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# CORRECTION METHODS FOR STEADY INCOMPRESSIBLE FLOWS*1)

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**Abstract**

Correction methods for the steady semi-periodic motion of incompressible fluid are investigated. The idea is similar to the influence matrix to solve the lack of vorticity boundary conditions. For any given boundary condition of the vorticity, the coupled vorticity-stream function formulation is solved. Then solve the governing equations with the correction boundary conditions to improve the solution. These equations are numerically solved by Fourier series truncation and finite difference method. The two numerical techniques are employed to treat the non-linear terms. The first method for small Reynolds number $R = 0 - 50$ has the same results as that in M. Anwar and S.C.R. Dennis' report. The second one for $R > 50$ obtains the reliable results.

*Key words*: Incompressible flow, vorticity, stream function, numerical solution.

## 1. Introduction

For semi-periodic incompressible fluid flows, S.C.R. Dennis and co-workers[1-4] solve the vorticity-stream function formulation of the governing equations by the series truncation and finite difference method. Since no boundary condition for the vorticity, they propose the vorticity integral conditions based on Green identity. These methods are effective. But the vorticity integral conditions are implicit. In this paper, the correction method with explicit boundary conditions is proposed. We investigate the steady two-dimensional semi-periodic flow near an infinite array of moving plane walls. This example is developed by M. Anwar and S.C.R. Dennis[3]. They get the numerical solutions by Fourier series and finite-difference approximations. Their series truncation method loses effectiveness for $R > 50$. In the computations by the correction method, we adopt the two numerical techniques to treat the non-linear terms for the various ranges of $R$. The first method is explicit. The vorticity transport equation with given boundary conditions and the Poisson equation for the stream function with Dirichlet boundary conditions are solved respectively. Then solve a homogeneous problem to correct the solutions. The numerical results for $R = 0 - 50$ are the same as that in [3]. The second method is to solve the coupled vorticity-stream function formulation with

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any given boundary condition of the vorticity. Then again solve the governing equations with the correction boundary conditions to improve the solution. The numerical results for $R > 50$ are reliable. Since the explicit boundary condition of the vorticity, difference equation of the coefficients of Fourier series can be solved by direct method in explicit method. This saves the computational work.

## 2. Governing Equations

The vorticity-stream function formulation of the steady state incompressible flow is as follows,

$$\begin{cases} \nabla^2\xi = R\left(\frac{\partial\psi}{\partial y}\frac{\partial\xi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\xi}{\partial y}\right), \\ \nabla^2\psi = -\xi, \end{cases} \tag{2.1}$$

where $\psi$ and $\xi$ are the dimensionless stream function and vorticity respectively, $R$ is the Reynolds number.

We consider the example of steady semi-periodic flow as in [3]. The flow is generated by the motion of an infinite array of walls along the y-direction. The velocity components of the moving wall are $u = 0$, $v = -\sin y$, $(-\infty \le y \le \infty)$. Since the flow is periodic and antisymmetrical for $y$, the boundary conditions are

$$\begin{aligned} &\psi = 0, \frac{\partial\psi}{\partial x} = \sin y, \text{ for } x = 0, \\ &\xi \to 0, \psi \to 0, \text{ as } x \to \infty, \\ &\psi = \xi = 0, \quad \text{for } \ y = 0 \text{ and } y = \pi. \end{aligned}$$

## 3. Method of Correction Solution

We expand $\xi$ and $\psi$ as Fourier series with respect to $y$,

$$\begin{cases} \xi(x,y) = \sum\limits_{n=1}^{\infty} g_n(x)\sin ny, \\ \psi(x,y) = \sum\limits_{n=1}^{\infty} f_n(x)\sin ny. \end{cases}$$

