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NONLINEAR INTEGER PROGRAMMING AND GLOBAL OPTIMIZATION*1)

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Abstract

Various approaches have been developed for solving a variety of continuous global optimization problems. But up to now, less work has been devoted to solving nonlinear integer programming problems due to the inherent difficulty. This paper manages to transform the general nonlinear integer programming problem into an "equivalent" special continuous global minimization problem. Thus any effective global optimization algorithm can be used to solve nonlinear integer programming problems. This result will also promote the research on global optimization. We present an interval Branch-and-Bound algorithm. Numerical experiments show that this approach is efficient.

Key words: Integer programming, Global minimization problem, Branch-bound algorithm.

1. Introduction

Although the general linear integer programming problem is NP-hard, much work has been devoted to it (See Numhauser and Wolsey [1988], Schrijver [1986]). The solution methods include the cutting plane, the Branch-and-Bound, the dynamic programming methods etc.. However, the general nonlinear integer programming problem is difficult to solve. Garey and Johnson [1979] pointed out that the integer programming over R^n with a linear objective function and quadratic constraints is undecidable. So if a nonlinear integer programming problem is handled, it is always solved over a bounded box. Due to the inherent difficulty of nonlinear integer programming, less work has been done (see e.g. Benson, Erenguc and Horst [1990], Chichinadze [1991]). But during the past 30 years, various approaches have been developed to construct algorithms for a variety of continuous global optimization problems (for detail, see Rinooy kan and Timmer [1988]). In this paper, we transform the general nonlinear integer programming problem into an "equivalent" special continuous global minimization problem which can be solved by any one of effective global optimization algorithms. So it is a reasonable way to handle nonlinear integer programming problems. The involved functions of the considered nonlinear integer programming problem are only required to be Lipschitz

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continuous or continuous. Hence this result is a generalization of Ge [1989], where the involved functions are assumed to be twice continuously differentiable. Moreover, our proof is simple. We present an interval Branch-and-Bound algorithm for the special continuous global optimization problem. Lower bounds are calculated by the rules of interval analysis (Ratschek and Rokne [1988]). Methods for local optimal solutions can be incorporated into the Branch-and-Bound scheme to find better incumbent solutions. At last, numerical experiments are presented to show that this approach is efficient.

2. Unconstrained Case

Consider the following problem

$$(UP)_I \begin{cases} \min & f(x) \\ \text{s.t. } x \in X_I, \end{cases}$$

where $f(x) : R^n \to R$ is a Lipschitz function with Lipschitz constant L over a set X, here $X \subset R^n$ is a bounded closed box whose vertices all are integral lattices, X_I is the set of integer points in X.

A continuous global optimization problem corresponding to $(UP)_I$ is

$$(UP_\mu) \begin{cases} \min & f(x) + \mu \sum_{i=1}^{n} |\sin \pi x_i|, \ x = (x_1, \cdots, x_n)^T, \\ \text{s.t.} & x \in X. \end{cases}$$

For developing the relationship between problems $(UP)_I$ and (UP_μ), we need the following lemmas.

Lemma 2.1.

$$|\sin x| = \sin x \geq \frac{2}{\pi} x, \qquad \textit{if } 0 \leq x \leq \frac{\pi}{2}$$
$$|\sin x| = \sin x \geq \frac{2}{\pi} |\pi - x|, \qquad \textit{if } \frac{\pi}{2} \leq x \leq \pi.$$

Proof. Construct a line through points $(0, 0)$, $(\pi/2, 1)$ and a line through points $(\pi/2, 1)$, $(\pi, 0)$. Their equations are

$$y = \frac{2}{\pi} x,$$
$$y = \frac{2}{\pi} (\pi - x).$$

Since $\sin x$ is concave over $[0, \pi]$, and

$$\sin 0 = 0,$$
$$\sin \frac{\pi}{2} = 1,$$
$$\sin \pi = 0,$$

we have

$$|\sin x| = \sin x \geq \frac{2}{\pi}x,\text{if } 0 \leq x \leq \frac{\pi}{2};$$
$$|\sin x| = \sin x \geq \frac{2}{\pi}|\pi - x|,\text{if } \frac{\pi}{2} \leq x \leq \pi,$$

and the proof is completed. □

Obviously, we have

Lemma 2.2. *Let X be a bounded closed box whose vertices all are integral lattices. For any $x = (x_1, \cdots, x_n)^T \in X$, there exists an integer point $x_I \in X$, such that*

$$\min_{y \in R_I^n} \|x - y\|_\infty = \|x - x_I\|_\infty, \tag{2.1}$$

where R_I^n is a set of integer points in R^n.

