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REAL-VALUED PERIODIC WAVELETS: CONSTRUCTION AND RELATION WITH FOURIER SERIES*[1)]

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Abstract

In this paper, we construct the real-valued periodic orthogonal wavelets. The method presented here is new. The decomposition and reconstruction formulas involve only 4 terms respectively. It demonstrates that the formulas are simpler than that in other kinds of periodic wavelets. Our wavelets are useful in applications since it is real valued. The relation between the periodic wavelets and the Fourier series is also discussed.

Key words: Periodic wavelet, Multiresolution, Fourier series, Linear independence.

1. Introduction

Wavelets have recently received a great deal of attention in such areas as signal processing and image processing ([12], [8]). Various methods to construct wavelets have been given ([14], [13], [9], [7]). It is well known that in mathematics and mathematic physics many periodic problems are encountered. In application areas, the input signals are usually finite length which may lead extra computations. To avoid this, various efforts have been made ([5], [10], [21]), among which periodization method is an important approach, i.e., the finite length input signal is first periodized, then a periodic wavelet is used which motivated an extensive study of periodic wavelets.

Y. Meyer ([14]) studied periodic multiresolutions by periodizing known wavelets. Perrier and Basdevant ([16]) stated the construction and algorithm of periodic wavelets, their algorithm makes heavy use of the fast Fourier transform(FFT). Chui and Mhasker[6]

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constructed the trigonometric wavelets. Plonka and Tasche ([17], [18]) studied p-periodic wavelets for general periodic scaling functions. Their algorithms ([19]) are based on Fourier technique. Chen Han-Lin made a full study of periodic wavelets when the scaling functions are derived from different kinds of spline functions (see [1], [2], [3], [4]). Each equation in the decomposition and reconstruction algorithms involves only two terms which does not depend on the regularity of the underlying wavelets. The discret Fourier transform is used implicitly. The approximation error estimations are also given. Koh, Lee and Tan ([11]) gave a general framework of periodic wavelets where two terms are obtained and the two-term algorithms operate on the frequency domain is also realized. Narcowich and Ward[15] investigated the periodic scaling functions and wavelets generated by continuously differentiable periodic functions with positive Fourier coefficients. They also discussed the localization of scaling functions and wavelets. The method of using the periodic wavelets, e.g., to denoise and to detect singularity, is also pointed out.

Our interest in this paper is to construct real-valued periodic orthogonal wavelets. The relation between the periodic wavelets and the Fourier series is also discussed. Our method to construct periodic wavelet is quite different from Narcowich and Ward's ([15]). The conditions of the underlying function φ is original.

This chapter is organized as follows. We will finish this section with some notations. The periodic scaling functions and nested subspaces will be constructed in Section 1. In Section 2, the dilation equations and periodic wavelets are discussed. Section 3 will devoted to the discussion of the relations between periodic wavelets and the Fourier series. Some examples will be given in Section 4.

We will use the following notations.

Let $T = Kh$ where K is a positive even integer, h a positive real number,$K = 2N$. We also use $N_j := 2^j N, K_j := 2^j K, h_j := T/K_j = h/2^j$. Note that $h_j K_j = hK = T$. $\overset{o}{L}_2$ $[0, T]$ represents the set of all periodic, square-summable functions defined on $[0, T]$, equipped with the inner product $< f, g >= \dfrac{1}{T}\int_0^T f(x)\overline{g(x)}dx$.

2. The Scaling Functions

In this this section, we will construct the scaling functions and discuss their properties. To do this, we suppose that a compactly supported real valued function $\varphi(x) \in L^2(\mathcal{R})$ satisfies

(i) For some $p \in \mathcal{Z}^+, 2p \leq N$ the support of $\varphi : supp\varphi \subset [-ph, ph]$

(ii) φ is refinable, i.e. there exists $\{c_k\} \in l^2$, s.t.

$$\varphi(x) = \sum_{k\in Z} c_k \varphi(2x - kh) \tag{2.1}$$

(iii)

$$\int_{\mathcal{R}} \varphi(x)dx \neq 0 \tag{2.2}$$

(iv) $\{\varphi(x-lh)\}_{l=-p+1}^{k+p-1}$ are linearly independent on $[0,T]$

We note that the summation in condition (2.1) is finite since φ is compactly supported (cf. [R]). Therefore we also have

$$\varphi(x) = \sum_{|k|\leq p} c_k \varphi(2x-kh) \tag{2.3}$$

Definion 2.1. *We denote the* 2^j*– dilation of* φ *as* φ^j*, i.e.* $\varphi^j(x) = \varphi(2^j x)$*. The* T*-periodization of* φ *is denoted by* Φ_α^j*.*

$$\Phi_\alpha^j(x) := \sum_{\lambda\in\mathcal{Z}} \varphi^j(x+\lambda T-\alpha h_j) \quad for \quad \alpha\in\mathcal{Z}, j\in\mathcal{Z}^+ \quad x\in[0,T]$$

