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A ESTIMATE OF THE RATE OF ENTROPY DISSIPATION OF HIGH RESOLUTION MUSCL TYPE GODUNOV SCHEMES*1)

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Abstract

In this paper, following the paper [7], we analysis the "sharp" estimate of the rate of entropy dissipation of the fully discrete MUSCL type Godunov schemes by the general compact theory introduced by Coquel–LeFloch [1, 2], and find: because of small viscosity of the above schemes, in the vincity of shock wave, the estimate of the above schemes is more easily obtained, but for rarefaction wave, we must impose a "sharp" condition on limiter function in order to keep its entropy dissipation and its convergence.

Key words: Hyperbolic conservation laws, MUSCL type godunov schemes, Entropy dissipation, shock wave, Rarefaction wave.

1. Introduction

Let us consider the Cauchy problems for nonlinear hyperbolic scalar conservation laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \tag{1.1}$$

$$u(x,0) = u_0(x) \tag{1.2}$$

where $f : \Re \longrightarrow \Re$ is Lipschitz continuous functions, and the initial data $u_0(x)$ is a given function in $L^1(R) \bigcap L^\infty(R)$. As it is well-known, this problem in general does not admit smooth solution, so that weak solutions in the sense of distributions must be consider.

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Moreover, an entropy condition must be added in order to ensure the uniqueness of the weak solutions of equation (1.1) and (1.2). The convergence of high resolution schemes has been investigated by many authers, such as Osher and Tadmor [3], Vila [5],and Coquel and LeFloch [1, 2]. However, some quantities depending on space mesh size are always introduced in their paper. In general, the difference schemes only depends on the ratio of the mesh size but the mesh size. So, the introduction of these quantities may be improper.

In this paper, we discuss a class of the fully discrete MUSCL type Godunov schemes based on the general theory introduced by Coquel-LeFloch [1, 2]. In section 2, we recall the Godunov schemes for scalar conservation laws and give its MUSCL type high resolution Godunov schemes. Section 3 deals with the rate of entropy dissipation of the schemes. We give a cubic estimate of the "sharp" entropy inequalities of the MUSCL type Godunov schemes in the case of shock wave. Moreover, we anaylsis the case of rarefaction wave, and find that a "sharp" condition must be imposed on the limiter functions in the case of rarefaction wave, in order to ensure entropy inequalities and convergence of the above schemes. The above limitations make these schemes fail to preserve the second order accuracy.

2. The Fully Discrete MUSCL Type Godunov Schemes

Let us consider finite difference schemes in conservative form for conservation laws (1) and (2)

$$u_j^{n+1} = u_j^n - \lambda(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}), \tag{2.1}$$

where $\lambda = \Delta t/\Delta x$ is the mesh ratio, and Δt and Δx are the variable meshsize in time and space directions, respectively. $h_{j+\frac{1}{2}}$ denotes the numerical flux

$$h_{j+\frac{1}{2}} = h(u_{j-s+1}, ..., u_{j+s}), h(u, ..., u) = f(u). \tag{2.2}$$

As well known, the weak solution of equation (1.1) and (1.2) is not unique. So let the function U(u) be any convex function,so–called the entropy function, and associated with entropy function F(u) satisfies $F'(u) = U'(u)f'(u)$. (U,F) is called an entropy pair. If the weak solution of equation (1.1) and (1.2) satisfies the following inequality:

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0, \tag{2.3}$$

in the sense of distribution to every entropy pair (U,F), the weak solution is the unque physical solution of equation (1.1) and (1.2). The inequality (2.3) is called the entropy inequality (or the entropy condition). Corresponding to the conservative scheme (2.1), the discrete entropy inequality is defined as

$$U(u_j^{n+1}) - U(u_j^n) + \lambda(H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}}) \leq 0, \tag{2.4}$$

where the discrete entropy flux

$$H_{j+\frac{1}{2}} = H(u_{j-s+1}, ..., u_{j+s}), H(u, ..., u) = F(u). \tag{2.5}$$

In this paper, the given flux f is assumed to be of C^2 class and uniformly convex. Our estimates will depend explicitly on the convexity modulus given by

$$\delta = \inf_u f''(u) \tag{2.6}$$

the infimum being taken on all u under consideration. For the sake of simplicity in the presentation (it is not a respection for our purpose and the extension to a general entropy in immediate) , we shall use in all this section and the next section the entropy (U, F) defined by

$$U(u) = \frac{u^2}{2}, \quad F(u) = \int_0^u vf'(v)dv, \ \forall u \in \Re \tag{2.7}$$