Theorem 2.1. *If $\mu > L/2$, then problem (UP_μ) and problem $(UP)_I$ have the same optimal solutions.*

Proof. Obviously, to prove this theorem, we only need to prove that any of the optimal solutions to problem (UP_μ) is an integer vector, which is also an optimal solution to problem $(UP)_I$. We prove it by contradiction. Denote by G the set of optimal solutions to problem (UP_μ). Suppose that there exists $x^* = (x_1^*, \cdots, x_n^*)^T \in G$ which is not integral. Let $x_I^* = (x_{1I}^*, \cdots, x_{nI}^*)^T$ be the closest integer point to x^*, i.e.,

$$\|x^* - x_I^*\|_\infty \leq \frac{1}{2}.$$

Then by Lemma 2.2, $x_I^* \in X$, and if $\mu > L/2$, by Lemma 2.1, we have

$$\begin{aligned}
&f(x_I^*) + \mu \sum_{i=1}^n |\sin \pi x_{iI}^*| - f(x^*) - \mu \sum_{i=1}^n |\sin \pi x_i^*| \\
=&f(x_I^*) - f(x^*) - \mu \sum_{i=1}^n |\sin \pi (x_i^* - x_{iI}^*)| \\
\leq&L \cdot \|x_I^* - x^*\|_\infty - \mu \sum_{i=1}^n |\sin \pi (x_i^* - x_{iI}^*)| \\
\leq&L \cdot \|x_I^* - x^*\|_\infty - \mu \cdot 2/\pi \cdot \pi \cdot \|x_I^* x^*\|_\infty < 0.
\end{aligned}$$

It indicates that x^* is not an optimal solution to problem (UP_μ), which contradicts the assumption that $x^* \in G$. Therefore, x^* is integral, and is also an optimal solution to problem $(UP)_I$. □

Note that the objective function of problem (UP_μ) is generally not continuously differentiable. For the possibility of using the gradient methods of global optimization, e.g. the Filled Function method (Ge [1990]), the Tunnelling method (Levy and Montalvo [1985]), to solve nonlinear integer programming problems, corresponding to problem $(UP)_I$, we construct the following global optimization problem with a continuously differentiable objective function if $f(x)$ itself is,

$$(UP_\mu)_1 \begin{cases} \min & f(x) + \mu \sum_{i=1}^n \sin^2 \pi x_i \\ \text{s.t.} & x \in X. \end{cases}$$

Then we have the following results.

Theorem 2.2. *Suppose that there exists $M > 0$, such that*

$$|f(x)| \leq M, \quad \textit{for all } x \in X.$$

Given a point $x = (x_1, \cdots, x_n)^T$, if there exists a component x_{i0} in x x_{i0} and its nearest integer number x_{i0}^I such that

$$\frac{1}{2} \geq |x_{i_0} - x_{i_0}^I| > \frac{1}{6},$$

then

$$f(x) + \mu \sum_{i=1}^{n} \sin^2 \pi x_i > M, \;\; \textit{if } \mu > 8M.$$

Proof. Under the assumption of Theorem 2.2, we have

$$f(x) + \mu \sum_{i=1}^{n} \sin^2 \pi x_i \geq -M + \mu \sin^2 \pi x_{i_0} > -M + 8M \sin^2 \frac{\pi}{6} = M.$$