Our construction will heavely depend on the following two functions

$$C_\alpha^j(x) = \sum_{\lambda=0}^{K_j-1} \cos\frac{2\pi\lambda\alpha}{K_j}\Phi_\lambda^j(x) \tag{2.4}$$

$$S_\alpha^j(x) = \sum_{\lambda=0}^{K_j-1} \sin\frac{2\pi\lambda\alpha}{K_j}\Phi_\lambda^j(x) \qquad \text{for} \qquad \alpha\in\mathcal{Z} \tag{2.5}$$

Which can be regarded as the Discrete Cosine Transform (DCT in abbreviation) and Discrete Sine Transform (DST) of $\{\Phi_\lambda^j(x)\}_{\lambda=0}^{K_j-1}$.

Definition 2.2. *A periodic multiresolution analysis (PMA) is a nested subspace sequence* $\{V_j\}_{j\geq 0}$ *satisfying*

i)

$$V_j \subseteq V_{j+1} \quad for\ any \quad j\geq 0 \tag{2.6}$$

ii)

$$\cup_{j\geq 0} V_j \quad is\ dense\ in \quad \overset{o}{L}_2\,[0,T] \tag{2.7}$$

iii) *For any* $j\geq 0$*, there exists a function* f_j *in* V_j *such that the* h_j*–shifts of* $f_j : \{f_j(\cdot - lh_j)\}_{l=0}^{K_j-1}$ *produce* V_j*, i.e.*

$$V_j = span\{f_j(\cdot - lh_j) : l=0,\cdots,K_j-q\}$$

To construct a PMA, we first note that :

Lemma 2.1.

$$\Phi_\lambda^j(x) = \sum_{|k|\leq p} c_k \Phi_{k+2\lambda}^{j+1}(x) \qquad for \qquad x\in[0,T]$$

This is a simple conclusion of Definition 2.1 and (2.1)

Therefore, if we define $V_j = \text{span}\{\Phi^j_\alpha : \alpha = 0, 1, \cdots, K_j - 1\}$, then $V_j \subseteq V_{j+1}$. To show that $\{V_j\}_{j\geq 0}$ is a PMRA, we need to verify that

Lemma 2.2. $\cup_{j\geq 0} V_j = \overset{o}{L}_2 [0, T]$

Proof. Let $V = \cup_{j\geq 0} V_j$, we shall show that $v^\perp = \{0\}$.

First, for f $\in V$; we have $f(x - h_j) \in V$ for any $j \geq 0$ which implies that V is a h_j–shift invariant space for any $j \geq 0$.

Suppose that $g(x) \in v^\perp$, then $0 =< f, g >=< f(\cdot - \lambda h_j), g >$ for $\lambda \in \mathcal{Z}, j \in \mathcal{Z}^+$

Let the Fourier coefficients of $f(x)$ and $g(x)$ be $\{s_\mu\}_{\mu\in\mathcal{Z}}$ and $\{\eta_\lambda\}_{\lambda\in\mathcal{Z}}$ respectively, i.e.

$$g(x) = \sum_{\lambda\in\mathcal{Z}} \eta_\lambda exp(-2\pi i\lambda x/T)$$
$$f(x) = \sum_{\mu\in\mathcal{Z}} s_\mu exp(-2\pi i\mu x/T)$$

Then

$$f(x - \lambda h_j) = \sum_{\mu\in\mathcal{Z}} s_\mu exp(-2\pi i\mu x/T) exp(2\pi i\mu\lambda h_j/T)$$

and

$$0 =< f(\cdot - \lambda h_j), g(\cdot) >= \sum_{\mu\in\mathcal{Z}} s_\mu \bar{\eta}_\mu exp(-2\pi i\mu\lambda h_j/T)$$
$$= \sum_{\nu=0}^{K_j-1} \sum_{\mu\in\mathcal{Z}} s_{\nu+\mu K_j} \bar{\eta}_{\nu+\mu K_j} exp(-2\pi i\nu\lambda h_j/T)$$

By the DCT theory, we get

$$\sum_{\mu\in\mathcal{Z}} s_{\nu+\mu K_j} \bar{\eta}_{\nu+\mu K_j} = 0 \qquad \text{for} \qquad j > 0, \quad \nu = 0, 1, \cdots, K_j - 1$$