By substituting the above series into (2.1), we can get a system of differential equations for Fourier coefficients $g_n$ and $f_n$,

$$\begin{cases} g_n'' - n^2 g_n = r_n, & n = 1, 2, \cdots, \\ f_n'' - n^2 f_n = -g_n, & n = 1, 2, \cdots, \end{cases} \tag{3.1}$$

where

$$r_n = \frac{R}{2}\sum_{p=1}^{\infty}\{(|n-p|\, f_{|n-p|} - (n+p)f_{n+p})g_p' - p(f_{n+p}' + sgn(n-p)f_{|n-p|}')g_p\},$$

and $sgn(n-p)$ denotes the sign of $(n-p)$, with $sgn(0) = 0$. The boundary conditions in terms of $f_n$ and $g_n$ are $f_n(0) = 0$, $f_n'(0) = \delta_n$, $f_n(\infty) = 0$, $g_n(\infty) = 0$, $n = 1, 2, \cdots$, where $\delta_1 = 1$, $\delta_n = 0$, $n = 2, 3, \cdots$.

This nonlinear problem can be solved by the iterative method. Then we can treat the nonlinear terms by explicit and implicit methods. The second order ordinary equations can be approximated by finite difference method.

**3.1. Explicit correction method**

In the iterative procedure, the nonlinear terms are treated explicitly. Then the iterative procedure is

$$\begin{cases} (g_n^{(k+1)})'' - n^2 g_n^{(k+1)} = r_n^{(k)}, & n = 1, 2, \cdots, \\ (f_n^{(k+1)})'' - n^2 f_n^{(k+1)} = -g_n^{(k+1)}, & n = 1, 2, \cdots. \end{cases}$$

The correction method is to decompose the problem into a nonhomogeneous one with given boundary conditions and a homogeneous one with correction boundary conditions. The nonhomogeneous problem is as follows

$$\begin{cases} (g_{*,n}^{(k+1)})'' - n^2 g_{*,n}^{(k+1)} = r_n^{(k)}, & n = 1, 2, \cdots, \\ (f_{*,n}^{(k+1)})'' - n^2 f_{*,n}^{(k+1)} = -g_{*,n}^{(k+1)}, & n = 1, 2, \cdots, \\ g_{*,n}^{(k+1)}(0) \text{ given }, g_{*,n}^{(k+1)}(\infty) = 0, & n = 1, 2, \cdots, \\ f_{*,n}^{(k+1)}(0) = 0, \quad f_{*,n}^{(k+1)}(\infty) = 0, & n = 1, 2, \cdots. \end{cases} \tag{3.2}$$

While the homogeneous correction problem is

$$\begin{cases} (g_{**,n}^{(k+1)})'' - n^2 g_{**,n}^{(k+1)} = 0, & n = 1, 2, \cdots, \\ (f_{**,n}^{(k+1)})'' - n^2 f_{**,n}^{(k+1)} = -g_{**,n}^{(k+1)}, & n = 1, 2, \cdots, \\ g_{**,n}^{(k+1)}(\infty) = 0, & n = 1, 2, \cdots, \\ f_{**,n}^{(k+1)}(0) = 0, \quad f_{**,n}^{(k+1)}(\infty) = 0, \quad (f_{**,n}^{(k+1)})'|_\infty = \delta_n - (f_{*,n}^{(k+1)})'|_\infty, & n = 1, 2, \cdots. \end{cases} \tag{3.3}$$

Then $g_n^{(k+1)} = g_{*,n}^{(k+1)} + g_{**,n}^{(k+1)}$ and $f_n^{(k+1)} = f_{*,n}^{(k+1)} + f_{**,n}^{(k+1)}$ are the $(k+1)-$th iterative approximations of $g_n$ and $f_n$.

