Journal of Computational Mathematics, Vol.17, No.3, 1999, 243–256.

A NONLINEAR GALERKIN METHOD WITH VARIABLE MODES FOR KURAMOTO-SIVASHINSKY EQUATION*1)

Yu-jiang Wu
(*Department of Mathematics, Shanghai University, Jiading, Shanghai 201800, China; Department of Mathematics, Lanzhou University, Lanzhou 730000, China*)

Abstract

This article proposes a kind of nonlinear Galerkin methods with variable modes for the long-term integration of Kuramoto-Sivashinsky equation. It consists of finding an appropriate or best number of modes in the correction of the method. Convergence results and error estimates are derived for this method. Numerical examples show also the efficiency and advantage of our method over the usual nonlinear Galerkin method and the classical Galerkin method.

Key words: Kuramoto-Sivaskinsky equation, Nonlinear Galerkin method, Approximation, Convergence

1. Introduction

The nonlinear Galerkin method was introduced by Marion and Temam[4], which is stemmed from the theory of inertial manifolds and dynamical system theory. The considerable increase in the computing power during last years makes it possible for the mathematicians to solve numerical problems for approximating various dissipative evolution equations on large interval of time. Indeed, the nonlinear Galerkin method has proven to be a powerful tool for such problems (See [9], [11] and references therein).

Recently, this method has been applied to the long time integration of Kuramoto-Sivashinsky equation[12]. Thanks to a newly established inequality for the nonlinear term of Kuramoto-Sivashinsky equation, we can extend the method to a nonlinear Galerkin method with variable modes. Here the method involves a changeable number for the small-scale components $z_s = z_{s(m)}$, when the unknown function is $u \approx u_m + z_s$. After the analysis of error estimates we give an optimal value of s or $\omega = m + s$ which reduces the order of the error of the method to the lowest.

This paper is organized as follows: Section 2 contains the description of the equation and some preliminary results. In Section 3 we describe the modification of nonlinear Galerkin method with variable modes and prove successively the convergence of the method. In Section 4 we state and prove the error estimates of the method and give the possible minimum modes for the method. Finally, in Section 5 we make comparisons of various numerical computations for two examples which show a significant gain in computing time for our method.

* Received October 5, 1995.
1) The project is supported partially by the Science Foundation of the State Education Commission of China.

2. The Equation and Its Functional Setting

The Kuramoto-Sivashinsky equation with an initial condition and a periodic boundary condition reads as follows (with dimension= 1 and period= l):

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^2 u}{\partial x^2} + u\dfrac{\partial u}{\partial x} = 0 & 0 < x < l, \quad t > 0 \qquad (2.1) \\ u(x,0) = u_0(x) & 0 \le x \le l \qquad (2.2) \\ u(x,t) = u(x+l,t) & t \ge 0 \qquad (2.3) \end{cases}$$

For the functional setting of the equation, we can rewrite this partial differential equation into an abstract evolution equation in a Hilbert space H with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. In this case, we have $H = \{u | u \in L^2(0,l),\ u(0,t) = u(l,t) = 0\}$. Thus the equations (2.1)–(2.3) become

$$\begin{cases} \dfrac{du}{dt} + Au + B(u) + Cu = f & (2.4) \\ u(0) = u_0 & (2.5) \end{cases}$$

Here, we set $A = \dfrac{\partial^4}{\partial x^4}$, $B(u) = u\dfrac{\partial u}{\partial x}$ and

$$Cu = \begin{cases} \dfrac{\partial^2 u}{\partial x^2} & l < 2\pi \\ \dfrac{\partial^2 u}{\partial x^2} + \phi\dfrac{\partial u}{\partial x} + \phi' u & l \ge 2\pi \end{cases}$$

$$f = \begin{cases} 0 & l < 2\pi \\ -\phi^{(4)} - \phi'' - \phi\phi' & l \ge 2\pi \end{cases}$$

where $\phi = \phi(x)$ is a function given in [5] to keep the coercivity property of the operator $A + C$.

Since A^{-1} is compact and self-adjoint, there exists an orthonormal basis of H which consists of the eigenvectors of A: $Aw_j = \lambda_j w_j$, $0 < \lambda_1 \le \lambda_2 \le \cdots$, $\lambda_j \to \infty$ as $j \to \infty$.

Given another Hilbert space V endowed with scalar product $((\cdot,\cdot))$ and norm $\|\cdot\|$, $V = H_p^2(0,l) \cap H$. We denote the domain of the operator A by $D(A) = H_p^4(0,l) \cap H$. And we know that $B(u) = B(u,u)$ is a bilinear operator from $V \times V$ into V', C is a linear operator from V into H and $f \in H$.

Define a trilinear form b on V by $b(u,v,w) = \langle B(u,v), w\rangle_{V',V} \quad \forall u,v,w \in V$, we recall the following well-known properties:

$$b(u,u,u) = 0 \quad \forall u \in V \tag{2.6}$$

$$|b(u,v,w)| \le c_1 |u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} \quad \forall u,v,w \in V \tag{2.7}$$

$$|Cu| \le c_2\|u\| \quad \forall u \in V \tag{2.8}$$

$$|B(u,v)| \le c_3|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2} \quad \forall u \in V, v \in D(A) \tag{2.9}$$

$$|B(u,v)| \le c_4|u|^{1/2}|Au|^{1/2}\|v\| \quad \forall u,v \in D(A) \tag{2.10}$$

$$|B(u,v)| \le c_5\Big(1 + \log\frac{|Au|^2}{\lambda_1\|u\|^2}\Big)^{1/2}\|u\|\|v\| \quad \forall u \in D(A), v \in V \tag{2.11}$$

$$|B(u,v)| \le c_6\Big(3 - \frac{\lambda_1\|u\|^2}{\tau|Au|^2}\Big)^{1/2}\|u\|\|v\| \quad (\tau > 0) \quad \forall u \in D(A),\ v \in V \tag{2.12}$$

By the construction, we have, for an $\alpha > 0$,

$$((A+C)u, u) \ge \alpha\|u\|^2, \quad \forall u \in D(A) \tag{2.13}$$

Let m denote a fixed positive integer. To such an m, we associate the orthogonal projection $P = P_m$ in H onto the subspace spanned by the first m eigenvectors of $A, w_1, w_2, \cdots, w_m$. For any integer $m' > m$, it results in the following lemma.