Since $\sum_{\mu\in\mathcal{Z}} s_\mu \eta_\mu$ is absolutely convergent. We have

$$s_\nu \bar{\eta}_\nu = -\sum_{\mu\in\mathcal{Z}\mu\neq 0} s_{\nu+\mu K_j} \bar{\eta}_{\nu+\mu K_j}$$

tends to zero as $j \to \infty$, hence

$$s_\nu \bar{\eta}_\nu = 0 \quad for \quad any \quad \nu \in \mathcal{Z}$$

Putting $f(x) = \Phi^j_0$ note that the support of $\varphi(x)$ is contained in one period,

$$s_\nu = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \Phi^j_0 exp(2\pi i x\nu/T) dx$$
$$= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi(x) exp(2\pi i x\nu 2^{-j}/T) dx \cdot 2^{-j}$$

We obtain

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\varphi(x)exp(2\pi ix\nu 2^{-j}/T)dx\cdot 2^{-j}\bar{\eta}_\nu=0$$

Let $j\to\infty$,then we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\varphi(x)dx\bar{\eta}_\nu=0$$

Recall that (2.2) and $supp\varphi\subset[-ph,ph]\subset[-\frac{T}{2},\frac{T}{2}]$. It follows that $\eta_\nu=0$ for $\nu\in\mathcal{Z}$ which implies that $g(x)\equiv 0$ and $V^\perp=\{0\}$. The Lemma follows.

Now, we turn to discuss the basis in V_j.

Lemma 2.3. *Suppose* $\Phi_l^j; l=0,1,\cdots,K_j-1$ *is defined by Definition 1, then* $\{\Phi_l^j\}_{l=-p+1}^{K_j+p-1}$ *is linearly independent in* $[0,T]$

Proof. To this end, suppose that

$$\sum_{l=-p+1}^{K_j+p-1}c_l\varphi^j(x-lh_j)=0\qquad\text{for}\qquad x\in[0,T]$$

A change of variable $y=2^jx$ yields that

$$\sum_{l=-p+1}^{K_j+p-1}c_l\varphi(y-lh)=0\qquad\text{for}\qquad y\in[0,2^jT]$$

if y is restricted to the subinterval $[mT,(m+1)T]$,then

$$\sum_{l=mK-p+1}^{mK+p-1+K}c_l\varphi(y-lh)=0\qquad\text{for}\qquad y\in[mT,(m+1)T]$$

which is equivalent to

$$\sum_{l=mK-p+1}^{mK+p-1+K}c_l\varphi(t+mkh-lh)=0\qquad\text{for}\qquad t\in[0,T]$$

therefore

$$\sum_{l=-p+1}^{K+p-1}c_{l+mk}\varphi(t-lh)=0\qquad\text{for}\qquad t\in[0,T]$$

By the linear independence of $\{\varphi(t-lh)\}_{l=-p+1}^{K+p-1}$, we obtain that

$$c_l=0\qquad\text{for}\qquad l=mK-p+1,\cdots,(m+1)K+p-1$$

when m varies from 0 to 2^j-1, we have

$$c_l=0\qquad\text{for}\qquad l=-p+1,\cdots,2^jK+p-1$$

which implies that $\{\varphi^j(t-lh_j)\}_{l=-p+1}^{K_j+p-1}$ is linearly independent on $[0,T]$. The proof of Lemma 2.3 is finished.

Lemma 2.3 gives a basis for V_j which is generally non-orthogonal. Now, we want to give another basis for V_j which is orthogonal. Before doing that, we prove the following lemma.

Lemma 2.4. *For* $\mu = 0, 1, \cdots, K_j - 1$

$$\Phi^j_\mu(x) = \frac{1}{K_j}\sum_{\alpha=0}^{K_j-1}(C^j_\alpha(x)\cos\frac{2\pi\mu\alpha}{K_j} + S^j_\alpha(x)\sin\frac{2\pi\mu\alpha}{K_j})$$

Proof. By the definitions of Φ^j_μ, C^j_α and S^j_α, recall the following trigonometric identity,

$$\sum_{l=0}^{K_j-1}\cos\frac{2\pi l\alpha_1}{K_j}\cos\frac{2\pi l\alpha_2}{K_j} = N_j$$

for

$$\alpha_1+\alpha_2 = 0 \quad (mod \quad K_j) \quad \text{and} \quad \alpha_1-\alpha_2 \neq 0 \quad (mod \quad K_j)$$

or

$$\alpha_1-\alpha_2 = 0 \quad (mod \quad K_j) \quad \text{and} \quad \alpha_1+\alpha_2 \neq 0 \quad (mod \quad K_j)$$

For $0 \le \mu \le N_j$, we have

$$\begin{aligned}\sum_{\alpha=0}^{K_j-1} C^j_\alpha(x)\cos\frac{2\pi\mu\alpha}{K_j} &= \sum_{\alpha=0}^{K_j-1}\cos\frac{2\pi\mu\alpha}{K_j}\sum_{\lambda=0}^{K_j-1}\cos\frac{2\pi\lambda\alpha}{K_j}\Phi^j_\lambda(x)\\ &= \sum_{\lambda=0}^{K_j-1}(\sum_{\alpha=0}^{K_j-1}\cos\frac{2\pi\mu\alpha_1}{K_j}\cos\frac{2\pi\lambda\alpha_2}{K_j})\Phi^j_\lambda(x)\\ &= N_j(\Phi^j_\mu(x)+\Phi^j_{K_j-\mu}(x))\end{aligned}$$