**3.2. Implicit correction method**

The iterative procedure is

$$\begin{cases} (g_n^{(k+1)})'' - n^2 g_n^{(k+1)} + \frac{R}{2}(2n) f_{2n}^{(k)} (g_n^{(k+1)})' + \frac{R}{2} n (f_{2n}^{(k)})' g_n^{(k+1)} = \tilde{r}_n^{(k)}, & n = 1, 2, \cdots, \\ (f_n^{(k+1)})'' - n^2 f_n^{(k+1)} = -g_n^{(k+1)}, & n = 1, 2, \cdots, \end{cases}$$

where

$$\tilde{r}_n^{(k)} = \frac{R}{2} \sum_{p=1, p\neq n}^{\infty} \{[|n-p|\, f_{|n-p|}^{(k)} - (n+p) f_{n+p}^{(k)}](g_p^{(k)})' - p[(f_{n+p}^{(k)})' + sgn(n-p)(f_{|n-p|}^{(k)})'] g_p^{(k)}\}.$$

This problem can also be solved by the correction method. First, the following problem is considered

$$\begin{cases} (g_{*,n}^{(k+1)})'' - n^2 g_{*,n}^{(k+1)} + \frac{R}{2}(2n) f_{2n}^{(k)} (g_{*,n}^{(k+1)})' + \frac{R}{2} n (f_{2n}^{(k)})' g_{*,n}^{(k+1)} = \tilde{r}_n^{(k)}, & n = 1, 2, \cdots, \\ (f_{*,n}^{(k+1)})'' - n^2 f_{*,n}^{(k+1)} = -g_{*,n}^{(k+1)}, & n = 1, 2, \cdots, \\ g_{*,n}^{(k+1)}(0) \text{ given}, \quad g_{*,n}^{(k+1)}(\infty) = 0, & n = 1, 2, \cdots, \\ f_{*,n}^{(k+1)}(0) = 0, \quad f_{*,n}^{(k+1)}(\infty) = 0, & n = 1, 2, \cdots. \end{cases} \tag{3.4}$$

Next, the correction problem is

$$\begin{cases} (g^{(k+1)}_{**,n})'' - n^2 g^{(k+1)}_{**,n} + \frac{R}{2}(2n) f^{(k)}_{2n} (g^{(k+1)}_{**,n})' + \frac{R}{2} n (f^{(k)}_{2n})' g^{(k+1)}_{**,n} = 0, & n = 1, 2, \cdots, \\ (f^{(k+1)}_{**,n})'' - n^2 f^{(k+1)}_{**,n} = -g^{(k+1)}_{**,n}, & n = 1, 2, \cdots, \\ g^{(k+1)}_{**,n}(\infty) = 0, & n = 1, 2, \cdots, \\ f^{(k+1)}_{**,n}(0) = 0, \quad f^{(k+1)}_{**,n}(\infty) = 0, \quad (f^{(k+1)}_{**,n})'|_\infty = \delta_n - (f^{(k+1)}_{*,n})'|_\infty, & n = 1, 2, \cdots. \end{cases} \tag{3.5}$$

Then $g^{(k+1)}_n = g^{(k+1)}_{*,n} + g^{(k+1)}_{**,n}$ and $f^{(k+1)}_n = f^{(k+1)}_{*,n} + f^{(k+1)}_{**,n}$.

## 4. Numerical Method

For computational convenience, the variable $x$ is transformed $z = \mathrm{e}^{-x}$. Then (3.1) is the following form

$$\begin{cases} z^2 g''_n + z g'_n - n^2 g_n = r_n, & n = 1, 2, \cdots, \\ z^2 f''_n + z f'_n - n^2 f_n = -g_n, & n = 1, 2, \cdots, \end{cases} \tag{4.1}$$

where

$$r_n = -\frac{Rz}{2} \sum_{p=1}^{\infty} \{(|n-p|\, f_{|n-p|} - (n+p) f_{n+p}) g'_p - p(f'_{n+p} + sgn(n-p) f'_{|n-p|}) g_p\}.$$

The boundary conditions become $f_n = g_n = 0$, for $z = 0$, $n = 1, 2, \cdots$, $f_n = 0$, $f'_n = -\delta_n$, for $z = 1$.

