

By the derivation above and in section 2, it is easy to know that the coefficients in the second order infinitesimal $O(h^2)$ are only dependent on the local values of derivatives of numerical solution U^n and the Lipschitz constant L of TVD Limiter Q. In this way, if U^n is smooth enough on $[x_{i-2}, x_{i+2}]$, there is a finite number $M > 0$, such that:

$$\begin{aligned} |Q_{i+1/2} - QB_{i+1/2}| \cdot |\Delta U_{i+1/2}| &\le M \cdot h^2 \\ |Q_{i-1/2} - QB_{i-1/2}| \cdot |\Delta U_{i-1/2}| &\le M \cdot h^2. \end{aligned} \tag{4.18}$$

It is obvious that the number of points in the smooth region of U^n is $\le N_1 = \frac{B_1}{h}$, so the possible increase of total variation:

$$\begin{aligned} \sum_{x_i \in C} |Q_{i+1/2} - QB_{i+1/2}| \cdot |\Delta U_{i+1/2}| &\le N_1 \cdot Mh^2 = B_4 \cdot h \\ \sum_{x_i \in C} |Q_{i-1/2} - QB_{i-1/2}| \cdot |\Delta U_{i-1/2}| &\le N_1 \cdot Mh^2 = B_4 \cdot h \end{aligned} \tag{4.19}$$

Here $x_i \in C$ means the point x_i is in the smooth region. $B_4 = B_1 \cdot M$ is independent of h.

Now assume the x_i is not in the smooth region. From principle (4.1), one of the two values: $|Q_{i-1/2} - QB_{i-1/2}|$ or $|Q_{i+1/2} - QB_{i+1/2}|$ must be 0, the other is correspond to the less ΔU and satisfy (4.13). From the assumption that the discontinuous points are isolated, this ΔU must be $O(h)$, so we have:

$$\begin{aligned} |Q_{i+1/2} - QB_{i+1/2}| \cdot |\Delta U_{i+1/2}| &= O(h) \\ |Q_{i-1/2} - QB_{i-1/2}| \cdot |\Delta U_{i-1/2}| &= O(h). \end{aligned} \tag{4.20}$$

Because the number of discontinuous points is finite, so there must be a B_5 independent of h such that:

$$\sum_{x_i \in DC} |Q_{i+1/2} - QB_{i+1/2}| \cdot |\Delta U_{i+1/2}| \leq B_5 \cdot h$$
$$\sum_{x_i \in DC} |Q_{i-1/2} - QB_{i-1/2}| \cdot |\Delta U_{i-1/2}| \leq B_5 \cdot h. \tag{4.21}$$

Here $x_i \in DC$ means the point x_i is a discontinuous point of U^n.

Substituting (4.19) and (4.21) into (4.11), we have:

$$TV(U^{n+1}) \leq TV(U^n) + 4a \cdot \lambda_2 \cdot (B_4 + B_5) \cdot h = TV(U^n) + B_6 \cdot h \tag{4.22}$$

here $B_6 = 4a \cdot \lambda_2 \cdot (B_4 + B_5)$, so when $n + 1 \leq N_2 = \frac{T}{\Delta t} = \frac{T}{h} \cdot \frac{h}{\Delta t} = \frac{T}{h} \cdot \frac{1}{\lambda}$, there should be:

$$TV(U^{n+1}) \leq TV(U^0) + N_2 \cdot B_6 \cdot h = TV(U^0) + \frac{TB_6}{\lambda} \leq TV(U^0) + \frac{TB_6}{\lambda_1} \tag{4.23}$$

The righthand side of the above inequality is independent of $\Delta t, h$, i.e. N_1, N_2. ■

Remark 1. More caref toul proof could give

$$|Q_{i+1/2} - QB_{i+1/2}| \cdot |\Delta U_{i+1/2}| \leq M \cdot \frac{h^2}{k}$$
$$|Q_{i-1/2} - QB_{i-1/2}| \cdot |\Delta U_{i-1/2}| \leq M \cdot \frac{h^2}{k} \tag{4.24}$$

instead of (4.18), so a better TV bound than (4.23) could be found.

Theorem 4.2. *If the exact solution of* (1.4) *is smooth enough, then the scheme* (1.5) *with the modified Limiter QB is unformly second order accuracy.*

Proof. Here the U denote the exact solution of (1.4).