Similarly, for $0 \le \mu \le N_j$, we have

$$\sum_{\alpha=0}^{K_j-1} S^j_\alpha(x)\sin\frac{2\pi\mu\alpha}{K_j} = N_j(\Phi^j_\mu(x)-\Phi^j_{K_j-\mu}(x))$$

It follows that for $0 \le \mu \le N_j$

$$\begin{aligned}\Phi^j_\mu(x) &= \frac{1}{K_j}\sum_{\alpha=0}^{K_j-1}(C^j_\alpha\cos\frac{2\pi\mu\alpha}{K_j} + S^j_\alpha(x)\sin\frac{2\pi\mu\alpha}{K_j})\\ \Phi^j_{K_j-\mu}(x) &= \frac{1}{K_j}\sum_{\alpha=0}^{K_j-1}(C^j_\alpha\cos\frac{2\pi\mu\alpha}{K_j} - S^j_\alpha(x)\sin\frac{2\pi\mu\alpha}{K_j})\end{aligned}$$

which is eqivalent to

$$\Phi_\mu^j(x) = \frac{1}{K_j}\sum_{\alpha=0}^{K_j-1}(C_\alpha^j \cos\frac{2\pi\mu\alpha}{K_j} + S_\alpha^j(x)\sin\frac{2\pi\mu\alpha}{K_j})$$

for $0 \le \mu \le K_j - 1$.
The proof of the lemma is completed.

Theorem 2.1. *Suppose* C_α^j, S_α^j *is defined by Definition 1,* $\mathcal{S}^j = \{C_\alpha^j : \alpha = 0, 1, \cdots, N_j,\ S_\alpha^j : \alpha = 1, 2, \cdots, N_j - 1\}$. *Then* $\mathcal{S}^j$ *is an orthogonal basis for* V_j.

Proof. First, we note that the periodicity of C_α^j and S_α^j.

$$C_\alpha^j = C_{\lambda K_j+\alpha}^j = C_{\lambda K_J-\alpha}^j, \quad S_\alpha^j = S_{\lambda K_j+\alpha}^j = -S_{\lambda K_J-\alpha}^j$$

for $\lambda \in \mathcal{Z}$
From the definition of C_α^j, S_α^j, we know that each element of $\mathcal{S}^j$ can be represented by the linear combination of $\{\Phi_\alpha^j\}_{\alpha=0}^{K_j-1}$. Lemma 2.4 and the periodicity of C_α^j and S_α^j imply that each element of $\mathcal{S}^j$ can be represented by linear combination of functions in $\mathcal{S}^j$. Since $\{\Phi_\alpha^j\}_{\alpha=0}^{K_j-1}$ is a basis for V_j. Therefore $\mathcal{S}^j$ is a basis for V_j.

Now, we need only to prove that different elements of $\mathcal{S}^j$ are orthogonal.

Only one equality

$$< C_{\alpha_1}^j, C_{\alpha_2}^j >= 0 \qquad \text{for} \qquad 0 \le \alpha_1, \alpha_2 \le N_j, \alpha_1 \ne \alpha_2$$

needs to proved, since others are similar.

By the definition of C_α^j, recall the periodicity of $\Phi_0^j(x)$ and $cosx$, for $\alpha_1 \ne \alpha_2, 0 \le \alpha_1, \alpha_2 \le N_j, \alpha_1 \ne \alpha_2$, we have