Lemma 2.1. *If $v \in (P_{m'} - P_m)H$, then it holds*

$$\frac{1}{\lambda_{m'} + c_2}|v| \le |(A+C)^{-1}v| \le \frac{1}{\alpha\lambda_{m+1}}|v|. \tag{2.14}$$

At the end of this section, we recall a result borrowed from [8]:

Lemma 2.2. (*Gronwall*) *If $y(t) \ge 0$ $(y(0) = 0)$, $x(t) \ge 0$, $g(t) \ge 0$ and $h(t) \ge 0$ satisfy $y'(t) + x(t) \le g(t)y(t) + h(t)$ $\forall t \ge 0$ then*

$$y(t) + \int_0^t x(\tau)d\tau \le \int_0^t h(\tau)\exp\Big(\int_\tau^t g(s)ds\Big)d\tau. \tag{2.15}$$

3. Nonlinear Galerkin Method with Variable Modes

We assume that $\omega = \omega(m)$ $(> m)$ is another integer associated with m. Let $s = s(m) = \omega(m) - m$. The precise values of ω and s will be given in Section 4. Using the eigenvectors w_j, $j \in \mathbf{N}$, of the operator A, this kind of nonlinear Galerkin methods (NLG) for an approximate solution of problem (2.4)–(2.5) is implemented as follows:

The approximate solution $u_m + z_s$ is of the form

$$u_m(t) = \sum_{j=1}^m g_{jm}(t)w_j, \quad z_s(t) = \sum_{j=m+1}^{\omega} h_{jm}(t)w_j \tag{3.1}$$

where $u_m : \mathbf{R}^+ \to W_m = \mathrm{span}\,\{w_1, w_2, \cdots, w_m\}$, $z_s : \mathbf{R}^+ \to \tilde{W}_s = \mathrm{span}\,\{w_{m+1}, w_{m+2}, \cdots, w_\omega\}$. The pair (u_m, z_s) satisfies

$$\frac{d}{dt}u_m + Au_m + Cu_m + P_m(B(u_m,u_m) + B(u_m,z_s) + B(z_s,u_m)) = P_m f \tag{3.2}$$

$$Az_s + Cz_s + (P_\omega - P_m)B(u_m,u_m) = (P_\omega - P_m)f \tag{3.3}$$

$$u_m(0) = P_m u_0 \tag{3.4}$$

The system (3.2)–(3.3) is equivalent to the following:

$$\begin{aligned} \frac{d}{dt}(u_m, v) + ((A+C)u_m, v) + b(u_m,u_m,v) + b(u_m,z_s,v) \\ + b(z_s,u_m,v) = (f,v) \quad \forall v \in W_m \end{aligned} \tag{3.5}$$

$$((A+C)z_s, \tilde{v}) + b(u_m, u_m, \tilde{v}) = (f, \tilde{v}) \quad \forall \tilde{v} \in \tilde{W}_s \tag{3.6}$$

Remark 3.1. If $s = m$, then we go back to the first NLG method presented in [4]. Later on, Temam pointed out that we might change the number m of modes to approximate z_m (see e.g. [9]). His proposal is to use $(d-1)m$ modes, $d > 2$. Here, our choice is more general because $\omega = \omega(m)$ is probably a nonlinear function of m rather than a linear one $(d-1)m$.

Now we go to analyze the approximate solution produced by our method. These properties lead successively to the convergence of the approximate solution. (In the following, absolute constant c may be different when it is used in different places.)

Proposition 3.2. *The approximate solution $u_m + z_{s(m)}$ produced by* (3.2)–(3.4) *satisfies*

(i) *u_m and z_s remain both in a bounded set of $L^\infty(\mathbf{R}^+; H)$ as $m \to \infty$;*

(ii) *u_m and z_s remain both in a bounded set of $L^2(0, T; V)$ as $m \to \infty$.*

Proof. Taking $v = u_m$ in (3.5) and $\tilde{v} = z_s$ in (3.6) we have

$$\frac{d}{dt}|u_m|^2 + \|u_m\|^2 + (Cu_m, u_m) + b(u_m, z_s, u_m) + b(z_s, u_m, u_m) = (f, u_m) \tag{3.7}$$

$$\|z_s\|^2 + (Cz_s, z_s) + b(u_m, u_m, z_s) = (f, z_s) \tag{3.8}$$

Summing up (3.7) and (3.8) and due to (2.13) and $b(u_m, z_s, u_m) + b(z_s, u_m, u_m) + b(u_m, u_m, z_s) = 0$, we obtain that

$$\frac{d}{dt}|u_m|^2 + \alpha(\|u_m\|^2 + \|z_s\|^2) \le \frac{|f|^2}{\alpha\lambda_1} \tag{3.9}$$

It follows also that

$$\frac{d}{dt}|u_m|^2 + \alpha\lambda_1|u_m|^2 \le \frac{|f|^2}{\alpha\lambda_1} \tag{3.10}$$

Besides, using (2.9) and (2.13) we deduce an estimate for z_s from (3.8)

$$|z_s| \le c\Big(\frac{\lambda_m}{\lambda_{m+1}}|u_m| + \frac{1}{\lambda_{m+1}}|f|\Big)$$

Hence, by using this estimate and by integrating (3.9) and (3.10) respectively we complete the proof. □

Proposition 3.3. *We have, as $m \to \infty$, that*

(i) *$z_{s(m)} \to 0$ in $L^2(0, T; H)$ strongly;*

(ii) *$z_{s(m)} \to 0$ in $L^2(0, T; V)$ weakly;*

(iii) *$z_{s(m)} \to 0$ in $L^\infty(\mathbf{R}^+; H)$ weak-star.*

Proof. We also infer from (3.8) another inequalitry

$$\lambda_{m+1}^{1/2}|z_s| \le c(|u_m|\|u_m\| + |f|).$$

Combining it with the results of Proposition 3.2, we find that

$$\lambda_{m+1}^{1/2} z_s \quad \textit{remain bounded in } L^2(0, T; H).$$

However, we know that $\lim\limits_{m\to\infty} \lambda_m = \infty$. Hence, (i) follows.