Let $M$ be a positive integer. The interval (0, 1) is divided into $M$ sub-intervals, each with the length $h = \frac{1}{M}$. All derivatives in (4.1) are approximated by central-difference quotients. The difference equations at $z_j = jh$ are

$$\begin{cases} (z_j^2 + \frac{h}{2} z_j) g_{j+1,n} - (2z_j^2 + n^2 h^2) g_{j,n} + (z_j^2 - \frac{h}{2} z_j) g_{j-1,n} = r_{j,n} h^2, \\ (z_j^2 + \frac{h}{2} z_j) f_{j+1,n} - (2z_j^2 + n^2 h^2) f_{j,n} + (z_j^2 - \frac{h}{2} z_j) f_{j-1,n} = -g_{j,n} h^2. \end{cases}$$

When the explicit correction method is used, (3.2) can be solved by the above scheme. For simplicity, we drop the superscripts $k, k+1$. The boundary conditions are

$$g_n(0) = 0, g_n(1) \text{ given }, f_n(0) = 0, f_n(1) = 0. \tag{4.2}$$

We usually take $g_n(1)$ to be the value of the $k-$th iteration. The exact solutions of (3.3) are $g_n = -2n f'_n(1) z^n$, $f_n = f'_n(1) z^n \ln z$. So the explicit correction method saves the computational work. In the implicit correction method adopted, the difference scheme of (3.4) is

$$\begin{cases} \left(z_j^2 + \frac{h}{2} z_j - \frac{h}{2} nRz_j f_{j,2n}\right) g_{j+1,n} - \left(2z_j^2 + n^2 h^2 + \frac{1}{2} nRz_j f'_{j,2n} h^2\right) g_{j,n} \\ \qquad + \left(z_j^2 - \frac{h}{2} z_j + \frac{h}{2} nRz_j f_{j,2n}\right) g_{j-1,n} = \tilde{r}_{j,n} h^2, \\ \left(z_j^2 + \frac{h}{2} z_j\right) f_{j+1,n} - (2z_j^2 + n^2 h^2) f_{j,n} + \left(z_j^2 - \frac{h}{2} z_j\right) f_{j-1,n} = -g_{j,n} h^2. \end{cases}$$

The boundary conditions are (4.2). While the difference scheme of (3.5) is

$$\begin{cases} \Big(z_j^2+\frac{h}{2}z_j-\frac{h}{2}nRz_jf_{j,2n}\Big)g_{j+1,n}-\Big(2z_j^2+n^2h^2+\frac{1}{2}nRz_jf'_{j,2n}h^2\Big)g_{j,n} \\ \quad +\Big(z_j^2-\frac{h}{2}z_j+\frac{h}{2}nRz_jf_{j,2n}\Big)g_{j-1,n}=0, \\ \Big(z_j^2+\frac{h}{2}z_j\Big)f_{j+1,n}-(2z_j^2+n^2h^2)f_{j,n}+\Big(z_j^2-\frac{h}{2}z_j\Big)f_{j-1,n}=-cg_{j,n}h^2, \\ g_n(0)=0, g_n(1)=1, f_n(0)=0, f_n(1)=0, \end{cases}$$

where $c$ is determined by the following formulation

$$c\int_0^1 z^{n-1}g_n\mathrm{d}z=-f'_n(1).$$

By the numerical integration,

$$c=-f'_n(1)/\Big\{\frac{h}{2}g_n(1)+h\sum_{j=1}^{M-1}g_{j,n}(jh)^{n-1}\Big\}.$$

Then $cg_{j,n}$, $f_{j,n}$ are the numerical solutions of (3.5).

For the iterative procedure converging, the relaxation is employed. If $g^{(k+1)}$, $f^{(k+1)}$ are obtained by correction method, the new values of $(k+1)$-th approximations are given by

$$\begin{cases} g^{(k+1)}=\omega g^{(k)}+(1-\omega)g^{(k+1)}, \\ f^{(k+1)}=\omega f^{(k)}+(1-\omega)f^{(k+1)}, \end{cases}$$

where $0\le\omega\le 1$ is a relaxation parameter.

## 5. Numerical Results

For the various ranges of $R$, results are computed by the explicit and implicit correction methods. The calculations are carried out with $h=0.025$. We take $N$ terms of the series. In Table 1, $N$ and $\omega$ are given for different $R$.

