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A GOLDSTEIN'S TYPE PROJECTION METHOD FOR A CLASS OF VARIANT VARIATIONAL INEQUALITIES*1)

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Abstract

Some optimization problems in mathematical programming can be translated to a variant variational inequality of the following form: Find a vector u^*, such that

$$Q(u^*) \in \Omega, \qquad \big(v - Q(u^*)\big)^T u^* \geq 0, \qquad \forall v \in \Omega.$$

This paper presents a simple iterative method for solving this class of variational inequalities. The method can be viewed as an extension of the Goldstein's projection method. Some results of preliminary numerical experiments are given to indicate its applications.

Key words: Variational inequality, Goldstein projection method.

1. Introduction

The classical variational inequality (VI) is to determine a vector u in a closed convex subset Ω of the n-dimentional Eucleaden space R^n such that

$$(v-u)^T F(u) \geq 0, \qquad \forall v \in \Omega, \tag{1}$$

where F is a mapping from R^n into itself. Let $\beta > 0$, since the early work of Eaves [3], it has been known that the variational inequality problem (VI) is equivalent to a projection equation

$$u = P_\Omega[u - \beta F(u)],$$

where $P_\Omega(\cdot)$ denotes the orthogonal projection map on Ω. In other words, to solve (VI) is equivalent to finding a zero point of the residue function

$$e(u, \beta) := u - P_\Omega[u - \beta F(u)].$$

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Among the existing methods (e.g., see [5–11,16,18–20]) for nonlinear variational inequality problems, the simplest is the Goldstein's projection method [6] which, starting with any $u^0 \in R^n$, iteratively updates u^{k+1} according to the formula

$$u^{k+1} = P_\Omega[u^k - \beta_k F(u^k)], \tag{2}$$

where β_k is a chosen positive stepsize. In contrast with Douglas–Rachford operator splitting method [2,12,13] for (VI), this projection method can be viewed as a simple explicit method, because u^{k+1} occurs only on the left-hand side of the equation in (2). Its convergence results can be found in [1,4] and [6].

In this paper, however, we consider a class of variant variational inequalities $(\mathrm{VI})_v$: Find an u, such that

$$Q(u) \in \Omega, \qquad (v - Q(u))^T u \geq 0, \qquad \forall v \in \Omega, \tag{3}$$

where $Q(u) : R^n \to R^n$ is a function and $\Omega \subset R^n$ is a closed convex set. The existence results on such a problem have been investigated recently by Pang and Yao [17].

There are some methods in literature ([12,14–16]), which can be used for solving $(\mathrm{VI})_v$. However, our interest in this paper is to develop the simplest method–Goldstein's type projection method for solving the variant problem (3). Throughout this paper we assume that the solution set of $(\mathrm{VI})_v$, denoted by S^*, is nonempty and the projection on Ω is simple to carry out. The Eucliden norm in this paper will be denoted by $\|\cdot\|$.

2. Motivation and the Method

As the classical variational inequality is equivalent to

$$u = P_\Omega[u - \beta F(u)]$$

with a $\beta > 0$, it is easy to prove that the variant variational inequality (3) is equivalent to the following projection equation (PE)

$$Q(u) = P_\Omega[Q(u) - \beta u]. \tag{4}$$

Let

$$r(u, \beta) := \frac{1}{\beta}(Q(u) - P_\Omega[Q(u) - \beta u]) \tag{5}$$

denote the scaled *residue* of the (PE). Then we have

$$u \in S^* \Longleftrightarrow r(u, \beta) = 0.$$

This tells us that to solve the variant variational inequality is equivalent to finding a zero point of $r(u, \beta)$. Note that the Goldstein's projection scheme (2) for (VI) can be viewed as

$$u^{k+1} = u^k - e(u^k, \beta_k). \tag{6}$$

A natural question is whether we can build a similar method for $(\mathrm{VI})_v$ based on $r(u,\beta)$. Thus, we consider the following iterative scheme:

Projection method for $(\mathbf{VI})_v$

Given $u^0 \in R^n$,

For $k = 0, 1, \ldots$, if $u^k \notin S^*$ then do:

$$u^{k+1} = u^k - r(u^k, \beta_k). \tag{7}$$

Remark. Scheme (7) can be also written as

$$u^{k+1} = -\frac{1}{\beta_k}\{(Q(u^k) - \beta_k u^k) - P_\Omega[Q(u^k) - \beta_k u^k]\}. \tag{8}$$

As in Goldstein's projection method for (VI), each iteration of the presented method for $(\mathrm{VI})_v$ consists of a function evaluation and a projection on Ω. Therefore, we say this method is a Goldstein's type method for $(\mathrm{VI})_v$.