$$\begin{aligned}
< C_{\alpha_1}^j, C_{\alpha_2}^j >&= \sum_{\lambda_1=0}^{K_j-1}\sum_{\lambda_2=0}^{K_j-1}\cos\frac{2\pi\lambda_1\alpha_1}{K_j}\cos\frac{2\pi\lambda_2\alpha_2}{K_j} < \Phi_{\lambda_1}^j, \Phi_{\lambda_2}^j > \\
&= \sum_{\lambda_1=0}^{K_j-1}\sum_{\lambda_2=0}^{K_j-1}\cos\frac{2\pi\lambda_1\alpha_1}{K_j}\cos\frac{2\pi\lambda_2\alpha_2}{K_j}\cdot\frac{2}{T}\int_0^T \Phi_0^j(y)\overline{\Phi_0^j(y+(\lambda_1-\lambda_2)h_j)}dy \\
&= \sum_{\lambda_1=0}^{K_j-1}\sum_{\mu=-\lambda_1}^{K_j-1-\lambda_1}\cos\frac{2\pi\lambda_1\alpha_1}{K_j}\cos\frac{2\pi(\lambda_1+\mu)\alpha_2}{K_j}\cdot\frac{2}{T}\int_0^T \Phi_0^j(y)\bar{\Phi}_\mu^j dy \\
&= \sum_{\lambda_1=0}^{K_j-1}\sum_{\mu=0}^{K_j-1}\cos\frac{2\pi\lambda_1\alpha_1}{K_j}\cos\frac{2\pi(\lambda_1+\mu)\alpha_2}{K_j} < \Phi_0^j, \Phi_\mu^j > \\
&= \sum_{\mu=0}^{K_j-1} < \Phi_0^j, \Phi_\mu^j > \cos\frac{2\pi\mu\alpha_2}{K_j}\cdot\sum_{\lambda_1=0}^{K_j-1}\cos\frac{2\pi\lambda_1\alpha_1}{K_j}\cos\frac{2\pi\lambda_1\alpha_2}{K_j} \\
&= 0
\end{aligned}$$

The theorem follows.

3. Scaling Relations and Periodic Wavelets

In this section, the scaling relations of the orthogonal basis are given and the periodic wavelets are constructed. We will note that the scaling relations are very simple, each equation has only four terms which is independent of the regularity of wavelets or scaling functions and if the underlying function φ is symmetric, then only two terms are involved.

Theorem 3.1. *Let C_α^j, S_α^j be defined as in Definition 2.1, $\varphi(x)$ satisfy the two-scale equation* (2.3)*, and*

$$\sigma_\alpha^j = \sum_{|\mu|\le p} c_\mu \cos\frac{2\pi\mu\alpha}{K_j}, \qquad \delta_\alpha^j = \sum_{|\mu|\le p} c_\mu \sin\frac{2\pi\mu\alpha}{K_j},$$

Then, we have the following refinable equations

$$C_\alpha^j(x) = \sigma_\alpha^{j+1} C_\alpha^{j+1}(x) + \delta_\alpha^{j+1} S_\alpha^{j+1}(x) + \sigma_{K_j-\alpha}^{j+1} C_{K_j-\alpha}^{j+1}(x) + \delta_{K_j-\alpha}^{j+1} S_{K_j-\alpha}^{j+1}(x), \quad (3.1)$$

for $0 \le \alpha \le N_j$

$$S_\alpha^j(x) = -\delta_\alpha^{j+1} C_\alpha^{j+1}(x) + \sigma_\alpha^{j+1} S_\alpha^{j+1}(x) + \delta_{K_j-\alpha}^{j+1} C_{K_j-\alpha}^{j+1}(x) - \sigma_{K_j-\alpha}^{j+1} S_{K_j-\alpha}^{j+1}(x), \quad (3.2)$$

for $1 \le \alpha \le N_j - 1$

Proof. Recall (2.3), Lemma 2.1, Lemma 2.4 and the definition 2.1, we have, for $1 \le \alpha \le N_j - 1$

$$\begin{aligned}
&C_\alpha^j(x)\\
&= \sum_{\lambda=0}^{K_j-1} \cos\frac{2\pi\lambda\alpha}{K_j} \Phi_\lambda^j(x)\\
&= \sum_{\lambda=0}^{K_j-1} \cos\frac{2\pi\lambda\alpha}{K_j} \sum_{|\mu|\le p} c_\mu \Phi_{\mu+2\lambda}^{j+1}(x)\\
&= \sum_{\lambda=0}^{K_j-1} \cos\frac{2\pi\lambda\alpha}{K_j} \sum_{|\mu|\le p} c_\mu \frac{1}{K_j} \sum_{\nu=0}^{K_{j+1}-1} (C_\nu^{j+1} \cos\frac{2\pi(\mu+2\lambda)\nu}{K_{j+1}} + S_\nu^{j+1} \sin\frac{2\pi(\mu+2\lambda)\nu}{K_{j+1}})\\
&= \frac{1}{K_j} \sum_{|\mu|\le p} c_\mu \sum_{\nu=0}^{K_{j+1}-1} (C_\nu^{j+1} \cos\frac{\pi\mu\nu}{K_j} + S_\nu^{j+1} \sin\frac{\pi\mu\nu}{K_j}) \sum_{\lambda=0}^{K_j-1} \cos\frac{2\pi\lambda\alpha}{K_j} \cos\frac{2\pi\lambda\nu}{K_j}\\
&= \frac{1}{2} \sum_{t=0}^{1} \{ C_{\alpha+tK_j}^{j+1} \sigma_{\alpha+tK_j}^{j+1} + S_{\alpha+tK_j}^{j+1} \delta_{\alpha+tK_j}^{j+1} + C_{(t+1)K_j-\alpha}^{j+1} \sigma_{(t+1)K_j-\alpha}^{j+1}\\
&\qquad + S_{(t+1)K_j-\alpha}^{j+1} \delta_{(t+1)K_j-\alpha}^{j+1} \}
\end{aligned}$$

Since $\sigma_\nu^j(\delta_\nu^j)$ also possesses periodicity (antiperiodicity), the equality (3.1) follows immediately.