Hence Theorem 2.2 holds. □

By Lemma 2.2, theorem 2.2 implies that for any one of the global optimal solutions to problem $(UP_\mu)_1$, say x^*, its nearest integer point, say x_I, is in X_I and satisfies that

$$\|x^* - x_I\|_\infty \leq 1/6, \quad \text{if } \mu > 8M.$$

Theorem 2.3. *Suppose that x_I^1, x_I^2 are two different integer points in X_I, $f(x_I^1) < f(x_I^2)$. If $\mu > \dfrac{L^2}{16(f(x_I^2) - f(x_I^1))}$, we have*

$$f(x) + \mu \sum_{i=1}^{n} \sin^2 \pi x_i > f(x_I^1), \;\; \textit{for all } x \in \{x : \|x - x_I^2\|_\infty \leq 1/6\} \cap X.$$

Proof. For all $x \in \{x : \|x - x_I^2\|_\infty \leq 1/6\} \cap X$, we have

$$\begin{aligned}
f(x) + \mu \sum_{i=1}^{n} \sin^2 \pi x_i \geq & f(x_I^2) - L \cdot \|x - x_I^2\|_\infty + \mu \sum_{i=1}^{n} \sin^2 \pi x_i \\
= & f(x_I^2) - L \cdot \|x - x_I^2\|_\infty + \mu \sum_{i=1}^{n} \sin^2 \pi (|x_i - x_{Ii}^2|) \\
\geq & f(x_I^2) - L \cdot \|x - x_I^2\|_\infty + \mu (2/\pi \cdot \pi \cdot \|x - x_I^2\|_\infty)^2 \text{ (by Lemma 2.1)} \\
= & f(x_I^2) - L \cdot \|x - x_I^2\|_\infty + 4\mu \cdot \|x - x_I^2\|_\infty^2. \qquad (2.2)
\end{aligned}$$

Clearly, a minimal solution x of (2.2) satisfies that

$$\|x - x_I^2\|_\infty = \frac{L}{8\mu}.$$

Moreover, the minimal value of (2.2) is

$$\begin{aligned} f(x_I^2) - L\cdot\frac{L}{8\mu} + 4\mu\cdot\frac{L^2}{64\mu^2} =& f(x_I^2) - \frac{L^2}{16\mu} \\ >& f(x_I^2) - \frac{L^2}{16\cdot\dfrac{L^2}{16(f(x_I^2)-f(x_I^1))}} \Big(\text{since} \mu > \frac{L^2}{16(f(x_I^2)-f(x_I^1))}\Big) \\ =& f(x_I^1). \end{aligned}$$

Hence Theorem 2.3 holds. □

Thus we can establish the relationship between problems $(UP)_I$ and $(UP_\mu)_1$ as follows.

Theorem 2.4. *Let m satisfy that*

$$0 < m \le \begin{cases} \min & f(x_I^2) - f(x_I^1) \\ s.t. & f(x_I^2) > f(x_I^1) \\ & x_I^2, x_I^1 \in X_I. \end{cases}$$

If $\mu > \max(8M, L^2/16m)$, then for any one of the global optimal solutions to problem $(UP_\mu)_1$, say x^*, there exists an integer point $x_I^* \in X_I$, such that $\|x^* - x_I^*\|_\infty \le 1/6$, and x_I^* is an optimal solution to problem $(UP)_I$.

Proof. By Theorem 2.2, if $\mu > 8M$, then for any one of the global optimal solutions to problem $(UP_\mu)_1$, say x^*, there exists some integer point $x_I^* \in X_I$, such that

$$\|x^* - x_I^*\|_\infty \le \frac{1}{6}.$$

Thus if $\mu > \max(8M, L^2/16m)$, x_I^* is an optimal solution to problem $(UP)_I$; otherwise, by Theorem 2.3, x^* is not a global optimal solution to problem $(UP_\mu)_1$. Hence Theorem 2.4 holds. □

Remark. If $f(x)$ is a polynomial function with integer coefficients, then we may take $m = 1$.