3. Some Lemmas

In this section, we prove some lemmas, which are useful for the convergence analysis of the projection method.

Lemma 1. *Let Ω be a closed convex set in R^n, then we have*

$$\|(v - P_\Omega(v)) - (w - P_\Omega(w))\| \leq \|v - w\|, \qquad \forall v, w \in R^n. \tag{9}$$

Proof. Using a well-known inequality of the projection mapping,

$$(v - P_\Omega(v))^T (u - P_\Omega(v)) \leq 0, \qquad \forall v \in R^n, u \in \Omega,$$

we can prove

$$(v - w)^T (P_\Omega(v) - P_\Omega(w)) \geq \|P_\Omega(v) - P_\Omega(w)\|^2, \qquad \forall v, w \in R^n.$$

It follows that

$$\begin{aligned}
&\|(v - P_\Omega(v)) - (w - P_\Omega(w))\|^2 \\
&\quad = \|v - w\|^2 - 2(v - w)^T (P_\Omega(v) - P_\Omega(w)) + \|P_\Omega(v) - P_\Omega(w)\|^2 \\
&\quad \leq \|v - w\|^2 - \|P_\Omega(v) - P_\Omega(w)\|^2
\end{aligned}$$

and the lemma is proved.

Lemma 2. *The sequence $\{u^k\}$ generated by the projection method for* $(\mathrm{VI})_v$ satisfies

$$\|u^{k+1} - u^*\| \leq \|(u^k - u^*) - \frac{1}{\beta_k}(Q(u^k) - Q(u^*))\| \tag{10}$$

and

$$\|r(u^{k+1},\beta_k)\| \le \|(u^{k+1}-u^k) - \frac{1}{\beta_k}(Q(u^{k+1}-Q(u^k))\|. \tag{11}$$

Proof. By using (8) and

$$u^* = u^* - r(u^*,\beta_k) = -\frac{1}{\beta_k}\{(Q(u^*)-\beta_k u^*) - P_\Omega[Q(u^*)-\beta_k u^*]\}$$

we get

$$\begin{aligned} u^{k+1}-u^* = &-\frac{1}{\beta_k}\{(Q(u^k)-\beta_k u^k) - P_\Omega[Q(u^k)-\beta_k u^k]\} \\ &+\frac{1}{\beta_k}\{(Q(u^*)-\beta_k u^*) - P_\Omega[Q(u^*)-\beta_k u^*]\}. \end{aligned}$$

Substituting $v = Q(u^k) - \beta_k u^k$ and $w = Q(u^*) - \beta_k u^*$ in (9) we get the assertion (10). Similarly, using (5) and (8) we get

$$\begin{aligned} r(u^{k+1},\beta_k) = &\frac{1}{\beta_k}\{(Q(u^{k+1})-\beta_k u^{k+1}) - P_\Omega[Q(u^{k+1}-\beta_k u^{k+1}]\} \\ &-\frac{1}{\beta_k}\{(Q(u^k)-\beta_k u^k) - P_\Omega[Q(u^k)-\beta_k u^k]\}. \end{aligned}$$

Then the assertion (11) follows immediately from Lemma 1.

4. Convergence

The projection method for $(\mathrm{VI})_v$ in this paper generates an infinite sequence $\{u^k\}$, which is not necessarily contained in the feasible set $(\{u : Q(u) \in \Omega\})$, but under suitable assumptions, will be asymptotically feasible and converge to a solution of $(\mathrm{VI})_v$. For $u^* \in S^*$ and an arbitrary start point u^0, we denote

$$S^0(u^*) := \{u \in R^n \mid \|u-u^*\| \le \|u^0-u^*\|\},$$

and use the following definitions as in literature [4] and [17].

Definition 1. *The function Q is said to be* Lipschitz continuous *on set $S^0(u^*)$ if there is a constant $L > 0$ such that*

$$u,v \in S^0(u^*) \qquad \Rightarrow \qquad \|(Q(u)-Q(v))\| \le L\|u-v\|.$$

Definition 2. *The function Q is said to be*
i) monotone *on the set $S^0(u^*)$ if*

$$u,v \in S^0(u^*) \qquad \Rightarrow \qquad (u-v)^T(Q(u)-Q(v)) \ge 0;$$

ii) strongly monotone *on the set $S^0(u^*)$ if there exists a constant $\alpha > 0$ such that*

$$u,v \in S^0(u^*) \qquad \Rightarrow \qquad (u-v)^T(Q(u)-Q(v)) \ge \alpha\|u-v\|^2.$$

Now we prove the following convergence theorem.