The proof of formula (3.2) is similar.

Theorem 3.1 establishes the relations between the basis for V_j and V_{j+1}. Now we define W_j as the orthogonal complement of V_j in V_{j+1} , that is, $W_j \perp V_j$ and $V_{j+1} = V_j + W_j$, we will denote this orthogonal sum by

$$V_{j+1} = V_j \bigoplus W_j \tag{3.3}$$

A simple conclusion of (2.6), (2.7) and (3.3) is that

$$W_j \perp W_r \qquad \text{for } j \neq r$$

and

$$\overset{o}{L}_2\,[0,T] = V_0 \bigoplus \bigoplus_{j\geq 0} W_j$$

Now, we construct an orthogonal basis for each W_j.

Theorem 3.2. *Let $\sigma_\alpha^j, \delta_\alpha^j$ be defined in Theorem 1, C_α^j, S_α^j be defined in Definition 2.1, for $1 \leq \alpha \leq N_j - 1$, we define A_α^j and B_α^j as follows*

$$A_\alpha^j(x) := \tilde{\sigma}_{K_j-\alpha}^{j+1}\tilde{C}_\alpha^{j+1} + \tilde{\delta}_{K_j-\alpha}^{j+1}\tilde{S}_\alpha^{j+1} - \tilde{\sigma}_\alpha^{j+1}\tilde{C}_{K_j-\alpha}^{j+1} - \tilde{\delta}_\alpha^{j+1}\tilde{C}_{K_j-\alpha}^{j+1}$$

$$B_\alpha^j(x) := \tilde{\delta}_{K_j-\alpha}^{j+1}\tilde{C}_\alpha^{j+1} - \tilde{\sigma}_{K_j-\alpha}^{j+1}\tilde{S}_\alpha^{j+1} + \tilde{\delta}_\alpha^{j+1}\tilde{C}_{K_j-\alpha}^{j+1} - \tilde{\sigma}_\alpha^{j+1}\tilde{S}_{K_j-\alpha}^{j+1}$$

and

$$A_0^j(x) := \tilde{\sigma}_{K_j}^{j+1}\tilde{C}_0^{j+1} - \tilde{\delta}_0^{j+1}\tilde{C}_{K_j}^{j+1} \qquad A_{N_j}^j(x) := 2(\tilde{\delta}_{N_j}^{j+1}\tilde{C}_{N_j}^{j+1} - \tilde{\sigma}_{N_j}^{j+1}\tilde{S}_{N_j}^{j+1})$$

where

$$\tilde{C}_\alpha^j = \frac{C_\alpha^j}{||C_\alpha^j||}, \qquad \tilde{S}_\alpha^j = \frac{S_\alpha^j}{||S_\alpha^j||}, \qquad \tilde{\sigma}_\alpha^j = \sigma_\alpha^j \cdot ||C_\alpha^j||, \qquad \tilde{\delta}_\alpha^j = \delta_\alpha^j \cdot ||S_\alpha^j||,$$

Then, $\{A_\alpha^j : 0 \leq \alpha \leq N_j; B_\alpha^j : 1 \leq \alpha \leq N_j - 1\}$ is an orthogonal basis for W_j. We call these A_α^j, B_α^j periodic wavelets.

Proof. From the definitions of A_α^j and B_α^j, we know that each element of $\mathcal{S}^j := \{A_\alpha^j : 0 \leq \alpha \leq N_j, B_\alpha^j : 1 \leq \alpha \leq N_j - 1\}$ belongs to V_{j+1}. A simple calculation shows that

$$< A_{\alpha_1}^j, C_{\alpha_2}^j > = < B_{\alpha_1}^j, C_{\alpha_2}^j > = < A_{\alpha_1}^j, S_{\alpha_2}^j > = < B_{\alpha_1}^j, S_{\alpha_2}^j > = 0$$

which implies that $\mathcal{S}^j \subset W_j$.