(ii) and (iii) are direct consequences of Proposition 3.2. $\square$

Proposition 3.4. *There exist an element u^* and a subsequence m' such that, as $m' \to \infty$,*

(i) $u_{m'} \to u^*$ *in* $L^2(0,T;V)$ *weakly;*

(ii) $u_{m'} \to u^*$ *in* $L^\infty(\mathbf{R}^+;H)$ *weak-star;*

(iii) $u_{m'} \to u^*$ *in* $L^2(0,T;H)$ *strongly;*

(iv) $\dfrac{du_{m'}}{dt} \to \dfrac{du^*}{dt}$ *in* $L^2(0,T;V')$ *weakly.*

Proof. Obviously, (i) and (ii) are consequences of Proposition 3.2.

By (2.12) and the inclusion $V \subset H \subset V'$, we know that $B(u_m,u_m)$, $B(u_m,z_s)$ and $B(z_s,u_m)$ are all bounded in $L^2(0,T;V')$. Due to (2.8) and Proposition 3.3, we derive that

$$\frac{du_m}{dt} \quad \text{remains bounded in} \quad L^2(0,T;V')$$

which implies (iv).

By virtue of a classical compactness theorem[7], we get (iii). $\square$

Proposition 3.5. *For the trilinear terms, we have, as $m \to \infty$, that*

(i) $b(u_{m'},u_{m'},v) \to b(u^*,u^*,v)$ *in* $L^1(0,T)$ *strongly;*

(ii) $b(u_{m'},z_{s(m')},v) \to 0$ *in* $L^1(0,T)$ *strongly;*

(iii) $b(z_{s(m')},u_{m'},v) \to 0$ *in* $L^1(0,T)$ *strongly.*

Proof. At first, we have simply that

$$b(u_{m'},u_{m'},v) - b(u^*,u^*,v) = b(u_{m'}-u^*,u_{m'},v) + b(u^*,u_{m'}-u^*,v)$$

By Sobolev imbedding theorem and Propositions 3.2 and 3.4, we derive that

$$\begin{aligned}|b(u_{m'}-u^*,u_{m'},v)| =& \Big|\int_0^l (u_{m'}-u^*)\frac{\partial u_{m'}}{\partial x} v\,dx\Big| \\ \le& \Big(\int_0^l |u_{m'}(x)-u^*(x)|^2 dx\Big)^{1/2}\Big(\int_0^l \Big|\frac{\partial u_{m'}}{\partial x}\Big|^4 dx\Big)^{1/4}\Big(\int_0^l v^4 dx\Big)^{1/4} \\ \le& c\|u_{m'}\|\|v\||u_{m'}-u^*| \le c|u_{m'}-u^*| \quad \forall t\in[0,T]\end{aligned}$$

and

$$\begin{aligned}|b(u^*,u_{m'}-u^*,v)| =& \Big|\int_0^l u^*\frac{\partial(u_{m'}-u^*)}{\partial x} v\,dx\Big| \le \int_0^l \Big|(u_{m'}-u^*)\frac{\partial(u^*v)}{\partial x}\Big| dx \\ \le& \Big(\int_0^l |u_{m'}(x)-u^*(x)|^2 dx\Big)^{1/2}\Big(\int_0^l \Big|\frac{\partial(u^*v)}{\partial x}\Big|^2 dx\Big)^{1/2} \\ \le& |u_{m'}-u^*|\Big\{\Big(\int_0^l \Big|v\frac{\partial u^*}{\partial x}\Big|^2 dx\Big)^{1/2} + \Big(\int_0^l \Big|u^*\frac{\partial v}{\partial x}\Big|^2 dx\Big)^{1/2}\Big\} \\ \le& c|u_{m'}-u^*| \quad \forall t\in[0,T].\end{aligned}$$

So we get

$$\int_0^T |b(u_{m'},u_{m'},v) - b(u^*,u^*,v)|dt \le c\int_0^T |u_{m'}(t)-u^*(t)|dt \to 0 \quad \text{as } m'\to\infty.$$

Recalling that (2.12) and Proposition 3.2 and Proposition 3.3, we find easily the proof of (ii) and (iii). □

As a consequence, we have

Proposition 3.6. *The limit u^* satisfies*

$$\begin{cases} \dfrac{d}{dt}(u^*,v)+((A+C)u^*,v)+b(u^*,u^*,v)=(f,v) \quad \text{for all } v\in V \\ u^*(0)=u_0 \end{cases}$$

Hence, $u^*=u$ is the solution of (2.4)–(2.5).

In view of the passage to limit, we find that $u^*=u$ is the solution of problem (2.4)–(2.5). Since this solution is unique, we know from [7] that the convergences in Proposition 3.4 and Proposition 3.5 hold for the whole sequence m.

Finally we consider as in [4] that the expression

$$X_m=\frac{1}{2}|u_m(T)-u(T)|^2+\int_0^T\{\|u_m-u\|^2+(C(u_m-u),u_m-u)+\|z_s\|^2+(Cz_s,z_s)\}dt.$$

We show exactly as in [4] that $X_m\to 0$ as $m\to\infty$. This shows that

$$\begin{aligned} &u_m(T)\to u(T) \text{ strongly in} H\\ &u_m\to u,\ z_{s(m)}\to 0 \text{ strongly in} L^2(0,T;V),\ \forall T\ge 0. \end{aligned}$$

Using the Lebesgue dominated convergence theorem we have furthermore

$$u_m\to u,\ z_{s(m)}\to 0 \text{ in } L^p(0,T;H) \text{ strongly, for all } T>0, \quad \text{and all } 1\le p<\infty.$$

Up to now, we obtain the convergence result of our nonlinear Galerkin method with variable modes. *i.e.*