For any u_L and u_R in $\Re$, let $\frac{x}{t} \longrightarrow w(\frac{x}{t}; u_L, u_R)$ denotes the unique entropy weak solution to the Riemann problem

$$u_t + f_x(u) = 0, t > 0, x \in \Re. \tag{2.8}$$

$$u(0,x) = \begin{cases} u_L, & if \quad x < 0, \\ u_R, & if \quad x > 0. \end{cases} \tag{2.9}$$

Since the function f is strictly convex,$w(*; u_L, u_R)$ is composed of either a shock wave $(u_L > u_R)$ or a rarefaction wave $(u_L \leq u_R)$.
Now,let us discuss the Godunov scheme

$$u_j^{n+1} = u_j^n - \lambda(h^G_{j+\frac{1}{2}} - h^G_{j-\frac{1}{2}}), \tag{2.10}$$

where

$$h^G(b, a) = f(w(0^+; b, a)) \tag{2.11}$$

The above scheme admits a following decomposition (Tadmor [4])

$$u_j^{n+1} = \frac{(u^R_{j-\frac{1}{2}} + u^L_{j+\frac{1}{2}})}{2}, \tag{2.12}$$

where

$$u^R_{j-\frac{1}{2}} = \frac{2}{\Delta x}\int_0^{\frac{\Delta x}{2}} w(\frac{x}{\Delta t}; u^n_{j-1}, u^n_j)dx, \quad u^L_{j+\frac{1}{2}} = \frac{2}{\Delta x}\int_{-\frac{\Delta x}{2}}^0 w(\frac{x}{\Delta t}; u^n_{j-1}, u^n_j)dx, \tag{2.13}$$

Under the CFL stability restriction on λ,

$$\lambda \max_u |f'(u)| \leq \frac{1}{2} \tag{2.14}$$

Coquel and LeFloch [2] have given a quadratic estimate of the estimate of the entropy dissipation in the Godunov scheme.

Now,we will anaysis the sharp entropy inequality of the following MUSCL type conversions of the Godunov scheme

$$u_j^{n+1} = u_j^n - \lambda(f(w(0^+;\tilde{u}_{j+\frac{1}{2}},\hat{u}_{j+\frac{1}{2}}) - f(w(0^-;\tilde{u}_{j-\frac{1}{2}},\hat{u}_{j-\frac{1}{2}})) \tag{2.15}$$

where

$$\begin{aligned} \hat{u}_{j+\frac{1}{2}} &= u_{j+1}^n - \frac{\phi(r_{j+1})}{2}(u_{j+2}^n - u_{j+1}^n), \\ \tilde{u}_{j+\frac{1}{2}} &= u_j^n + \frac{\phi(r_j)}{2}(u_{j+1}^n - u_j^n), \\ r_j &= \frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n}. \end{aligned} \tag{2.16}$$

and $\phi(r)$ is the Limiter function (see [5] for details).

Lemma 2.1. *If the limiter function $\phi(r)$ satisfies*

$$0 \le \{\frac{\phi(r)}{r}, \phi(r)\} \le 2, \tag{2.17}$$

then

$$\begin{cases} u_{j+1} \le \hat{u}_{j+\frac{1}{2}} \le \tilde{u}_{j+\frac{1}{2}} \le u_j, & \text{if} u_{j+1} \le u_j, \\ u_j \le \tilde{u}_{j+\frac{1}{2}} \le \hat{u}_{j+\frac{1}{2}} \le u_{j+1}, & \text{if} u_{j+1} \ge u_j. \end{cases} \tag{2.18}$$