Theorem 1. *If the mapping Q is Lipschitz continuous (with a constant $L > 0$) and strongly monotone (with a constant modulus $\alpha > 0$) on $S^0(u^*)$, and the stepsize β_k satisfies*

$$\beta_k > \frac{L^2}{2\alpha}, \tag{12}$$

then the sequence $\{u^k\}$ generated by the projection method is contained in $S^0(u^*)$, moreover, $\{u^k\}$ satisfies

$$\|u^{k+1} - u^*\| \le c_k \cdot \|u^k - u^*\| \tag{13}$$

and

$$\|r(u^{k+1}, \beta_k)\| \le c_k \cdot \|r(u^k, \beta_k)\|, \tag{14}$$

where

$$c_k = \sqrt{1 - \frac{2\alpha}{\beta_k} + \frac{L^2}{\beta_k^2}}.$$

Proof. The proof follows from induction. Assume that $u^k \in S^0(u^*)$, it follows from the assumptions and (10) that

$$\begin{aligned}
\|u^{k+1} - u^*\|^2 &\le \|(u^k - u^*) - \frac{1}{\beta_k}(Q(u^k) - Q(u^*))\|^2 \\
&= \|u^k - u^*\|^2 - \frac{2}{\beta_k}(u^k - u^*)^T(Q(u^k) - Q(u^*)) + \frac{1}{\beta_k^2}\|Q(u^k) - Q(u^*)\|^2 \\
&\le \Big(1 - \frac{2\alpha}{\beta_k} + \frac{L^2}{\beta_k^2}\Big)\|u^k - u^*\|^2.
\end{aligned}$$

Since $\beta_k > \frac{L^2}{2\alpha}$, we have $0 \le c_k < 1$ and $u^{k+1} \in S^0(u^*)$. Similarly, from (11) we get

$$\begin{aligned}
\|r(u^{k+1}, \beta_k)\|^2 &\le \|(u^{k+1} - u^k) - \frac{1}{\beta_k}(Q(u^{k+1} - Q(u^k))\| \\
&\le \Big(1 - \frac{2\alpha}{\beta_k} + \frac{L^2}{\beta_k^2}\Big)\|u^{k+1} - u^k\|^2 \\
&= \Big(1 - \frac{2\alpha}{\beta_k} + \frac{L^2}{\beta_k^2}\Big)\|r(u^k, \beta_k)\|^2.
\end{aligned}$$

From Theorem 1 we get directly

Corollary 1. *If the assumptions of Theorem 1 is satisfied and the stepsize in projection method for* $(\mathrm{VI})_v$ satisfies

$$\beta_L \le \beta_k \le \beta_U$$

with $\beta_L > \frac{L^2}{2\alpha}$, then the method is globally linear convergent. Moreover, if we take a constant stepsize $\beta_k \equiv \beta_c > \frac{L^2}{2\alpha}$, then both $\{\|u^k - u^*\|\}$ and $\{r(u^k, \beta_c)\}$ globally and linearly converge to zero.

In the case that Q is the gradient of a function, say q, we have the following:

Theorem 2. *Let $q(u) : R^n \to R$ be a twice continuous differentiable function on $S^0(u^*)$ and $\Lambda(\nabla^2 q(u))$ denote the set of the eigenvalues of the Hessian matrix of q, moreover,*

$$\lambda_{\min} := \inf\{\lambda \in \Lambda(\nabla^2 q(u)) \mid u \in S^0(u^*)\}$$

and

$$\lambda_{\max} := \sup\{\lambda \in \Lambda(\nabla^2 q(u)) \mid u \in S^0(u^*)\}.$$

If $Q(u) = \nabla q(u)$ is monotone on $S^0(u^*)$, then for all $\beta_k > \frac{\lambda_{\max}}{2}$ the sequence $\{u^k\}$ produced by the projection method for (VI)$_v$ *satisfies*

$$\|u^{k+1} - u^*\| \le d_k \cdot \|u^k - u^*\| \tag{15}$$

and

$$\|r(u^{k+1}, \beta_k)\| \le d_k \cdot \|r(u^k, \beta_k)\|, \tag{16}$$

where

$$d_k = \max\{\ |1 - \frac{\lambda_{\min}}{\beta_k}|,\ |1 - \frac{\lambda_{\max}}{\beta_k}|\ \}.$$