Theorem 3.7. *For u_0 given in H, the approximate solution u_m+z_s determined by* (3.2)–(3.4) *converges, as $m\to\infty$, to the solution of Kuramoto-Sivashinsky equation* (2.4)–(2.5) *in the following sense:*

(i) *$u_m\to u$, $z_{s(m)}\to 0$ in $L^2(0,T;V)$ and $L^p(0,T;H)$ strongly for all $T>0$, and all $1\le p<\infty$;*

(ii) *$u_m\to u$, $z_{s(m)}\to 0$ in $L^\infty(\mathbf{R}^+;H)$ weak-star.*

4. Error Estimation

In order to analyze the error of our methods, let us decompose orthogonally the space H into $H=PH\oplus QH$, where operator $Q=Q_m=I-P$. We associate to any orbit u of (2.4) in H its projectories $p=Pu$, $q=Qu$. Projecting (2.4) on PH and QH (noting that P,Q commute with A and the powers of A, and that C is a power of A or a linear combination of powers of A), we obtain a coupled system for p,q

$$\frac{dp}{dt}+Ap+Cp+PB(p+q)=Pf \tag{4.1}$$

$$\frac{dq}{dt}+Aq+Cq+QB(p+q)=Qf \tag{4.2}$$

We suppose that the initial datum u_0 in (2.5) satisfies $|u_0| \le R_0$, $\|u_0\| \le R_1$, for certain constants R_0, R_1. It is well-known that there exists a time t_* depending on R_0, R_1 and the other data α, $|f|$ and λ_1 such that for $t \ge t_*$, $|u(t)| \le M_0$, $\|u(t)\| \le M_1$, where M_0, M_1 are independent of u_0, but depend on the other data.

Upon taking the scalar product of (4.2) with q and using (2.6) we find

$$\frac{1}{2}\frac{d}{dt}|q|^2 + ((A+C)q,q) = (Qf,q) - (B(p,p),q) - (B(q,p),q) - (B(p,q),q)$$

Making use of (2.7), (2.12) and (2.13) and noticing that there exists a constant $c' > 0$ such that

$$\Big(3 - \frac{\lambda_1\|u\|^2}{\tau|Au|^2}\Big)^{1/2} \le c', \quad \forall u \in D(A)$$

We know that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|q|^2 + \alpha\|q\|^2 \le & |Qf||q| + c_6\Big(3 - \frac{\lambda_1\|p\|^2}{\tau|Ap|^2}\Big)^{1/2}\|p\|^2|q| + c_1\|p\|\|q\||q| + c_6c'\|p\|\|q\||q| \\
\le & \Big(|Qf| + c_6\Big(3 - \frac{\lambda_1}{\tau\lambda_{m+1}}\Big)^{1/2}M_1^2 + c_1M_1^2 + c_6c'M_1^2\Big)|q| \\
\le & c_1'\lambda_{m+1}^{-1/2}(|Qf| + M_1^2)\|q\| \le \frac{\alpha}{2}\|q\|^2 + \frac{c_2'}{\alpha\lambda_{m+1}}(|Qf|^2 + M_1^4) \\
\frac{d}{dt}|q|^2 + \alpha\lambda_{m+1}|q|^2 \le & \frac{c_3'}{\alpha\lambda_{m+1}}(|Qf|^2 + M_1^4)
\end{aligned}$$

By integration in time we find

$$|q(t)|^2 \le |q(t_*)|^2 \mathrm{e}^{-\alpha\lambda_{m+1}(t-t_*)} + \frac{c_3'}{\alpha^2\lambda_{m+1}^2}(|Qf|^2 + M_1^4)$$

If we write $\kappa' = \sqrt{\dfrac{2c_3'}{\alpha{\lambda_1}^2}(|f|^2 + M_1^4)}$, we can obtain

$$|q(t)| \le \kappa'\Big(\frac{\lambda_1}{\lambda_{m+1}}\Big), \quad \text{for } t \ge t_1', \quad t_1' = \max\Big\{t_*, t_* + \frac{1}{\alpha\lambda_{m+1}}\log\frac{2M_0\lambda_{m+1}^2}{\kappa_0{\lambda_1}^2}\Big\}. \tag{4.3}$$

Similarly, we can prove

$$\|q(t)\| \le \kappa'\Big(\frac{\lambda_1}{\lambda_{m+1}}\Big)^{1/2}, \quad |q'(t)| \le \kappa'\Big(\frac{\lambda_1}{\lambda_{m+1}}\Big)^{1/2}, \quad |Aq(t)| \le \kappa', \quad \text{for large } t. \tag{4.4}$$

Let us define a mapping

$$\begin{aligned}
&\Phi : PH \to H \\
&\Phi(p) = (A+C)^{-1}(I - P_m)(f - B(p)) \quad \forall p \in PH
\end{aligned}$$

and denote by $\mathcal{M}$ the graph of Φ. According to (4.3) and (4.4), we set the induced trajectories $u_m + \bar{z}_m, \bar{z}_m = \Phi(u_m)$ associated to $u(t)$ as in [10] and [2], then there exists a $\kappa > 0$ such that

$$\text{dist}\,(u, \mathcal{M}) \le |u - (u_m + \bar{z}_m)| \le \kappa\Big(\frac{\lambda_1}{\lambda_{m+1}}\Big)^{3/2} \tag{4.5}$$

Let us define another mapping

$$\Psi : PH \to H$$
$$\Psi(p) = (A+C)^{-1}(P_\omega - P_m)(f - B(p)) \quad \forall p \in PH$$

and denote by $\mathcal{N}$ the graph of Ψ. Thus, we obtain the first error estimation of our NLG method as follows:

Theorem 4.1. *There exist constants κ_1 and κ_2 such that*

$$\operatorname{dist}(u, \mathcal{N}) \le |u - (u_m + z_s)| \le \kappa_1 \Big(\frac{1}{\lambda_{m+1}}\Big)^{3/2} + \kappa_2 \Big(\frac{1}{\lambda_{\omega+1}}\Big) \tag{4.6}$$