If define

$$\bar{u}_{j-\frac{1}{2}}^R = \frac{2}{\Delta x}\int_0^{\frac{\Delta x}{2}} w(\frac{x}{\Delta t};\tilde{u}_{j-\frac{1}{2}},\hat{u}_{j-\frac{1}{2}})dx, \bar{u}_{j+\frac{1}{2}}^L = \frac{2}{\Delta x}\int_{-\frac{\Delta x}{2}}^0 w(\frac{x}{\Delta t};\tilde{u}_{j+\frac{1}{2}},\hat{u}_{j+\frac{1}{2}})dx, \tag{2.19}$$

then we have

Lemma 2.2. *Under the CFL condition* (2.14)*, for each n in N and each i in Z, we have*

$$U(\bar{u}_{j-\frac{1}{2}}^R) - U(\hat{u}_{j-\frac{1}{2}}) + 2\lambda(F(\hat{u}_{j-\frac{1}{2}}) - F_{j-\frac{1}{2}}^G) \tag{2.20}$$

$$= 2\lambda J_{j-\frac{1}{2}}^R - \frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} (w(\frac{x}{\Delta t};\tilde{u}_{j-\frac{1}{2}},\hat{u}_{j-\frac{1}{2}}) - \bar{u}_{j-\frac{1}{2}}^R)^2 dx \equiv A,$$

$$U(\bar{u}_{j+\frac{1}{2}}^L) - U(\tilde{u}_{j+\frac{1}{2}}) - 2\lambda(F(\tilde{u}_{j+\frac{1}{2}}) - F_{j+\frac{1}{2}}^G) \tag{2.21}$$

$$= 2\lambda J_{j-\frac{1}{2}}^R - \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^0 (w(\frac{x}{\Delta t};\tilde{u}_{j+\frac{1}{2}},\hat{u}_{j+\frac{1}{2}}) - \bar{u}_{j-\frac{1}{2}}^R)^2 dx \equiv B,$$

where the terms $J^R_{j+\frac{1}{2}}$ and $J^L_{j+\frac{1}{2}}$ equal zero except when the Riemann solution $w(*;\tilde{u}_{j+\frac{1}{2}}, \hat{u}_{j+\frac{1}{2}})$ contains a shock wave with speed $\sigma_{j+\frac{1}{2}}$ and, in this latter case,they are given by

$$J^R_{j+\frac{1}{2}} = \begin{cases} F(\tilde{u}_{j+\frac{1}{2}}) - F(\hat{u}_{j+\frac{1}{2}}) - \sigma_{j+\frac{1}{2}}(U(\tilde{u}_{j+\frac{1}{2}}) - U(\hat{u}_{j+\frac{1}{2}})), & \text{if } \sigma_{j+\frac{1}{2}} > 0, \\ 0, \quad otherwise \end{cases} \tag{2.22}$$

$$J^L_{j+\frac{1}{2}} = \begin{cases} F(\tilde{u}_{j+\frac{1}{2}}) - F(\hat{u}_{j+\frac{1}{2}}) - \sigma_{j+\frac{1}{2}}(U(\tilde{u}_{j+\frac{1}{2}}) - U(\hat{u}_{j+\frac{1}{2}})), & \text{if } \sigma_{j+\frac{1}{2}} \leq 0, \\ 0, \quad otherwise \end{cases} \tag{2.23}$$

As first order Godunov schemes (2.10), we can write equation (2.15) in the following form:

$$u_j^{n+1} = \frac{(\bar{\bar{u}}^R_{j-\frac{1}{2}} + \bar{\bar{u}}^L_{j+\frac{1}{2}})}{2}, \tag{2.24}$$

where

$$\bar{\bar{u}}^R_{j-\frac{1}{2}} = \bar{u}^R_{j-\frac{1}{2}} + u_j + \hat{u}_{j-\frac{1}{2}} + 2\lambda(f(\hat{u}_{j-\frac{1}{2}}) - f(u_j)), = \hat{u}^R_{j-\frac{1}{2}} + (1 - 2\lambda\hat{\sigma}_j)(u_j - \hat{u}_{j-\frac{1}{2}}), \tag{2.25}$$

$$\bar{\bar{u}}^L_{j+\frac{1}{2}} = \bar{u}^L_{j+\frac{1}{2}} + u_j + \tilde{u}_{j+\frac{1}{2}} + 2\lambda(f(u_j) - f(\hat{u}_{j-\frac{1}{2}})) = \tilde{u}^L_{j+\frac{1}{2}} + (1 + 2\lambda\tilde{\sigma}_j)(u_j - \tilde{u}_{j+\frac{1}{2}}) \tag{2.26}$$

$$\hat{\sigma}_j = \frac{(f(\hat{u}_{j-\frac{1}{2}}) - f(u_j))}{\hat{u}_{j-\frac{1}{2}} - u_j)},$$

$$\tilde{\sigma}_j = \frac{(f(\tilde{u}_{j-\frac{1}{2}}) - f(u_j))}{\tilde{u}_{j-\frac{1}{2}} - u_j)},$$

Next,we will anaysis the entropy dissipation of MUSCL type Godunov schemes (2.15) from the above decomposition forms, based on the general theory introduced by Coquel and LeFloch [2].