From (4.2), (4.4), (4.5), the main obstacle of second order accuracy is the R_i depend on k which is the step number from the present point x_i to the nearst extremum point x_{i-k}, and this k can varry from 0 to $O(N_1) = O(\frac{1}{h})$, so we can not say the scheme is uniformly second order only from the elimination of the coefficient of $O(h^2)$ term in (4.4). That elimination only give the second order accuracy if $k = 0$.

Now assume $k > 0$.

When $U'_{i-k} = 0$, From (2.11), we have:

$$\begin{aligned}\Delta U_{i+1/2} =& h \cdot U'_i + \frac{h^2}{2} U''_i + O(h^3) \\ =& h[U'_{i-k} + khU''_{i-k} + O((kh)^2)] + \frac{h^2}{2} U''_i + O(h^3) \\ =& kh^2 \cdot U''_{i-k} + O(k^2h^3)) + O(h^2)) = O(kh^2).\end{aligned} \tag{4.25}$$

From (4.14) and above, it holds:

$$|Q_{i+1/2} - 1/2| \leq L \cdot \Big|\frac{\Delta U_{i-1/2} - \Delta U_{i+1/2}}{\Delta U_{i+1/2}}\Big| = O\Big(\frac{1}{k}\Big) \tag{4.26}$$

If the Limiter $QB_{i-1/2}$ is formed by (4.7), we have:

$$|QB_{i+1/2} - QB_{i-1/2}| = |Q_{i+1/2} - QB_{i-1/2}| = \Big|\frac{2(Q_{i+1/2} - 1/2)}{2k-1}\Big| = O\Big(\frac{1}{k^2}\Big). \tag{4.27}$$

Substituting the above and (4.3) into (2.12):

$$\begin{aligned}
U_i^{n+1} =& U_i^n - a\cdot\lambda\Big[h\cdot U_i' + (QB_{i+1/2} - QB_{i-1/2})\cdot hU_i' \\
&+ (QB_{i+1/2} + QB_{i-1/2} - 1)\frac{h^2}{2}\cdot U_i'' + O(h^3)\Big] \\
=& U_i^n - a\cdot\lambda\Big[h\cdot U_i' + (QB_{i+1/2} - QB_{i-1/2})\cdot kh^2\cdot U_i'' + h\cdot(QB_{i+1/2} \\
&- QB_{i-1/2})\cdot O((kh)^2) + (QB_{i+1/2} + QB_{i-1/2} - 1)\frac{h^2}{2}\cdot U_i'' + O(h^3)\Big] \\
=& U_i^n - a\cdot\lambda\Big[h\cdot U_i' + ((2k+1)\cdot QB_{i+1/2} + (1-2k)\cdot QB_{i-1/2} - 1)\cdot\frac{h^2}{2}\cdot U_i'' \\
&+ h\cdot O\Big(\frac{1}{k^2}\Big)\cdot O((kh)^2) + O(h^3)\Big] \\
=& U_i^n - a\cdot\lambda[h\cdot U_i' + O(h^3)].
\end{aligned} \tag{4.28}$$

This means the scheme is second order independent of k, i.e. uniformly second order accuracy. (When $k < 0$ or used (4.6), (4.8) and (4.9) to form QB, the proof are similar) ■

5. The Numerical Examples and the Conclusion

Although in the above sections we have only discussed the scheme (1.5) which is for the scalar differential equation (1.4), this scheme can be generalized by some well known methods such as that in [1] and used for the equation system (1.1).

The first example (In fact, the background problem) is solving a three dimensional N-S equation around a blunt revolution body with the symmetric section shown in Fig.2.

Here the angle of attack $\alpha = 20°$ and the Mach number of free stream is 0.9. A mesh of O–O type with the points number $77\times 40\times 45$ is used for this outer flow field. Our task is to obtain the distribution of pressure on the body's surface. When we use van Leer flux vector splitting plus the Limiters from Minimod to Superbee, the fluctuating phenomenon occurs and the numerical solutions do not converge even after more than 10000 time steps. This phenomenon can be seen from Fig.3 and Fig.4 which give the pressure distributions on the lee side of the above symmetric section of that blunt body.