3. Constrained Case

Now we consider the following constrained problem

$$(P)_I \begin{cases} \min & f(x) \\ \text{s.t.} & x \in S_I, \end{cases}$$

where $S_I = \{x \in X_I : g_i(x) \le 0,\ i = 1, \cdots, m\}$, X_I is the set of integer points in a bounded box X, $f(x), g_i(x)$, $i = 1, \cdots, m : R^n \to R^1$ are continuous functions over X, S_I is not empty. Construct a function $p(x)$ such that for all $x \in X_I$, $p(x) \ge 0$, and $p(x) = 0$, $x \in X_I$ if and only if $x \in S_I$. This kind of $p(x)$ may be $p(x) = \sum_{i=1}^m \max(0, g_i(x))$ or $p(x) = \sum_{i=1}^m (\max(0, g_i(x)))^2$ if $p(x)$ is wanted to be differentiable when $g_i(x)$, $i = 1, \cdots, m$ are. Then a penalty problem corresponding to $(P)_I$ is

$$(P_\mu)_I \begin{cases} \min & f(x) + \mu \cdot p(x) \\ \text{s.t.} & x \in X_I. \end{cases}$$

Let $m_I \leq \min\limits_{x \in X_I \backslash S_I} p(x)$, $m_I > 0$, $\bar{f} \geq \max\limits_{x \in X_I} f(x)$, $\underline{f} \leq \min\limits_{x \in X_I} f(x)$. We have the following theorem.

Theorem 3.1. *If* $\mu > \dfrac{\overline{f} - \underline{f}}{m_I}$, *then problems* $(P)_I$ *and* $(P_\mu)_I$ *have the same optimal solutions.*

Proof. Denote by G_I and $G_{\mu I}$ the sets of optimal solutions to problems $(P)_I$ and $(P_\mu)_I$ respectively. For $\mu > \dfrac{\overline{f} - \underline{f}}{m_I}$, to prove $G_I = G_{\mu I}$, we only need to prove that $G_I \subseteq G_{\mu I}$ and $G_{\mu I} \subseteq G_I$.

(i) $G_I \subseteq G_{\mu I}$. For all $x^* \in G_I$, $p(x^*) = 0$, and for all $x \in S_I$,

$$f(x) + \mu p(x) = f(x) \geq f(x^*) = f(x^*) + \mu p(x^*).$$

Moreover, for all $y \in X_I \backslash S_I$, $p(y) \geq m_I$, and

$$f(y) + \mu p(y) \geq \underline{f} + \mu \cdot m_I > \underline{f} + \frac{\overline{f} - \underline{f}}{m_I} \cdot m_I = \overline{f} \geq f(x^*).$$

So $x^* \in G_{\mu I}$, and $G_I \subseteq G_{\mu I}$.

(ii) $G_{\mu I} \subseteq G_I$. For all x^* must be a feasible integer point of $(P)_I$; otherwise if $x^* \in X_I \backslash S_I$, then for any $x_0 \in S_I$,

$$\overline{f} \geq f(x_0) = f(x_0) + \mu p(x_0) \geq f(x^*) + \mu \cdot p(x^*) \geq \underline{f} + \mu \cdot m_I > \underline{f} + \frac{\overline{f} - \underline{f}}{m_I} \cdot m_I = \overline{f}. \quad (3.1)$$

Inequalities (3.1) show that x^* is not an optimal solution to problem $(P_\mu)_I$, which contradicts the assumption that $x^* in G_{\mu I}$. Therefore, $x^* \in S_I$ and

$$f(x^*) + \mu \cdot p(x^*) = f(x^*) \leq f(x) + \mu \cdot p(x) = f(x), \quad \text{for all } x \in S_I.$$

Hence, $x^* \in G_I$, and $G_{\mu I} \subseteq G_I$. $\square$

Remark 1. In Theorem 3.1, m_I exists theoretically,a nd generally is very difficult to calculate. But if $g_i(x)$, $i = 1, \cdots, m$ are polynomial functions with integer coefficients, then we can set $m_I = 1$, and Theorem 3.1 holds if $\mu > \overline{f} - \underline{f}$.

Remark 2. According to Theorem 3.1, we can always transform the constrained problem $(P)_I$ into an unconstrained one $(P_\mu)I$ using the penalty function method. Thus the constrained problem $(P)_I$ can be handled by the methods discussed in section 2.