Proof. Thanks to (4.5), (2.12) and (2.14), we deduce that

$$\begin{aligned}
\operatorname{dist}(u, \mathcal{N}) \le& |u - (u_m + z_s)| \le |u - (u_m + \bar{z}_m)| + |(u_m + \bar{z}_m) - (u_m + z_s)| \\
\le& \kappa \Big(\frac{\lambda_1}{\lambda_{m+1}}\Big)^{3/2} + |\Phi(u_m) - \Psi(u_m)| \\
\le& \kappa \Big(\frac{\lambda_1}{\lambda_{m+1}}\Big)^{3/2} + |(A+C)^{-1}(I - P_\omega)(f - B(u_m))| \\
\le& \kappa \Big(\frac{\lambda_1}{\lambda_{m+1}}\Big)^{3/2} + \frac{1}{\alpha}\frac{1}{\lambda_{\omega+1}}(|f| + c_5' M_1^2) \le \kappa_1 \Big(\frac{1}{\lambda_{m+1}}\Big)^{3/2} + \kappa_2 \Big(\frac{1}{\lambda_{\omega+1}}\Big). \qquad \square
\end{aligned}$$

Remark 4.2. It is well-known that the eigenvalues of the one-dimensinal Kuramoto-Sivashinsky equation are

$$\lambda_j = \Big(\frac{2\pi j}{l}\Big)^4, \quad j = 1, 2, \cdots \tag{4.7}$$

If we hope that the nonlinear Galerkin method with variable modes can attain to its better precision, (4.6) suggests that we choose the number of modes, $\omega(m)$, so large that $\kappa_2\Big(\frac{1}{\lambda_{\omega+1}}\Big) \le \kappa_1\Big(\frac{1}{\lambda_{m+1}}\Big)^{3/2}$. Combining with (4.7) we should at least set $O(\omega(m)) \ge \frac{3}{2}$ as $m \to \infty$. *i.e.* we should take

$$\omega(m) \ge m\sqrt{m} \tag{4.8}$$

We give a result derived from Parseval identity. (See e.g. [1])

Lemma 4.3. *If $u \in H_0^\sigma(0, l)$, then there exists a $c > 0$ such that*

$$|u - P_m u| \le c m^{-\sigma} \Big|\frac{d^\sigma u}{dx^\sigma}\Big| . \tag{4.9}$$

This lemma is frequently used in the process of error analysis. Recalling the Section 2 (also [5]), it is reasonable to devide the error estimation process into two steps: $l < 2\pi$ and $l \ge 2\pi$.

I. Case 1: $l < 2\pi$.

In this case, (3.2) and (3.3) reduce to

$$\frac{\partial u_m}{\partial t} + \frac{\partial^4 u_m}{\partial x^4} + \frac{\partial^2 u_m}{\partial x^2} + P_m\Big(u_m \frac{\partial u_m}{\partial x} + z_s \frac{\partial u_m}{\partial x} + u_m \frac{\partial z_s}{\partial x}\Big) = 0 \tag{4.10}$$

$$\frac{\partial^4 z_s}{\partial x^4} + \frac{\partial^2 z_s}{\partial x^2} + (P_\omega - P_m)\Big(u_m \frac{\partial u_m}{\partial x}\Big) = 0 \tag{4.11}$$

The results of the following two lemmas are straightforward.

Lemma 4.4. *There exists a $\beta > 0$ such that, for any $w \in D(A)$, we have*

$$((A+C)w, Aw) \geq \beta |Aw|^2 \tag{4.12}$$

Proof. Direct computation shows $\beta \leq 1 - \left(\frac{l}{2\pi}\right)^2$. □

Lemma 4.5. *For the approximate solution $u_m + z_s$ of (4.10)–(4.11), we have*

$$u_m \in L^\infty(\mathbf{R}^+; H) \cap L^2(\mathbf{R}^+; V)$$
$$z_s \in L^\infty(\mathbf{R}^+; H) \cap L^2(\mathbf{R}^+; V)$$

Proof. Thanks to Proposition 3.2, all we need to do now is to show that u_m and z_s are both in $L^2(\mathbf{R}^+; V)$. However, this is easy. Because of (3.9) and $f = 0$, we integrate it from 0 to t and get

$$|u_m(t)|^2 + \alpha \int_0^t (\|u_m\|^2 + \|z_s\|^2) d\tau \leq |u_0|^2 \quad \forall t \geq 0$$
$$\int_0^t (\|u_m\|^2 + \|z_s\|^2) d\tau \leq \frac{1}{\alpha} |u_0|^2 \quad \forall t \geq 0$$

This completes the proof. □

From now on, we denote

$$u - (u_m + z_s) = \rho(x,t) + \theta(x,t) \tag{4.13}$$

where

$$\rho(x,t) = u - P_\omega u \quad \theta(x,t) = \theta_1(x,t) + \theta_2(x,t)$$
$$\theta_1(x,t) = P_m u - u_m \quad \theta_2(x,t) = P_\omega u - P_m u - z_s$$

We have the error estimation theorems

Theorem 4.6. *For u_0 given in H, it follows that*

(i) $|u - (u_m + z_s)| = O((\omega(m)^{-\sigma} + \lambda_{m+1}^{-1} m^{-\sigma})$ *uniformly for $t > 0$*

(ii) $\int_0^\infty \|u - (u_m + z_s)\|^2 dt = O((\omega(m))^{-2\sigma} + \lambda_{m+1}^{-2} m^{-2\sigma})$

Proof. By (4.13), we know that $\theta(x,t)$ satisfies

$$\frac{\partial \theta_1}{\partial t} + A\theta_1 + C\theta_1 + P_m\Big(u\frac{\partial u}{\partial x} - \Big(u_m \frac{\partial u_m}{\partial x} + z_s \frac{\partial u_m}{\partial x} + u_m \frac{\partial z_s}{\partial x}\Big)\Big) = 0 \tag{4.14}$$

$$\frac{\partial \theta_2}{\partial t} + A\theta_2 + C\theta_2 + (P_\omega - P_m)\Big(u\frac{\partial u}{\partial x} - u_m \frac{\partial u_m}{\partial x}\Big) + \frac{\partial z_s}{\partial t} = 0 \tag{4.15}$$