3. The Estimate of Entropy Dissipation of a Shock Wave in MUSCL Type Godunov Schemes

In this section, we will only consider the following case: $u^n_{j+1} - u^n_j$ and $u^n_j - u^n_{j-1}$ have the same sign. Because, otherwise, from the equation(2.16), (2.17), (2.25), and (2.26), we can find that

$$\bar{\bar{u}}^R_{j-\frac{1}{2}} = \bar{u}^R_{j-\frac{1}{2}};$$

$$\bar{\bar{u}}^L_{j+\frac{1}{2}} = \bar{u}^L_{j+\frac{1}{2}}. \tag{3.1}$$

So, in this case, we have the same results as [2] in the case of shock (or rarefaction) wave.First,let us discuss the case of shock wave $(u_{j+1} < u_j)$. By Lemma 2.1, we have

Lemma 3.1. *Under the CFL condition* (2.14),

$$U(\bar{u}^R_{j-\frac{1}{2}}) - U(u_j) + 2\lambda(F(u_j) - F(\hat{u}^G_{j-\frac{1}{2}}) \tag{3.2}$$

$$= A + 2\lambda\hat{J}^R + U(\bar{\bar{u}}^R_{j-\frac{1}{2}}) - U(\bar{u}^R_{j-\frac{1}{2}}) - (1-2\lambda\hat{\sigma}_j)(U(u^n_j) - U(\hat{u}_{j-\frac{1}{2}})),$$

$$U(\bar{u}^L_{j-\frac{1}{2}}) - U(u_j) - 2\lambda(F(u_j) - F(\tilde{u}^G_{j+\frac{1}{2}}) \tag{3.3}$$

$$= B + 2\lambda\tilde{J}^L + U(\bar{\bar{u}}^L_{j+\frac{1}{2}}) - U(\bar{u}^L_{j+\frac{1}{2}}) - (1+2\lambda\tilde{\sigma}_j)(U(u^n_j) - U(\tilde{u}_{j+\frac{1}{2}})).$$

where

$$\hat{J}^R = F(u_j) - F(\hat{u}_{j-\frac{1}{2}}) - \hat{\sigma_j}(U(u_j) - U(\hat{u}_{j-\frac{1}{2}})), \tag{3.4}$$

$$\tilde{J}^L = F(\tilde{u}_{j+\frac{1}{2}}) - F(u_j) - \tilde{\sigma_j}(U(\tilde{u}_{j+\frac{1}{2}}) - U(u_j)), \tag{3.5}$$

If let $\sigma_{j-\frac{1}{2},+} = \max(0., \sigma_{j-\frac{1}{2}}), \sigma_{j+\frac{1}{2},-} = \min(0., \sigma_{j+\frac{1}{2}})$,then

$$RHS(3.2) = A + 2\lambda\hat{J}^R - \lambda\sigma_{j-\frac{1}{2},+}(1-2\lambda\ \hat{\sigma}_j) \tag{3.6}$$

$$\frac{\phi(r_j)}{r_j}(1 - \frac{\phi(r_{j-1})}{2} - \frac{\phi(r_j)}{2r_j})(u^n_j - u^n_{j-1})^2 - \frac{\lambda\hat{\sigma}_j(1-2\lambda\hat{\sigma}_j)}{4}(\frac{\phi(r_j)}{r_j})^2(u^n_j - u^n_{j-1})^2,$$

$$RHS(3.3) = B + 2\lambda\tilde{J}^L + \lambda\sigma_{j+\frac{1}{2},-}(1+2\lambda\tilde{\sigma}_j) \tag{3.7}$$

$$\phi(r_j)(1 - \frac{\phi(r_j)}{2} - \frac{\phi(r_{j+1})}{2r_{j+1}})(u^n_{j+1} - u^n_j)^2 + \frac{\lambda\tilde{\sigma}_j(1+2\lambda\tilde{\sigma}_j)}{4}(\phi(r_j)^2(u^n_{j+1} - u^n_j)^2.$$