Proof. Under the assumptions follows directly

$$\begin{aligned}
\|u^{k+1} - u^*\| &\le \|(u^k - u^*) - \frac{1}{\beta_k}(Q(u^k) - Q(u^*))\| \\
&= \|(u^k - u^*) - \frac{1}{\beta_k}(\nabla^2 q(u^* + t(u^k - u^*))(u^k - u^*)\| \\
&\le \|I - \frac{1}{\beta_k}\nabla^2 q(u^* + t(u^k - u^*))\| \cdot \|u^k - u^*\| \\
&= d_k \cdot \|u^k - u^*\|,
\end{aligned}$$

with a $t \in (0,1)$. Q is monotone means that $\lambda_{\min} \ge 0$. Since $\beta_k > \frac{\lambda_{\max}}{2}$, we have $0 \le d_k \le 1$ and $u^{k+1} \in S^0(u^*)$. Similarly, from (11) we get

$$\begin{aligned}
\|r(u^{k+1}, \beta_k)\|^2 &\le \|(u^{k+1} - u^k) - \frac{1}{\beta_k}(Q(u^{k+1} - Q(u^k))\| \\
&= \|(u^{k+1} - u^k) - \frac{1}{\beta_k}(\nabla^2 q(u^k + t(u^{k+1} - u^k))(u^{k+1} - u^k)\| \\
&\le \|I - \frac{1}{\beta_k}\nabla^2 q(u^k + t(u^{k+1} - u^k))\| \cdot \|r(u^k, \beta_k)\| \\
&= d_k \cdot \|r(u^k, \beta_k)\|.
\end{aligned}$$

As a consequence of Theorem 2 we have

Corollary 2. *If the assumptions of Theorem 2 is satisfied and $\lambda_{\min} > 0$, and the stepsize in projection method for* (VI)$_v$ *satisfies*

$$\beta_L \le \beta_k \le \beta_U$$

with $\beta_L > \frac{\lambda_{\max}}{2}$*, then the method is globally linear convergent. Moreover, if we take a constant stepsize* $\beta_k \equiv \beta_c > \frac{\lambda_{\max}}{2}$*, then both* $\{\|u^k - u^*\|\}$ *and* $\{r(u^k, \beta_c)\}$ *globally and linearly converge to zero.*

5. An Example of Applications

Many optimization problems in mathematical programming are equivalent to a classical variational inequality. However, some of the problems may be formulated to *low dimensional* variant problems of the form (3), and thereby can be solved by the proposed method in this paper advantageously. As an example we consider the following least distance problem:

$$\begin{aligned} \min \quad & \frac{1}{2}\|x - c\|^2 \\ \text{s.t.} \quad & Ax \in B \end{aligned} \tag{17}$$

where $A \in R^{m \times n}$, $c \in R^n$ and $B \subset R^m$ is a closed convex set. We assume that the projection to B is simple to carry out. This problem can be written as

$$\begin{aligned} \min \quad & \frac{1}{2}\|x - c\|^2 \\ \text{s.t.} \quad & Ax - \xi = 0 \\ & \xi \in B \end{aligned} \tag{18}$$

The Lagrangean function of problem (18) is

$$L(x, \xi, y) = \frac{1}{2}x^T x - c^T x - y^T(Ax - \xi),$$

which defined on $R^n \times B \times R^m$. Under proper regularity assumption, there is a triplet $(x^*, \xi^*, y^*) \in R^n \times B \times R^m$, which is a saddle point of the Lagrangean function, i.e.,

$$L(x^*, \xi^*, y) \le L(x^*, \xi^*, y^*) \le L_{\xi \in B}(x, \xi, y^*). \tag{19}$$

From above inequalities we know that (x^*, ξ^*, y^*) is a solution of the following $(2m+n)$-dimensional variational inequality:

$$\begin{cases} x^* = A^T y^* + c, \\ (\xi - \xi^*)^T y^* \ge 0, \qquad \forall \xi \in B, \\ Ax^* = \xi^*. \end{cases} \tag{20}$$

Substituting the first and the third equation in the second of system (20), we get a m-dimensional variant variational inequality

$$(AA^T y^* + Ac) \in B, \qquad (\xi - (AA^T y^* + Ac))^T y^* \ge 0, \qquad \forall \xi \in B. \tag{21}$$

Therefore, we can solve problem (17) by solving the $(\mathrm{VI})_v$ problem (21), after obtaining a solution of (21), say y^*, we get $x^* = A^T y^* + c$, which is the solution of the least distance problem.