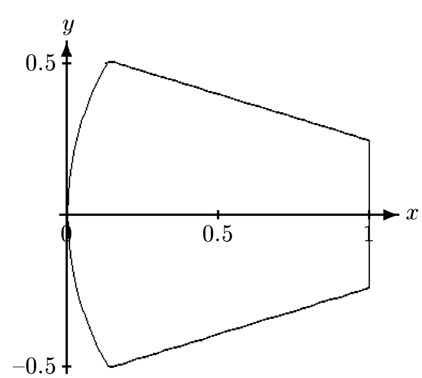


Fig.2

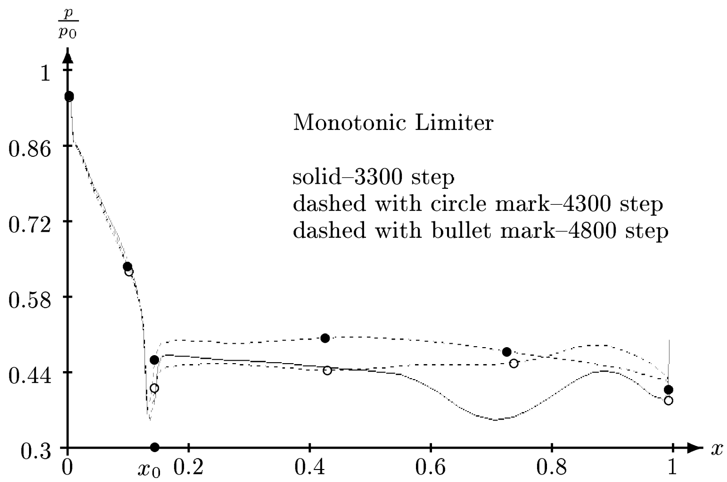


Fig.3 Lee side

The different lines in the same figure above show the results after different time steps. It is clear that with the limiters above, we can't get a convergent solution (The Minimod Limiter gives a similar result and with the Superbee limiter the solution vibrated so badly that the result omitted here); so we tried and formed the limiter in section 3. The result is shown in the following figure:

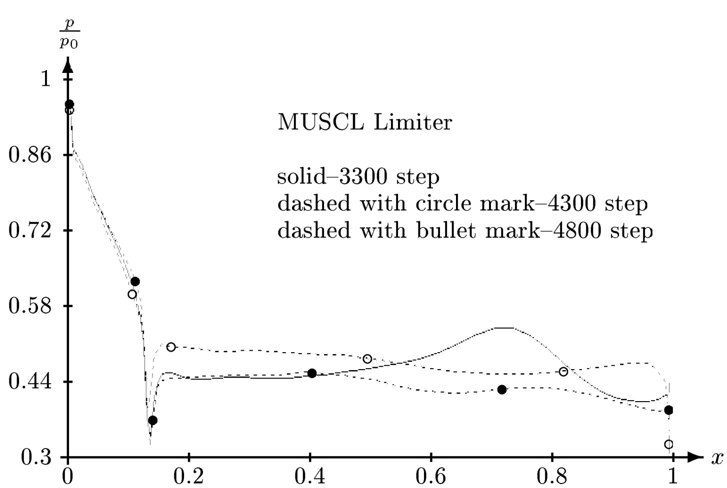


Fig.4 Lee side

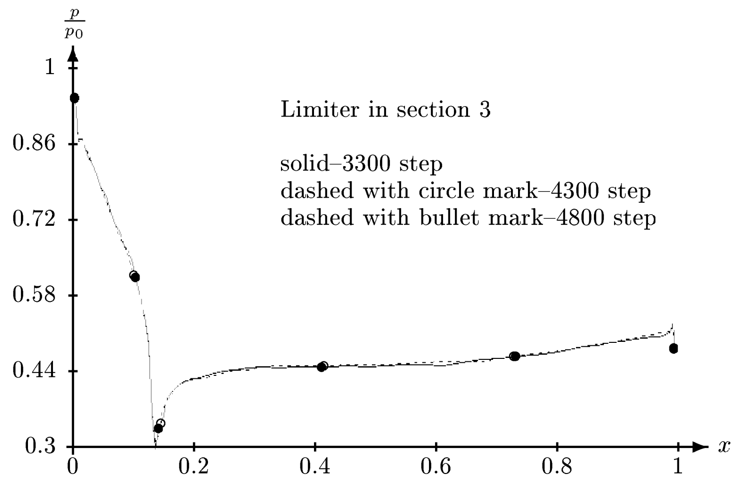


Fig.5 Lee side

It could be seen that the distributions of pressure remains unchanged even after thousands of time steps. The numerical calculation with this limiter indeed gives a successfully converging result.