Denote the fourth term of equation (3.6),or(3.7) as (I),(II),respectively. Since (see [2]),

$$\frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}}(w(\frac{x}{\Delta t}; \tilde{u}_{j-\frac{1}{2}}, \hat{u}_{j-\frac{1}{2}}) - \bar{u}^R_{j-\frac{1}{2}})^2 dx \tag{3.8}$$

$$= \lambda\sigma_{j-\frac{1}{2},+}(1-2\lambda\sigma_{j-\frac{1}{2},+})(\tilde{u}_{j-\frac{1}{2}} - \hat{u}_{j-\frac{1}{2}})^2,$$

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{0}(w(\frac{x}{\Delta t}; \tilde{u}_{j+\frac{1}{2}}, \hat{u}_{j+\frac{1}{2}}) - \bar{u}^L_{j-\frac{1}{2}})^2 dx \tag{3.9}$$

$$= -\lambda\sigma_{j+\frac{1}{2},-}(1+2\lambda\sigma_{j-\frac{1}{2},-})(\tilde{u}_{j+\frac{1}{2}} - \hat{u}_{j+\frac{1}{2}})^2,$$

Now, we have the following conclusion:

(1) If $\hat{\sigma} \geq 0 \geq \tilde{\sigma}$,then

$$RHS(3.2) \leq 2\lambda(J^R_{j-\frac{1}{2}} + \hat{J}^R),$$

$$RHS(3.3) \leq 2\lambda(J^L_{j-\frac{1}{2}} + \hat{J}^L),$$

under the CFL condition (2.14).

(2) If $0 \geq \hat{\sigma} \geq \tilde{\sigma}$,then

$$RHS(3.2) \leq 2\lambda(J^R_{j-\frac{1}{2}} + \hat{J}^R) + (I),$$

$$RHS(3.3) \leq 2\lambda(J^L_{j-\frac{1}{2}} + \hat{J}^L) - (I),$$

under the CFL condition

$$\lambda\sigma_{j+\frac{1}{2},+} \leq \frac{(1-\frac{\phi(r_{j+1})}{2r_{j+1}})^2 - (\frac{\phi(r_j)}{2})^2}{2((1-\frac{\phi(r_{j+1})}{2r_{j+1}})^2 + (\frac{\phi(r_j)}{2})^2)}. \tag{3.10}$$

(3) If $0 \leq \hat{\sigma} \leq \tilde{\sigma}$,then

$$RHS(3.2) \leq 2\lambda(J^R_{j-\frac{1}{2}} + \hat{J}^R) - (II)$$

$$RHS(3.3) \leq 2\lambda(J^L_{j-\frac{1}{2}} + \hat{J}^L) + (II)$$

under the CFL condition

$$\lambda\sigma_{j-\frac{1}{2},+} \leq \frac{(1-\frac{\phi(r_{j-1})}{2})^2 - (\frac{\phi(r_j)}{2r_j})^2}{2((1-\frac{\phi(r_{j-1})}{2})^2 + (\frac{\phi(r_j)}{2r_j})^2)}. \tag{3.11}$$

Therefore,

Lemma 3.2. *Conder the schemes* (2.15) *under the CFL condition*

$$\lambda sup_u \mid f'(u) \mid \leq \frac{\theta}{2}, 0 \leq \theta \leq 1, \tag{3.12}$$

Suppose that the Riemann solution $w(*; \tilde{u}_{j-\frac{1}{2}}, \hat{u}_{j-\frac{1}{2}})$ *consists of a shock wave with speed* $\sigma_{j-\frac{1}{2}}$ *,and that the Limiter* $\phi(r)$ *satisfies*

$$0 \leq \{\phi(r), \frac{\phi(r)}{r}\} \leq (1-\theta), \tag{3.13}$$

Then we have

$$U(u_j^{n+1}) - U(u_j^n) + \lambda(F^G_{j+\frac{1}{2}} - F^G_{j-\frac{1}{2}}) \leq -\lambda\delta(1-\theta)(\mid u_j^n - u_{j-1}^n \mid^3 + \mid u_j^n - u_{j-1}^n \mid^3) \tag{3.14}$$