4. General Integer Program

In this section, we consider the following general problem

$$(P)_I \begin{cases} \min & f(x) \\ \text{s.t.} & x \in S_I, \end{cases}$$

where $S_I = \{x \in X_I : g_i(x) \leq 0,\ i = 1, \cdots, m\}$, X_I is the set of integer points in a bounded closed box X whose vertices all are integral lattices, $f(x)$, $g_i(x)$, $i = 1, \cdots, m$ are continuous functions, S_I is not empty. Let $p(x) = \sum_{i=1}^{m} \max(0, g_i(x))$, $F(x,\mu) = f(x) + \mu \cdot p(x)$, $0 < m_I \leq \min_{x \in X_I \setminus S_I} p(x)$, $\overline{f} \geq \max_{x \in X_I} f(x)$, $\underline{f} \leq \min_{x \in X_I} f(x)$. By Theorem 3.1, if $\mu > \frac{\overline{f} - \underline{f}}{m_I}$, $\min_{x \in X_I} F(x,\mu) = \min_{x \in S_I} f(x)$. In order to describe the relationship between problem $(P)_I$ and its corresponding continuous one, we need the following definition.

Definition 4.1. *Suppose $\varepsilon > 0$ is sufficiently small. $x^0 = (x_1^0, \cdots, x_n^0)^T$ is called an ε-integer point, if for anyi $\in \{1, \cdots, n\}$, there exists an integer k_i, such that*

$$|x_i^0 - k_i| \leq \varepsilon, \quad i = 1, \cdots, n.$$

Since $F(x,\mu) = f(x) + \mu \cdot p(x)$ is continuous, there exists a positive number M_μ such that $|F(x,\mu)| \leq M_\mu$, $\forall x \in X$. Let $p_1(x) = \sum_{i=1}^{n} |\sin \pi x_i|$, $\mu > \frac{\overline{f} - \underline{f}}{m_I}$. Now we discuss the relationship between optimal solutions of problem $\min_{x \in X_I} F(x,\mu)$ and problem $\min_{x \in X} F(x,\mu) + \mu_1 \cdot p_1(x)$.

Lemma 4.1. *If x^0 is not an $\varepsilon/2$-integer point, then $p_1(x^0) > \varepsilon$, $0 < \varepsilon < 1/2$.*

Proof. Since x^0 is not an $\varepsilon/2$-integer point, there exists an index $i_0 \in \{1, \cdots, n\}$, such that for any integer k,

$$|x_{i_0}^0 - k| > \frac{\varepsilon}{2}.$$

Especially, let k satisfy the following inequalities

$$\frac{\varepsilon}{2} < |x_{i_0}^0 - k| \leq \frac{1}{2}.$$

Thus by Lemma 2.1, we have

$$|\sin(x_{i_0}^0 - k)\pi| \geq \frac{2}{\pi} \cdot |x_{i_0}^0 - k| \cdot \pi > \varepsilon.$$

And

$$p_1(x^0) = \sum_{i=1}^{n} |\sin x_i^0 \pi| > \varepsilon,$$

which completes the proof. □

Let $S_\varepsilon = \{x \in X : 0 < p_1(x) \leq \varepsilon\}$. Denote by G_{μ_1} the set of optimal solutions to problem

$$\begin{cases} \min & F(x,\mu) + \mu_1 \cdot p_1(x) \\ \text{s.t.} & x \in X. \end{cases}$$

We have

Theorem 4.1. *If $\mu_1 > 2M_\mu/\varepsilon$, then $G_\mu \subseteq X_I \cup S_\varepsilon$.*

Proof. If there exists some $x^* \in G_{\mu_1}$, such that $x^* \in X \backslash (X_I \cup S_\varepsilon)$, then $p_1(x^*) > \varepsilon$, and

$$F(x^*, u) + \mu_1 \cdot p_1(x^*) > -M_\mu + \frac{2M_\mu}{\varepsilon} \cdot \varepsilon = M_\mu. \tag{4.1}$$

But for all $x^0 \in X_I$,

$$F(x^0, \mu) + \mu_1 \cdot p_1(x^0) = F(x^0, \mu) \leq M_\mu. \tag{4.2}$$

(4.1) and (4.2) show that x^* is not a global optimal solution to problem

$$\begin{cases} \min & F(x, \mu) + \mu_1 \cdot p_1(x) \\ \text{s.t.} & x \in X, \end{cases}$$

which contradicts the assumption that $x^* \in G_{\mu_1}$. Therefore, $G_{\mu_1} \subseteq X_I \cup S_\varepsilon$. □

Theorem 4.1 means that any one of the global optimal solutions to problem

$$\min_{x \in X} F(x, \mu) + \mu_1 \cdot p_1(x)$$

is an $\varepsilon/2$-integer point.