Taking inner product of (4.14) with θ_1 and of (4.15) with θ_2, we obtain respectively

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\theta_1|^2 + \alpha\|\theta_1\|^2 \leq& \frac{1}{2}\Big(\sup_x |u + u_m + z_s||u - (u_m + z_s)|\Big|\frac{\partial \theta_1}{\partial x}\Big| + \int_0^l |z_s|^2 \Big|\frac{\partial \theta_1}{\partial x}\Big| dx\Big)\\ \leq& \frac{\alpha}{2}\|\theta_1\|^2 + \frac{c}{\alpha}\sup_x |u + u_m + z_s|^2 (|\rho|^2 + |\theta|^2) + \frac{c}{\alpha}\int_0^l |z_s|^4 dx\end{aligned}$$

and

$$\frac{1}{2}\frac{d}{dt}|\theta_2|^2+\alpha\|\theta_2\|^2 \leq \frac{1}{2}\sup_x|u+u_m||u-u_m|\Big|\frac{\partial\theta_2}{\partial x}\Big|+\Big|\frac{\partial z_s}{\partial t}\Big||\theta_2|$$
$$\leq \frac{\alpha}{2}\|\theta_2\|^2+\frac{c}{\alpha}\sup_x|u+u_m|^2(|\rho|^2+|\theta|^2+|z_s|^2)+\frac{c}{\alpha}\Big|\frac{\partial z_s}{\partial t}\Big|^2$$

Adding them together, we have

$$\begin{aligned}&\frac{d}{dt}(|\theta_1|^2+|\theta_2|^2)+\alpha(\|\theta_1\|^2+\|\theta_2\|^2)\\&\leq\frac{c}{\alpha}(\|u+u_m+z_s\|^2+\|u+u_m\|^2+\|z_s\|^2)(|\rho|^2+|\theta_1|^2\\&\quad+|\theta_2|^2+|z_s|^2)+\frac{c}{\alpha}\Big|\frac{\partial z_s}{\partial t}\Big|^2\end{aligned}\tag{4.16}$$

By Lemma 2.1 and Lemma 4.3 (see also [12] for the detail), we have

$$|z_s|=O(\lambda_{m+1}^{-1}m^{-\sigma})\quad\Big|\frac{\partial z_s}{\partial t}\Big|=O(\lambda_{m+1}^{-1}m^{-\sigma})\tag{4.17}$$

Applying Gronwall inequality (Lemma 2.2) to (4.16), we obtain

$$\begin{aligned}&|\theta_1(t)|^2+|\theta_2(t)|^2+\alpha\int_0^t(\|\theta_1\|^2+\|\theta_2\|^2)d\tau\\&=O((\omega(m))^{-2\sigma}+\lambda_{m+1}^{-2}m^{-2\sigma})\quad\text{uniformly for } t>0\end{aligned}$$

Note that $|\rho|=O(\omega^{-\sigma})$, we complete the proof. □

Theorem 4.7. *For u_0 given in V, it follows that*

$$\text{(i)}\ \|u-(u_m+z_s)\|=O((\omega(m))^{-\sigma}+\lambda_{m+1}^{-1}m^{-\sigma})\ \textit{uniformly for } t>0\tag{4.18}$$

$$\text{(ii)}\int_0^\infty|A(u-(u_m+z_s))|^2d\tau=O((\omega(m))^{-2\sigma}+\lambda_{m+1}^{-2}m^{-2\sigma})\tag{4.19}$$

$$\text{(iii)}\sup_x|u-(u_m+z_s)|=O((\omega(m))^{-\sigma}+\lambda_{m+1}^{-1}m^{-\sigma})\ \textit{uniformly for } t>0\tag{4.20}$$

Proof. Similarly, we take inner product of (4.16) with $A\theta_1$ and of (4.17) with $A\theta_2$. Thanks to Lemma 4.4, we obtain

$$\frac{d}{dt}\|\theta_1\|^2+\beta|A\theta_1|^2\leq\frac{c}{\beta}\max\{\|u_m+z_s\|^2,\|u\|^2,\|z_s\|^2\}(\|\rho\|^2+\|\theta\|^2+\|z_s\|^2)$$

and

$$\frac{d}{dt}\|\theta_2\|_2+\beta|A\theta_2|^2\leq\frac{c}{\beta}\Big(\max\{\|u_m\|^2,\|u\|^2\}(\|\rho\|^2+\|\theta\|^2+\|z_s\|^2)+\Big|\frac{\partial z_s}{\partial t}\Big|^2\Big)$$

Adding the corresponding inequalities, we get

$$\frac{d}{dt}(\|\theta_1\|^2+\|\theta_2\|^2)+\beta(|A\theta_1|^2+|A\theta_2|^2)$$

$$\leq \frac{c}{\beta}\Big(\max\{\|u_m\|^2, \|u\|^2, \|z_s\|^2\}(\|\rho\|^2 + \|\theta\|^2 + \|z_s\|^2) + \Big|\frac{\partial z_s}{\partial t}\Big|^2\Big)$$

and

$$\begin{aligned}&\|\theta_1(t)\|^2 + \|\theta_2(t)\|^2 + \beta\int_0^l (|A\theta_1(\tau)|^2 + |A\theta_2(\tau)|^2)d\tau\\ &=O((\omega(m)^{-2\sigma} + \lambda_{m+1}^{-2} m^{-2\sigma})\text{uniformly for } t > 0.\end{aligned}$$

This shows (i) and (ii).

By virtue of Sobolev imbedding theorem, *(iii)* is also verified. Hence, the proof is concluded. □

II. Case 2: $l \geq 2\pi$

In this case the Lemma 4.4 and Lemma 4.5 do not hold true. Therefore, the conclusion about the time t is weakened.