Next, we consider the case of rarefaction wave. Denote

$$\sigma^L_{j-\frac{1}{2}} = f\prime(\tilde{u}_{j-\frac{1}{2}}), \sigma^r_{j-\frac{1}{2}} = f\prime(\hat{u}_{j-\frac{1}{2}}), \tag{3.15}$$

$$\sigma_{j-\frac{1}{2}} = \frac{f(\hat{u}_{j-\frac{1}{2}}) - f(\tilde{u}_{j-\frac{1}{2}})}{\hat{u}_{j-\frac{1}{2}} - \tilde{u}_{j-\frac{1}{2}}} \tag{3.16}$$

in this case, $J^R_{j-\frac{1}{2}}=0, J^L_{j+\frac{1}{2}}=0$ and

$$\frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}}(w(\frac{x}{\Delta t};\tilde{u}_{j-\frac{1}{2}},\hat{u}_{j-\frac{1}{2}})-\bar{u}^R_{j-\frac{1}{2}})^2dx \le \tag{3.17}$$

$$-\lambda S^R_{j-\frac{1}{2}}\{\frac{\delta}{12}((1-S^L_{j-\frac{1}{2}}) \mid \hat{u}_{j-\frac{1}{2}}-u_* \mid^3 + S^L_{j-\frac{1}{2}} \mid \hat{u}_{j-\frac{1}{2}}-\tilde{u}_{j-\frac{1}{2}} \mid^3)\}$$
$$-\lambda S^R_{j-\frac{1}{2}}\{S^L_{j-\frac{1}{2}}\sigma^L_{j-\frac{1}{2}}(1-2\lambda\sigma_{j-\frac{1}{2}})^2$$
$$+2\lambda(1-2\lambda\sigma^R_{j-\frac{1}{2}})\sigma_{j-\frac{1}{2}}{}^2\} \mid \hat{u}_{j-\frac{1}{2}}-\tilde{u}_{j-\frac{1}{2}} \mid^2$$

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{0}(w(\frac{x}{\Delta t};\tilde{u}_{j+\frac{1}{2}},\hat{u}_{j+\frac{1}{2}})-\bar{u}^L_{j-\frac{1}{2}})^2dx \le -\lambda(1-S^L_{j+\frac{1}{2}})(\frac{\delta}{12}\{S^R_{j+\frac{1}{2}} \mid \tilde{u}_{j+\frac{1}{2}}-u_* \mid^3 \tag{3.18}$$
$$+(1-S^R_{j+\frac{1}{2}}) \mid \hat{u}_{j+\frac{1}{2}}-\tilde{u}_{j+\frac{1}{2}} \mid^3)\}$$
$$-\lambda(1-S^L_{j+\frac{1}{2}})\{(1+2\lambda\sigma^L_{j+\frac{1}{2}})2\lambda\sigma^2_{j+\frac{1}{2}}-(1-S^R_{j+\frac{1}{2}})(1+2\lambda\sigma_{j+\frac{1}{2}})^2\sigma^R_{j+\frac{1}{2}}\}(\hat{u}_{j+\frac{1}{2}}-\tilde{u}_{j+\frac{1}{2}})^2$$

where S^R, S^L defined by

$$S^\alpha=\begin{cases}1, & \text{if} \quad \sigma^\alpha>0,\\ 0, & \text{if} \quad otherwise, \quad \alpha=L \quad or \quad R\end{cases} \tag{3.19}$$

and, u_* is the sonic point(i.e. $f\prime(u_*)=0$).

As the above case, we can find :

(1) $\sigma^L_{j-\frac{1}{2}}>0$, then in order to hold the following "entropy" inequalities

$$RHS(3.3) \le (II)+2\lambda\tilde{J}^L,$$

$$RHS(3.2) \le -(II)-2\lambda\tilde{J}^L,$$

we must have the "sharp" condition:

$$(\frac{\phi(r_j)}{2r_j})^2 \le \frac{\lambda\sigma^L_{j-\frac{1}{2}}(1-2\lambda\tilde{\sigma})}{\lambda\tilde{\sigma}(1+2\lambda\tilde{\sigma})}\{(1-\frac{\phi(r_{j-1})}{2}+\frac{\phi(r_j)}{2r_j})^2+(\frac{\phi(r_j)}{2r_j})^2\} \tag{3.20}$$

and

$$\{\phi(r),\frac{\phi(r)}{r}\} \le \frac{\sqrt[3]{2}}{(1+\sqrt[3]{2})} \tag{3.21}$$