Theorem 4.2. *Let $\delta > 0$, and*

$$\delta \leq \begin{cases} \min & F(x_I, \mu) - f^* \\ s.t. & F(x_I, \mu) > f^* \\ & x_I \in X_I, \end{cases}$$

where f^* is the optimal value of problem $\min_{x \in X_I} F(x, \mu)$. There exists a small positive number $\varepsilon(\delta)$ such that if $\mu_1 > \dfrac{M_\mu}{\varepsilon(\delta)}$, then for any one of the global optimal solutions to problem $\min_{x \in X} F(x, \mu) + \mu_1 \cdot p_1(x)$, say x^*, its nearest integer point is an optimal solution to problem $\min_{x \in X_I} F(x, \mu)$.

Proof. For any $x_I \in X_I$, for $\delta > 0$, there exists some positive number $\varepsilon(\delta) < 1/2$, such that

$$|F(x, \mu) - F(x_I, \mu)| < \delta, \quad \forall x \in \{x \in X : \|x - x_I\|_\infty \leq \varepsilon(\delta)\},$$

since $F(x, \mu)$ is a continuous function. Thus

$$F(x, \mu) > F(x_I, \mu) - \delta, \quad \forall x \in \{x \in X : \|x - x_I\|_\infty \leq \varepsilon(\delta)\}.$$

Furthermore, if an integer point, say $y_I \in X_I$, is not an optimal solution to problem $\min_{x \in X_I} F(x, \mu)$, then

$$\begin{aligned} F(x, \mu) > F(y_I, \mu) - \delta =& (F(y_I, \mu) - f^*) - \delta + f^* \geq f^*, \\ & \forall x \in \{x \in X : \|x - y_I\|_\infty \leq \varepsilon(\delta)\}, \end{aligned}$$

where f^* is the optimal value of problem $\min_{x \in X_I} F(x, \mu)$. Thus

$$F(x,\mu) + \mu_1 \cdot p_1(x) \geq F(x,\mu) > f^*,$$

for all $x \in \{x \in X : \|x - y_I\|_\infty \leq \varepsilon(\delta),\ F(y_I,\mu) > f^*,\ y_I \in X_I\}$. But by Theorem 4.1, if $\mu_1 > M_\mu/\varepsilon(\delta)$, then $G_{\mu_1} \subseteq X_I \cup S_{2\varepsilon(\delta)}$, and

$$F(x^0,\mu) + \mu_1 \cdot p_1(x^0) \leq f^*, \quad \text{for all } x^0 \in G_{\mu_1}.$$

Hence for any $x^0 = (x_1^0, \cdots, x_n^0)^T \in G_{\mu_1}$, its nearest integer point, say $k = (k_1, \cdots, k_n)^T$, satisfying that

$$|x_i^0 - k_i| \leq \varepsilon(\delta), \quad i = 1, \cdots, n$$

is in X by Lemma 2.2 and is an optimal solution to problem $\min_{x \in X_I} F(x,\mu)$. $\square$

5. Interval Branch-and-Bound Algorithm

It has been turned out that interval analysis provides a natural framework for constructing inclusion functions for a class of functions which can be given in explicit analytical form. And interval Branch-and-Bound method has found good applications in global optimization (see Ratschek and Rokne [1988]). So in this section, we present an interval Branch-and-Bound algorithm for problem $(UP_\mu)_1$ in section 2 to find an optimal solution to problem $(UP)_I$, provided that $f(x)$ is stated in explicit analytical form.