But,

$$< A^j_{\alpha_1}, A^j_{\alpha_2} >= 0 \qquad \text{for} \quad \alpha_1 \neq \alpha_2, \quad 0 \le \alpha_1, \alpha_2 \le N_j$$

$$< B^j_{\alpha_1}, B^j_{\alpha_2} >= 0 \qquad \text{for} \quad \alpha_1 \neq \alpha_2, \quad 1 \le \alpha_1, \alpha_2 \le N_j - 1$$

$$< A^j_{\alpha_1}, B^j_{\alpha_2} >= 0 \qquad \text{for} \quad 0 \le \alpha_1 \le N_j \quad 1 \le \alpha_2 \le N_j - 1$$

and

$$< A^j_\alpha, A^j_\alpha >= \|\tilde{\sigma}^{j+1}_{K_j-\alpha}\|^2 + \|\tilde{\delta}^{j+1}_{K_j-\alpha}\|^2 + \|\tilde{\sigma}^{j+1}_\alpha\|^2 + \|\tilde{\delta}^{j+1}_\alpha\|^2 \neq 0$$
$$\text{for } 0 \le \alpha \le N_j$$

$$< B^j_\alpha, B^j_\alpha >= \|\tilde{\delta}^{j+1}_{K_j-\alpha}\|^2 + \|\tilde{\sigma}^{j+1}_{K_j-\alpha}\|^2 + \|\tilde{\delta}^{j+1}_\alpha\|^2 + \|\tilde{\sigma}^{j+1}_\alpha\|^2 \neq 0$$
$$\text{for } 1 \le \alpha \le N_j - 1$$

which show that $\mathcal{S}^j$ is an orthogonal basis for W_j, and the proof of the theorem is finished.

In general, the two-scale equations involve four terms , but, when the underlying function φ is symmetric, i.e. $\varphi(x) = \varphi(-x)$, then, there are only two terms in the scaling relations and the same in the construction of the basis for W_j, that is, we have the following Theorem.

Theorem 3.3. *If $\varphi(x) = \varphi(-x)$, and δ^j_α is defined as in Theorem 3.1, then* $\delta^j_\alpha = 0$.

Proof. By (2.3) and the linear independence of $\{\varphi(\cdot - \ell h)\}_{\ell=-p+1}^{K+p-1}$ on $[0, T]$, we have,

$$\varphi(x) = \varphi(-x) = \sum_{|\mu| \le p} c_\mu \varphi(-2x - \mu h) = \sum_{|\mu| \le p} c_{-\mu} \varphi(2x - \mu h)$$

$$\sum_{|\mu| \le p} (c_\mu - c_{-\mu}) \varphi(2x - \mu h) = 0 \quad \text{ for } x \in \mathbb{R}$$

which shows that $c_\mu = c_{-\mu}$ for $|\mu| \le p$.
Hence $\delta^j_\alpha = 0$, the result follows.

4. Periodic Wavelets and Fourier Series

In this section, we will show that in some special cases, the scaling functions C^j_α and S^j_α will converge to cosine and sine functions respectively which implies that the scaling functions constructed in this paper have some stationary properties.

To this end, we suppose that $\varphi(x)$ is continuous, $supp\varphi \subset [-\frac{T}{2}, \frac{T}{2}]$ and satisfy the partition of unity,

$$\sum_{k\in\mathbf{Z}} \varphi(x+kh) = 1 \quad \text{for } x \in \mathbb{R}$$

Define operator $A^j : C[0,T] \to C[0,T]$ by

$$A^j f(x) = \sum_{\mu=0}^{K_j-1} f(\mu h_j)\Phi_\mu^j(x)$$

where $\Phi_\mu^j(x) = \sum_{\lambda\in\mathbf{Z}} \varphi(2^j(x+\lambda T) - \mu h), C[0,T]$ is the continuous function space on $[0,T]$. Then, we have the following theorem.

Theorem 4.1.

$$\lim_{j\to\infty} ||A^j f - f||_\infty = 0$$

Proof. We note first that $\sum_{\mu=0}^{K_j-1} \Phi_\mu^j(x) = 1$ for $x \in [0,T]$, therefore,

$$\begin{aligned} |A^j f(x) - f(x)| &\le \sum_{\mu=0}^{K_j-1} |f(x) - f(\mu h_j)| \cdot |\Phi_\mu^j(x)| \\ &= \sum_{|\mu-[\frac{x}{h_j}]|\le \frac{K}{2}+1} |f(x) - f(\mu h_j)| \cdot |\Phi_\mu^j(x)| \\ &\le M \sum_{|\mu-[\frac{x}{h_j}]|\le \frac{K}{2}+1} |f(x) - f(\mu h_j)| \\ &\le M(K+2) max_{|x-t|\le(\frac{K}{2}+1)h_j} |f(x) - f(t)| \end{aligned}$$

which shows that

$$\lim_{j\to\infty} ||A^j f - f||_\infty = 0.$$

Corollary 4.1. *If $\varphi(x)$ is continuous, and satisfies the conditions in Section 2. C_α^j, S_α^j are defined by (2.4), (2.5), then*

$$\lim_{j\to+\infty} C_\alpha^j(x) = \cos\frac{2\pi\alpha x}{T} \quad \textit{for} \quad \alpha = 0, 1, \cdots$$

$$\lim_{j\to+\infty} S_\alpha^j(x) = \sin\frac{2\pi\alpha x}{T} \quad \textit{for} \quad \alpha = 1, 2, \cdots$$

Remarks:

1. Corallary 4.1 shows that, for $g \in C[0,T]$, let $P_j g$ be the projection of g on V_j, then, $< P_j g, C_\alpha^j >, < P_j g, S_\alpha^j >$ are the "step" approximation of the Fourier coefficients of $g(x)$.