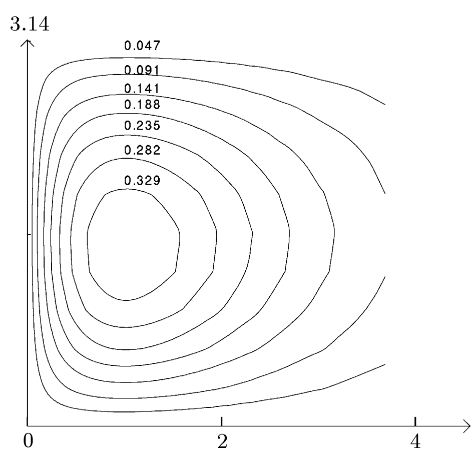


Fig.1. Streamlines for $R=10$

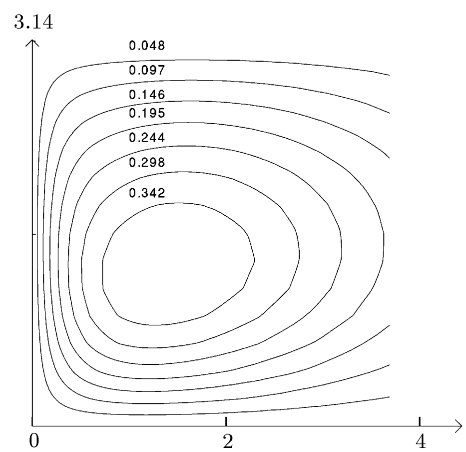


Fig.2. Streamlines for $R=40$

The results are obtained by the explicit correction method for $R = 10$ and $R = 40$. Curves of the stream-function $\psi$ are shown in Fig.1 and Fig.2. For the cases $R = 70, 120$, the implicit correction method is used. Fig.3-4 describe the results of the constant stream-function $\psi$. The results are nearly the same as that by finite difference method in [3]. These show that the correction methods are effective and feasible.

**Table 1.** $N$ and $\omega$ for different $R$

<table>
<tr><th>R</th><th>N</th><th>ω</th></tr>
<tr><td>0</td><td>5</td><td rowspan="4">0.1</td></tr>
<tr><td>10</td><td>10</td></tr>
<tr><td>40</td><td>14</td></tr>
<tr><td>70</td><td>18</td></tr>
<tr><td>120</td><td>22</td><td>0.01</td></tr>
</table>

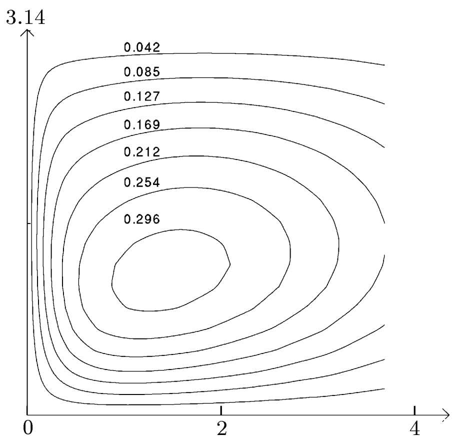


Fig.3. Streamlines for $R = 70$

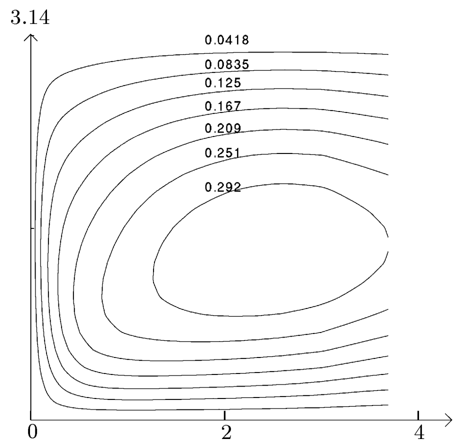


Fig.4. Streamlines for $R = 120$

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