Suppose that Y is a bounded closed box, $Y \subseteq X$. Denote by $F(Y)$ an inclusion function for $f(x)$ over Y, here $F(Y)$ can be calculated by interval mathematics methods (see Ratschek and Rokne [1988]). Denote by $ubF(Y)$, $lbF(Y)$ the upper and lower boundaries of $F(Y)$ respectively, and by $w(Y)$ the width of box Y, i.e., $w(Y) = \max_{1 \leq i \leq n} w(Y_i)$, here Y_i is an interval in R^1, $Y = Y_1 \times Y_2 \times \cdots \times Y_n$. Obviously, $lbF(F)$ is also a lower bound of $f(x_I)$, $x_I \in Y_I$, here Y_I is the set of integer points in Y.

For problem $(UP_\mu)_1$, let μ satisfy the condition of Theorem 2.4, where M, L, m are previously calculated; otherwise, let μ be large enough.

The Branch-and-Bound approach can be stated in general terms as follows. We bisect the box Y into two sub-boxes. Over each sub-box, calculate a lower bound on the objective function $f(x)+\mu p_0(x)$, here $p_0(x) = \sum_{i=1}^{n} \sin^2 \pi x_i$. For the sub-box with the smaller lower bound of $f(x) + \mu p_0(x)$, take a point in it, say x_0. Then $f(x_0) + \mu p_0(x_0)$ provides an upper bound on the optimal value of problem $(UP_\mu)_1$. Using these bounds we discard certain sub-boxes over which the integer optimal value of $f(x)$ is not lower than $f(x_0) + \mu p_0(x_0)$. Obviously, we hope to find an upper bound on the optimal value of problem $(UP_\mu)_1$ as small as possible, since we think that the earlier a smaller upper bound can be found, the more the computational cost is reduced. Now we describe the algorithm in detail.

Interval Branch-and-BOUND Algorithm

Step 0. Let $Y := X$, set list $L := \{(Y)\}$, $f^* = +\infty$.

Step 1. Calculate $lbF(Y)$.

Step 2. Choose a coordinate direction k Parallel to which Y has an edge of the maximum length. Bisect Y normal to direction k Obtaining boxes V_1, V_2 such that

$$Y = V_1 \cup V_2, \ (\text{int } V_1) \cap (\text{int } V_2) = \emptyset.$$

Step 3. Remove y from list L.

Step 4. For V_l, $l = 1, 2$,

4.1. If $w(V_l) < 1$, then there exists at most one integer point in V_l. Denote the integer point in V_l, if exists, by x_0, let $f^* := \min\{f^*, f(x_0)\}$, and let x^* be the incumbent solution providing f^*. Omit V_l.

4.2. Else calculate $lbF(V_l)$. For the box with the smaller lower bound of $f(x)$, take an integer point in it, say y_0. If $f(x)$ is continuously differentiable, using y_0 as an initial point, we can apply methods for local optimal solutions to problem $(UP_\mu)_1$ over the box V_l to find a lower function value than $f(y_0) + \mu p_0(y_0)$. If done, denote the solution also by y_0. Let $f^* := \min\{f^*, f(y_0) + \mu p_0(y_0)\}$, and let x^* be the incumbent solution providing f^*. Enter V_l into list L such that the widths of the boxes in list L decrease.

Step 5. For any box Z in list L, if $lbF(Z) \geq f^*$, delete Z from list L.

Step 6. If L is empty, end, the closest integer point to x^* is an optimal solution to problem $(UP)_I$; else denote the box with the maximum width in list L by Y, go to Step 2.

Remark. According to Ratschek and Rokne [1988], the inclusion function $F(Z)$ for $f(x)$ over a box Z is required to satisfy the following condition

$$ubF(Z) - lbF(Z) \to 0, \quad \text{if } w(Z) \to 0.$$

The algorithm has the following properties.