2. From the proof of Theorem 4.1, we know that, if $f(x)$ is smooth, and $|f'(x)| \leq M_1$, then

$$|A^j f(x) - f(x)| \leq M \cdot M_1 \frac{(K+2)^2}{2} h_j = M_2 2^{-j}$$

where M_2 is a constant independent of j, which shows that the approximation order is $O(2^{-j})$.

5. Example

In this section, we will use the above procedure to construct real value wavelets with B-spline. We point out that if $\psi(x)$ is symmetric, the final scaling relations will be simpler. Therefore we will use centered B-spline of degree 3.

Suppose $h = 1, T = 10$ and $K = 10$.

The B-spline functions are defined as follows:

$$N_0(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$$
$$N_m(x) = (N_{m-1} + N_1)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} N_{m-1}(x-t)dt, \quad m \geq 1$$

Hence

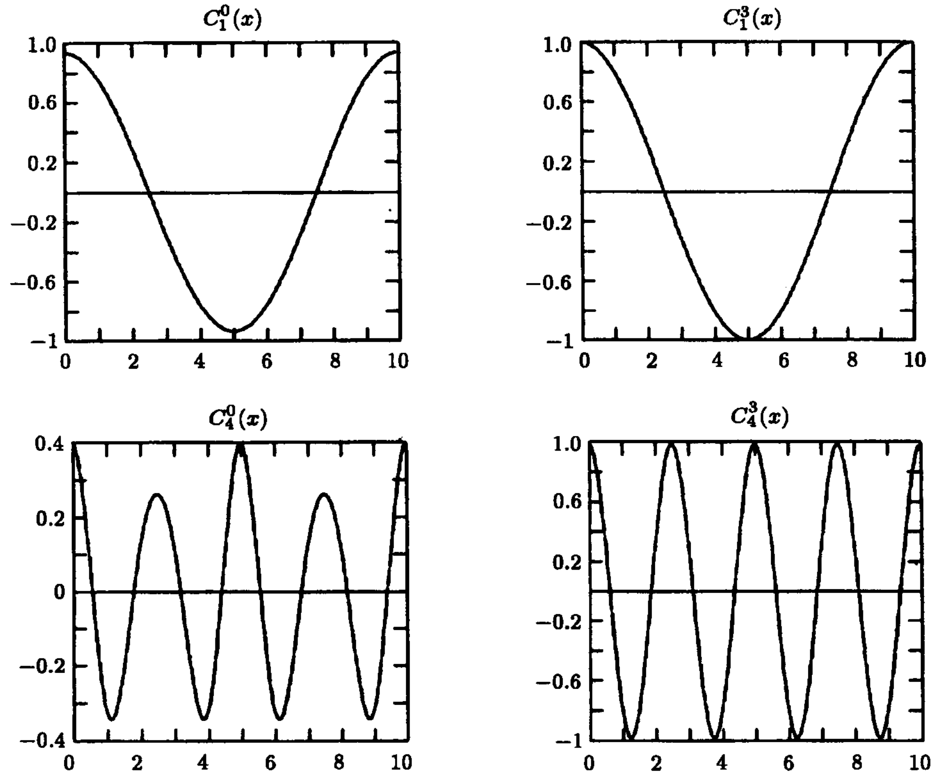
$$N_3(x) = \frac{1}{6}\sum_{j=0}^{4}(-1)^j \binom{4}{j} (x - j + 2)^3$$

and

$$N_3(x) = 2^{-3} \sum_{k=-2}^{2} \binom{4}{k+2} N_3(2x - k)$$

Putting $\psi(x) = N_3(x)$. By using the definitions in Section 2, we obtain $C_\alpha^j, S_\alpha^j, A_\alpha^j, B_\alpha^j$, for different j and $\varphi(x)$.

Here we only give the pictures of $C_1^0(x), C_1^3(x), C_4^0(x), C_4^3(x)$. From the figures we can find $C_1^0(x)$ and $C_1^3(x)$ give good approximations of $\cos(\frac{\pi x}{5})$ while $C_4^0(x)$ is a bad approximation of $\cos(\frac{4\pi x}{5})$. But $C_4^3(x)$ approximates $\cos(\frac{4\pi x}{5})$ very well.



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