Now we turn to the numerical examples for the section 4, i.e. comparison between the results obtained using a TVD limiter and those using the 'uniform second order' limiter constructed from that TVD limiter following the procedure in section 4. For this purpose, a shock tube problem (discontinuous initial value problem with one dimensional Euler equation as governing equation) is solved. The results are shown in the following figures:

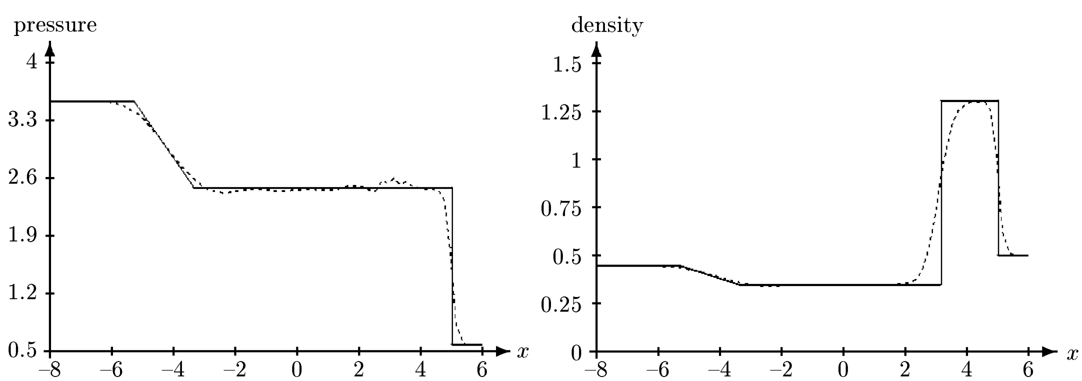
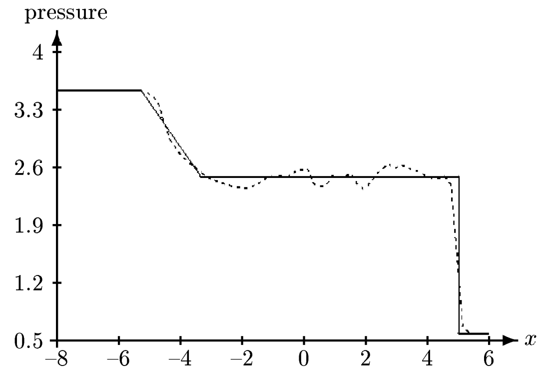


Fig.6



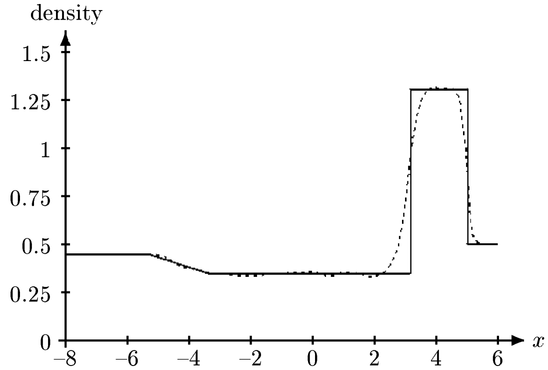


Fig.7

Here the solid curve represents the exact solution.

In Fig.6, the Minimod limiter is used while the Fig.7 is the result with the limiter formed from Minimod limiter following the formulas and principle in section 4. Although the new limiter makes the discontinuity a little bit of sharper, the vibration on the pressure curve is stronger.

The result using the TVD limiter discussed in section 2 and that using the limiter constructed from that TVD limiter are also compared and show the similar difference as in Fig.6 and Fig.7.

It seems to us that the limiters in section 4 produce less numerical viscosity than the limiters from which they constructed. Although they did not make remarkable improvement in above numerical experiments, it is worth doing some further investigations on them.

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