Theorem 5.1. *Since X is a bounded box, and by Steps 2–5 of the algorithm, it is obvious that the algorithm terminates after finite steps.*

Theorem 5.2. *After the algorithm terminates, the closest integer point to the incumbent solution x^* is an optimal solution to problem $(UP)_I$.*

Proof. Denote by T the set of boxes having been deleted in Step 5 in the algorithm. After the algorithm terminates, list L is empty, and we get f^*, x^* and

$$lbF(Z) \geq f^*, \quad \forall Z \in T.$$

Thus

$$f(x_I) \geq f(x^*) + \mu p_0(x^*) = f^*, \quad \forall x_I \in Z_I, \ Z \in T.$$

Moreover, for any box having been omitted in Step 4.1 of the algorithm, there exists at most one integer point in it, and at this integer point, the function value of $f(x)$ is not lower than f^*. So far all $x_I \in X_I$, $f(x_I) \geq f^*$. Furthermore, by Theorem 2.4, it is obvious that the closest integer point to the incumbent solution x^* is an integer optimal solution to problem $(UP)_I$. $\square$

6. Numerical Experiments

In this section, we convert four integer programming problems into unconstrained continuous ones by the penalty function method discussed in section 3.1, the theory discussed in this paper, and use the interval Branch-and-Bound algorithm developed in section 5 to solve them. For the first problem, the numerical result is compared with that of Ge [1989] by the Filled Function method. For the last two problems, the numerical results are compared with those of Conley [1980] by the Monte-carlo method.

Problem 1. (Ge [1989]).

$$\begin{aligned} \min \quad & x_1 + 10x_2 \\ \text{s.t.} \quad & 66x_1 + 14x_2 \geq 1430 \\ & -82x_1 + 28x_2 \geq 1306 \\ & 0 \leq x_i \leq 100, \ x_i : \text{integer}, \ i = 1, 2. \end{aligned}$$

Its minimal solution is $(7, 70)^T$.

Problem 2.

$$\begin{aligned} \min \quad & (x_1 - 10)^2 + (x_2 - 20)^2 \\ \text{s.t.} \quad & 2x_1 - x_2 = 0 \\ & 0 \leq x_i \leq 200, \ x_i : \ \text{integer}, \ i = 1, 2. \end{aligned}$$

Its minimal solution is $(10, 20)^T$.

Let

FF: denote the number of function evaluations by the Filled Function method (Ge [1990]).

BB: denote the number of function evaluations by the Branch-and-Bound algorithm developed in Section 5.

computational results

problem	1	2
FF	404	
BB	203	159

Problem 3 (Conley [1980], p.102).

$$\begin{aligned} \min \quad & x_2^2 + x_3^2 + 17x_5^2 + x_{10}^5 + x_5 x_{10} - x_1^2 - x_1 x_2 - x_1 x_3 \\ & - 8x_4^2 - 6x_6^3 - x_4 x_5 x_6 x_7 - x_8^3 - x_9^4 - 18 x_3 x_6 x_7 \\ \text{s.t.} \quad & 0 \leq x_i \leq 99, \quad x_i \text{ integer}, \ i = 1, \cdots, 10. \end{aligned}$$

Conley looked at 10^6 possible solutions, and found the solution (70, 66, 66, 98, 97, 95, 95, 9, 99, 9)^T with minimum -1.975×10^8. Our interval Branch and Bound algorithm calculates 37447 boxes (including single integer points) and finds the solution (99, 50, 99, 99, 99, 99, 99, 99, 99, 0)^T with the minimum -2.16×10^8. The number of function evaluations is about 74894.

Problem 4 (Conley [1980], P.119).

$$\begin{aligned}\min\quad & 6x_1^2+18x_2^2+7x_3^2-2x_1-16x_2-31x_3-12x_1x_2x_3\\ \text{s.t.}\quad & x_1+x_2+2x_3\le 2000\\ & x_1+17x_2\le 8000\\ & x_2+5x_3\le 4000\\ & x_1+7x_2+2x_3\ge 200\\ & x_1+x_2+x_3\ge 200\\ & x_1^2+x_2x_3\ge 900\\ & 0\le x_i\le 999,\quad x_i \text{ integer},\ i=1,2,3.\end{aligned}$$

Conley looked at a sample of 8×10^5 points, and took the solution $(720,424,428)^T$ with minimum $-1,560,310,784$. Our interval Branch and Bound algorithm calculates 5401 boxes (including single integer points) and finds the solution $(758, 426, 408)^T$ with the minimum -1.573×10^9. The number of function evaluations is about 10802.

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