6. Preliminary Numerical Results

This section tests the least distance problem described in the last section with the presented method. We form the test problem as follows: The matrix A is constructed synthetically such that it has a prescribed distribution of its singular values. This is accomplished by setting

$$A := U\Sigma V^T,$$

where

$$U = I_m - 2\frac{uu^T}{\|u\|_2^2},$$
$$V = I_n - 2\frac{vv^T}{\|v\|_2^2},$$

are Householder matrices and

$$\Sigma = \mathrm{diag}(\sigma_k)$$

is a $m \times n$ diagonal matrix. The vectors u, v and c contain pseudorandomnumbers:

$$\begin{aligned}
u_1 &= 13846 \\
u_i &= (31416u_{i-1} + 13846) \bmod 46261 \quad i = 2, \ldots, m \\
v_1 &= 13846 \\
v_j &= (42108v_{j-1} + 13846) \bmod 46273 \quad j = 2, \ldots, n \\
c_1 &= 13846 \\
c_i &= (45278b_{i-1} + 13846) \bmod 46219 \quad i = 2, \ldots, m.
\end{aligned}$$

The closed convex set B in (17) is defined as

$$B := \{z \in R^m \mid \|z\| \le a\}$$

with a prescribted a. In the test problems we set $\sigma_k = \cos\frac{k\pi}{l+1} + 1, k = 1, \ldots, l = \min\{m, n\}$. The singular values of matrix A tend to cluster at the endpoints of the interval $[0, 2]$.

We take $y^0 = 0$ as starting point and the iteration formula

$$y^{k+1} = y^k - \frac{1}{\beta}\{(AA^Ty^k + Ac) - P_B[(AA^Ty^k + Ac) - \beta y^k]\}$$

with constant steplength $\beta = 2.5(> \frac{\lambda_{\max}(AA^T)}{2} \approx \frac{4}{2})$. Note that in the case $\|Ac\| > a$, $\|AA^Ty^* + Ac\| = a$ (otherwise $y^* = 0$ is the trivial solution). Therefore, we test the problem with different $a < \|Ac\|$ and the stopping criterion is if both

$$\left|\frac{\|AA^Ty + Ac\| - a}{a}\right| \leq \varepsilon \qquad \text{and} \qquad \frac{\|r(y,1)\|}{a} \leq \varepsilon$$

are satisfied for tolerance $\varepsilon = 5 \cdot 10^{-6}$.

The code was written in FORTRAN. The calculations have been performed on a 486 personal computer (without Weitek coprecessor). In following tables k denotes the iteration number by the simple projection method until the convergence criterium was met.

$m = 500, \quad n = 1000$	
a	k
$0.05 \cdot \|Ac\|$	593
$0.10 \cdot \|Ac\|$	208
$0.15 \cdot \|Ac\|$	112
$0.20 \cdot \|Ac\|$	72
$0.25 \cdot \|Ac\|$	51
$0.30 \cdot \|Ac\|$	38
$0.35 \cdot \|Ac\|$	29
$0.40 \cdot \|Ac\|$	24
$0.45 \cdot \|Ac\|$	19
$0.50 \cdot \|Ac\|$	16
$0.55 \cdot \|Ac\|$	14
$0.60 \cdot \|Ac\|$	11

$m = 1000, \quad n = 500$	
a	k
$0.05 \cdot \|Ac\|$	681
$0.10 \cdot \|Ac\|$	231
$0.15 \cdot \|Ac\|$	123
$0.20 \cdot \|Ac\|$	78
$0.25 \cdot \|Ac\|$	54
$0.30 \cdot \|Ac\|$	40
$0.35 \cdot \|Ac\|$	31
$0.40 \cdot \|Ac\|$	25
$0.45 \cdot \|Ac\|$	20
$0.50 \cdot \|Ac\|$	17
$0.55 \cdot \|Ac\|$	14
$0.60 \cdot \|Ac\|$	12

$m = 1000, \quad n = 1000$	
a	k
$0.05 \cdot \|Ac\|$	535
$0.10 \cdot \|Ac\|$	190
$0.15 \cdot \|Ac\|$	103
$0.20 \cdot \|Ac\|$	67
$0.25 \cdot \|Ac\|$	48
$0.30 \cdot \|Ac\|$	36
$0.35 \cdot \|Ac\|$	28
$0.40 \cdot \|Ac\|$	25
$0.45 \cdot \|Ac\|$	19
$0.50 \cdot \|Ac\|$	15
$0.55 \cdot \|Ac\|$	13
$0.60 \cdot \|Ac\|$	11

Conclusion remark. The main advantage of the presented method is its simplicity. Our preliminary numerical results show, that the method may be efficient for some large problems. However, we would like to point out, as the Goldstein's method for (VI), the method convergens under strict conditions and it is easy to construct a small example for which the presented method runs very poorly. For more efficient (but expensive) methods, we refer the readers to consult the papers ([12,15-16]).

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