Theorem 4.8. *For u_0 given in H, it follows that*

$$\text{(i) } |u - (u_m + z_s)| = O((\omega(m))^{-\sigma} + \lambda_{m+1}^{1} m^{-\sigma}) \quad \textit{for } 0 < t \leq T; \tag{4.21}$$

$$\text{(ii) } \int_0^T \|u - (u_m + z_s)\|^2 dt = O((\omega(m))^{-2\sigma} + \lambda_{m+1}^{-2} m^{-2\sigma}) \tag{4.22}$$

Theorem 4.9. *For u_0 given in V, it follows that*

$$\text{(i)} \|u - (u_m + z_s)\| = O((\omega(m))^{-\sigma} + \lambda_{m+1}^{-1} m^{-\sigma}) \quad \textit{for } 0 < t \leq T; \tag{4.23}$$

$$\text{(ii)} \int_0^T |A(u - (u_m + z_s))(\tau)|^2 dt = O((\omega(m))^{-2\sigma} + \lambda_{m+1}^{-2} m^{-2\sigma}) \tag{4.24}$$

$$\text{(iii)} \sup_x |u - (u_m + z_s)| = O((\omega(m))^{-\sigma} + \lambda_{m+1}^{-1} m^{-\sigma}) \quad \textit{for } 0 < t \leq T. \tag{4.25}$$

Remark 4.10. Thanks to (4.7), the above theorems suggest that we choose

$$\omega(m) \geq m^{1+\frac{4}{\sigma}} \tag{4.26}$$

in which case the order of error reduces to $O(m^{-4-\sigma})$.

Combining with the Theorem 4.1 and the discussion there, we eventually find the least number of modes

$$\omega(m) = \omega_{\min} \overset{\triangle}{=} \gamma \cdot \max\{m[\sqrt{m}], m[m^{\frac{4}{\sigma}}]\}. \tag{4.27}$$

where γ is a positive constant. According to this choice, we use $s = \omega_{\min} - m$ modes to approximate the small structures component. The approximation involves a total of $\omega_{\min}$ modes, which brings our NLG method to the possibly highest precision.

5. Numerical Experiment

We describe here the results of computational tests performed with our nonlinear Galerkin method with variable modes (3.2)–(3.4). Comparisons are also made with the usual nonlinear Galerkin method and the classical Galerkin method. In all cases the time discretization is implicit in the linear terms and explicit in the nonlinear terms. Two examples will be discussed in the following part.

Example 5.1. We change a little bit of the equation (2.1) by adding a nonhomogeneous term $\tilde{f}$ on the right-hand side of (2.1). We use also a viscosity ν to replace A by νA in the equation. All the theoretic results are also true. In this situation the solution u is a priori chosen and $\tilde{f}$ is determined from (2.1). Hence the exact solution of the equation is known and it is easy to test safely the accuracy.

Our NLG method with variable modes here becomes:

$$\frac{\partial u_m}{\partial t} + (\nu A + C)u_m + P_m\Big(u_m\frac{\partial u_m}{\partial x} + z_s\frac{\partial u_m}{\partial x} + u_m\frac{\partial z_s}{\partial x}\Big) = P_m\tilde{f} \tag{5.1}$$

$$(\nu A + C)z_s + (P_{m+s} - P_m)\Big(u_m\frac{\partial u_m}{\partial x}\Big) = (P_{m+s} - P_m)\tilde{f} \tag{5.2}$$

$$u_m(0) = P_m u_0 \tag{5.3}$$

The exact solution $u = u(x,t)$ is:

$$u = g(t)\cdot\Big(\sin\tau x + \mathrm{e}^{-\sqrt{N}}\sin N\tau x\Big) \quad (\tau = 2\pi/l,\ N = 150)$$
$$g = g(t) = \frac{1}{1 + \frac{1}{2(\tau-\nu\tau^3)}\cdot(1 - \mathrm{e}^{(\tau^2-\nu\tau^4)t})}\cdot\mathrm{e}^{(\tau^2-\nu\tau^4)t}$$

Therefore,

$$\begin{aligned}\tilde{f} =& \tilde{f}(x,t) = (g' + \nu\tau^4 g - \tau^2 g)\sin\tau x + \frac{1}{2}\tau g^2\sin 2\tau x \\ &- \frac{1}{2}(N-1)\mathrm{e}^{-\sqrt{N}}\tau g^2\sin(N-1)\tau x + (g' + \nu\tau^4 g N^4 - \tau^2 g N^2)\mathrm{e}^{-\sqrt{N}}\sin N\tau x \\ &+ \frac{1}{2}(N+1)\mathrm{e}^{-\sqrt{N}}\tau g^2\sin(N+1)\tau x + \frac{1}{2}N\mathrm{e}^{-2\sqrt{N}}\tau g^2\sin 2N\tau x\end{aligned}$$

The time step is set to be $\triangle t = 10^{-3}$, the viscosity $\nu = 0.48$ and $l = 9\pi/5$. With $m = 64$ and $s = 106$ we employed five methods to compute.

a) NLG method with variable modes (64+106 modes).

b) NLG method (85+85 modes).

c) Galerkin method (170 modes).

d) NLG method (64+64 modes).

e) Galerkin method (128 modes).

Computational results show that the accuracy is about the same for methods a), b) and c), which is higher than that of d) and e). In like manner, the accuracy is also about the same for both d) and e). However, Table 5.1 shows the gain in computing time for the method a): the gain over b) is approximately 5%, over c) is approximately 12%. In less modes cases, both d) and e) use about the same CPU time. (See, e.g., Table 5.2).