(2) $\sigma^R_{j-\frac{1}{2}}>0>\sigma^L_{j-\frac{1}{2}}$, then in order to hold the following inequalities

$$RHS(3.3) \le (II)+2\lambda\tilde{J}^L$$

$$RHS(3.2) \le -(II)-2\lambda\tilde{J}^L$$

we must have the 'sharp' condition:

$$2(\lambda\sigma_{j-\frac{1}{2}})^2(1-2\lambda\tilde{\sigma})(1-\frac{\phi(r_{j-1})}{2}-\frac{\phi(r_j)}{2r_j})^2 \tag{3.22}$$

$$-\lambda\sigma^*(1-2\lambda\hat{\sigma})\frac{\phi(r_j)}{r_j}(1-\frac{\phi(r_{j-1})}{2}-\frac{\phi(r_j)}{2r_j})+\frac{\lambda\hat{\sigma}(1-2\lambda\tilde{\sigma})}{4}(\frac{\phi(r_j)}{r_j})^2$$

$$-\frac{\lambda\tilde{\sigma}(1+2\lambda\tilde{\sigma})}{4}(\frac{\phi(r_j)}{r_j})^2 \geq 0.$$

and

$$\{\phi(r), \frac{\phi(r)}{r}\} \leq \frac{(1-s)\sqrt[3]{2}}{(1+(1-s)\sqrt[3]{2})}, \ 0 < s < 1 \tag{3.23}$$

Here, we assume $u_* = s\hat{u}_{j+\frac{1}{2}} + (1-s)\tilde{u}_{j-\frac{1}{2}}$.

The case (3) $\sigma^R_{j+\frac{1}{2}} < 0$, and (4) $\sigma^L_{j+\frac{1}{2}} < 0 < \sigma^R_{j+\frac{1}{2}}$, are in similar to the above case (1) and (2), respectively.

The case (5) $\sigma^L_{j+\frac{1}{2}} > 0 > \sigma^R_{j-\frac{1}{2}}$, then

$$RHS(3.2) = (I) + 2\lambda\hat{J}^R > 0,$$

and

$$RHS(3.3) = (II) + 2\lambda\tilde{J}^L > 0.$$

So, we can not give negative estimate of entropy dissipation,because $\tilde{\sigma} \geq \hat{\sigma}$. Therefore, for rarefaction wave,in order to make the above fully discrete schemes to satisfy discrete entropy inequality, we must make $\phi(r) = 0$,i.e.make the above schemes to degenate 1–order accuracy. In this kind of modifiment, we can prove the convergence of the modified schemes (see [1, 2, 7]).

4. Conclusion

In the last sections, we have discussed the estimates of the rate of entropy dissipation in fully discrete MUSCL type Godunov schemes by using the theory of Coquel and LeFloch [1, 2] for nonlinear hyperbolic conservation laws. we have proven: because of small viscoity of Godunov scheme, the scheme can not obtain the "good" estimate of entropy dissipation in vincity of rarefactiion wave. But under some "sharp" condition and modifiments for Limiter function, we can proved the convergence of the scheme(2.15)–(2.16). Unfortunately, in this case, this modified MUSCL type Godunov scheme will not preserve the second order accuracy under these conditions. It should be further researched how to discretizate properly the entropy flux such that the discrete entropy conditions can be better consistent with the nonlinear stability of the difference schemes for hyperbolic conservation laws.

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References

[1] F. Coqual and Ph. LeFloch, Convergence of finite difference schemes for conservation laws in several space dimensions: general theory, *SIMA J. Numer. Anal.,* **30**(1993), 675-700.

[2] F. Coqual and Ph. LeFloch, Convergence of finite difference schemes for conservation laws in several space dimensions: the corrected anitidiffusion flux approach, *Math. Comp.* **57**(1991), 169-210.

[3] S. Osher and E. Tadmor, On the convergence of difference approximations to conservation Laws, *Math. Comp.,* **50**(1988), 19-51.

[4] E. Tadmor, Numerical viscosity and the entropy condition for conservative difference schemes, *Math. Comp.,* **43**(1984), 369-381.

[5] J. Vila, High order schemes and entropy condition for nonlinear hyperbolic systems of conservation Laws, Math.Comp., **50**(1988), 53-73.

[6] N. Zhao and H.Z. Tang, High resolution schemes and discrete entropy conditions for 2–D linear conservation Laws, J. Comp. Math., **13**(1995), 281-289.

[7] N. Zhao and H.M. Wu, Convergence of high resolution MUSCL type MLF schemes for conservation laws, (to appear), 1995.