Table 5.1 Computing Time in Second (more modes)

$t =$	methods		
	a)	b)	c)
$100 \cdot \triangle t$	9.36	9.70	10.54
$200 \cdot \triangle t$	18.71	19.54	21.19
$300 \cdot \triangle t$	27.85	29.22	31.80
$400 \cdot \triangle t$	37.16	39.02	42.35
$500 \cdot \triangle t$	46.69	48.90	52.98
$600 \cdot \triangle t$	56.04	58.57	63.50
$700 \cdot \triangle t$	65.34	68.06	74.14
$800 \cdot \triangle t$	74.42	78.19	84.74
$900 \cdot \triangle t$	84.12	87.89	95.81
$1000 \cdot \triangle t$	93.25	97.54	106.22

Table 5.2 Computing Time in Second (less modes)

$t =$	methods	
	d)	e)
$100 \cdot \triangle t$	5.83	5.83
$200 \cdot \triangle t$	11.73	11.76
$300 \cdot \triangle t$	17.55	17.65
$400 \cdot \triangle t$	23.63	23.47
$500 \cdot \triangle t$	29.50	29.56
$600 \cdot \triangle t$	35.43	35.53
$700 \cdot \triangle t$	41.21	41.56
$800 \cdot \triangle t$	47.28	47.51
$900 \cdot \triangle t$	53.08	53.03
$1000 \cdot \triangle t$	58.79	59.09

Example 5.2

We use the equation (2.1)–(2.3) without nonhomogeneous term on the right-hand side. In this case we do not know what the exact solution $u(x,t)$ is although it exists. Meanwhile we use also five methods to compute the approximate solution of the equation. Here we set: $\triangle t = 10^{-3}$, $\nu = 4, l = 9\pi/5; m = 64$ and $s = 106$.

a) NLG method with variable modes (64+106 modes).
b) NLG method (85+85 modes).
c) Galerkin method (170 modes).
d) NLG method (64+64 modes).
e) Galerkin method (128 modes).

Table 5.3 Computing Time in Second (more modes)

$t =$	methods		
	a)	b)	c)
$500 \cdot \triangle t$	37.67	39.93	44.54
$1000 \cdot \triangle t$	76.12	79.90	88.87
$1500 \cdot \triangle t$	113.43	120.20	133.25
$2000 \cdot \triangle t$	150.97	159.48	178.23
$2500 \cdot \triangle t$	188.61	199.08	222.47
$3000 \cdot \triangle t$	227.73	239.05	266.25
$3500 \cdot \triangle t$	263.91	278.08	311.53
$4000 \cdot \triangle t$	301.08	318.43	354.72
$4500 \cdot \triangle t$	343.46	357.95	399.26
$5000 \cdot \triangle t$	376.48	398.16	444.34

Table 5.4 Computing Time in Second (less modes)

$t =$	methods	
	d)	e)
$500 \cdot \triangle t$	23.58	26.51
$1000 \cdot \triangle t$	47.42	52.81
$1500 \cdot \triangle t$	71.57	78.88
$2000 \cdot \triangle t$	93.95	105.14
$2500 \cdot \triangle t$	118.59	131.75
$3000 \cdot \triangle t$	141.72	157.61
$3500 \cdot \triangle t$	164.99	182.99
$4000 \cdot \triangle t$	189.09	210.24
$4500 \cdot \triangle t$	212.74	236.54
$5000 \cdot \triangle t$	236.65	262.10

Note that here we use a variation of (3.2)–(3.4) to implement practically the nonlinear Galerkin method (with or without variable modes) in the cases a), b) and d). The variation reads:

$$\frac{\partial u_m}{\partial t} + (\nu A + C)u_m + P_m\Big(u_m \frac{\partial u_m}{\partial x} + z_s \frac{\partial u_m}{\partial x} + u_m \frac{\partial z_s}{\partial x}\Big) = 0 \tag{5.4}$$

$$\frac{\partial z_s}{\partial t} + (\nu A + C)z_s + (P_{m+s} - P_m)\Big(u_m \frac{\partial u_m}{\partial x}\Big) = 0 \tag{5.5}$$

$$u_m(0) = P_m u_0 \tag{5.6}$$

The computational results show that the gain in computing time for the method a) over b) is approximately 5.5%; the gain for a) over c) is approximately 15%. In less modes cases, the gain for the method d) over e) is approximately 10%. (See Table 5.3 and Table 5.4.)

Acknowledgement. The author would like to express his sincere thanks to Professor Guo Benyu and Professor Yang Zhonghua for their fruitful advice and encouragement. He thanks also the referees for their valuable suggestions leading to an improved presentation of this paper.

References

[1] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer-Verlag, Berlin, Heidelberg, 1988.

[2] C. Foias, O. Manley, R. Temam, Modelling of the interaction of small and large eddies in two dimensional turbulent flows, *Math. Model. Numer. Anal.,* **1** (1988), 93–118.

[3] C. Foias, O. Manley, R. Temam, Y. Trève, Asymptic analysis of the Navier-Stokes equations, *Physica D*, **9** (1983), 157–188.

[4] M. Marion, R. Temam, Nonlinear Galerkin methods, *SIAM J. Numer. Anal.*, **26** (1989), 1139–1157.

[5] B. Nicolaenko, B. Scheurer, R. Temam, Some global dynamical properties of the Kuramoto-Sivashinsky equation: nonlinear stability and attractors, *Physica D*, **16** (1985), 155–183.

[6] B. Nicolaenko, B. Scheurer, R. Temam, Some global dynamical properties of a class of pattern formation equations, *Comm. Part. Diff. Equ.*, **14** (1989), 245–297.

[7] R. Temam, Navier-Stokes Equations, 3^{rd} edition, North-Holland, Amsterdam, New York, 1984.

[8] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Appl. Math. Sci.68, Springer-Verlag, Berlin, New York, 1988.

[9] R. Temam, Dynamical systems, turbulence and the numerical solution of the Navier-Stokes equations, in *D. L. Dwoyer & R. Voigt(eds.), The Proceedings of the 11th International Conference on Numerical Methods in Fluid Dynamics, Lecture Notes in Physics*, Springer-Verlag, 1989.

[10] R. Temam, Induced trajectories and approximate inertial manifolds, *Math. Model. Numer. Anal.*, **3** (1989), 541–561.

[11] Y.-J. Wu, Studies on the approximate inertial manifolds and the numerical methods, *Advances in Mechanics*, **2** (1994), 145–153.

[12] Z.-H. Yang *et al*, Fully discrete nonlinear Galerkin methods for Kuramoto-Sivashinsky equation and their error estimates, *J. of Shanghai Univ (English Edition)*,**1** (1997), 20-27.