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THE LIMITING CASE OF THIELE'S INTERPOLATING CONTINUED FRACTION EXPANSION[*1)]

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Abstract

By means of the determinantal formulae for inverse and reciprocal differences with coincident data points, the limiting case of Thiele's interpolating continued fraction expansion is studied in this paper and given numerical example shows that the limiting Thiele's continued fraction expansion can be determined once for all instead of carrying out computations for each step to obtain each convergent as done in [3].

Key words: Continued fraction, Inverse difference, Reciprocal difference, Expansion.

1. Introduction

When we talk about the interpolation by polynomials, it is natural for us to have at heart the Lagrange interpolation, the Hermite interpolation and the Newton interpolation. Of these interpolants, the Newton interpolating polynomial is probably most favourite because of its advantages in carrying out computations and performing limits. As we know, a Newton interpolating polynomial is established on the basis of the divided differences, whose recursive calculation makes it possible that the Newton interpolant can be obtained by adding new support points one at a time after having interpolated the previous ones. Furthermore, the relation between divided differences and derivatives allows the support points in a Newton interpolant to coincide with one another, which is beyond the conditions required by the Lagrange polynomials. It is interesting to notice that the Newton's interpolating polynomial possesses its nonlinear counterpart, i.e., the Thiele's interpolating continued fraction, which is established in terms of inverse differences. Like divided differences, inverse differences are defined in a recursive manner and allow the occurrence of repeated support points. However, implementing the limit process for inverse differences is much more complicated than for divided differences. Although both the Thiele's method and Viscovatov's method are available for the computation of the limiting case of inverse differences, they expose the shortcomings that computations have to be carried out for each step to obtain each convergent.

In this paper, starting from the Newton-Padé approximants, we offer a new kind of determinantal representations for inverse and reciprocal differences which allow the coincidence of support points. A numerical example is given to support our argument that our method is more reliable in some cases than the Thiele's method and Viscovatov's method.

Suppose $f(x)$ is a function defined on a subset G of the complex plane and $X = \{x_i | i \in \mathbb{N}\}$ is the set of points belonging to G. Let

$$C_{ij} = \begin{cases} f[x_i, \dots, x_j], & \text{for } i \le j \\ 0, & \text{for } i > j \end{cases} \tag{1.1}$$

where $f[x_i, \dots, x_j]$ denotes the divided difference of the function $f(x)$ at points $x_i, \dots, x_j$, and let

$$\omega_i(x) = (x - x_0)(x - x_1) \cdots (x - x_{i-1}) \tag{1.2}$$

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with $\omega_0(x)=1$, then $f(x)$ can formally be expanded as the following Newton series

$$f(x)=\sum_{i=0}^{\infty} C_{0i}\omega_i(x) \tag{1.3}$$

It is known to all that one may find two polynomials

$$P_{m,n}(x)=\sum_{i=0}^{m} a_i\omega_i(x) \tag{1.4}$$

and

$$Q_{m,n}(x)=\sum_{i=0}^{n} b_i\omega_i(x) \tag{1.5}$$

such that

$$f(x)Q_{m,n}(x)-P_{m,n}(x)=\sum_{i\geq m+n+1} d_i\omega_i(x), \tag{1.6}$$

which is the very Newton-Padé approximation problem of order (m,n) for $f(x)$ (see [1], [3] or [9]). It is not difficult to verify that if the rank of the matrix

$$\begin{bmatrix} C_{0,m+1} & C_{1,m+1} & \cdots & C_{n,m+1} \\ C_{0,m+2} & C_{1,m+2} & \cdots & C_{n,m+2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,m+n} & C_{1,m+n} & \cdots & C_{n,m+n} \end{bmatrix} \tag{1.7}$$

is maximal, then $P_{m,n}(x)$ and $Q_{m,n}(x)$ possess the following determinant representation

$$P_{m,n}(x)=C\begin{vmatrix} F_{0,m}(x) & F_{1,m}(x) & \cdots & F_{n,m}(x) \\ C_{0,m+1} & C_{1,m+1} & \cdots & C_{n,m+1} \\ C_{0,m+2} & C_{1,m+2} & \cdots & C_{n,m+2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,m+n} & C_{1,m+n} & \cdots & C_{n,m+n} \end{vmatrix} \tag{1.8}$$

$$Q_{m,n}(x)=C\begin{vmatrix} \omega_0(x) & \omega_1(x) & \cdots & \omega_n(x) \\ C_{0,m+1} & C_{1,m+1} & \cdots & C_{n,m+1} \\ C_{0,m+2} & C_{1,m+2} & \cdots & C_{n,m+2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,m+n} & C_{1,m+n} & \cdots & C_{n,m+n} \end{vmatrix} \tag{1.9}$$

where

$$F_{i,j}(x)=\begin{cases} \sum_{k=i}^{j} C_{i,k}\omega_k(x), & \text{for } i\leq j \\ 0, & \text{for } i>j \end{cases} \tag{1.10}$$

and C is some nonzero normalization constant.

2. Thiele's Continued Fraction

Thiele's interpolating continued fraction is a continued fraction of the following form (see [7] and [8])

$$\varphi_0[x_0]+\frac{x-x_0}{|\,\varphi_1[x_0,x_1]}+\frac{x-x_1}{|\,\varphi_2[x_0,x_1,x_2]}+\cdots \tag{2.1}$$

where $\varphi_i[x_0,x_1,\ldots,x_i], i=0,1,2,\ldots$, are the inverse differences defined as follows

$$\begin{aligned} \varphi_0[x] &= f(x) \\ \varphi_1[x_0,x_1] &= \frac{x_1-x_0}{\varphi_0[x_1]-\varphi_0[x_0]} \\ \varphi_k[x_0,x_1,\ldots,x_k] &= \frac{x_k-x_{k-1}}{\varphi_{k-1}[x_0,x_1,\ldots,x_{k-2},x_k]-\varphi_{k-1}[x_0,x_1,\ldots,x_{k-2},x_{k-1}]} \end{aligned} \tag{2.2}$$

for every $x_0,x_1,\ldots,x_k$ and x in G.

Lemma 1. ([2]) *denote by $R_n(x)$ the n^{th} convergent of (2.1), i.e.,*

$$R_n(x) = \varphi_0[x_0] + \frac{x-x_0|}{|\varphi_1[x_0,x_1]} + \frac{x-x_1|}{|\varphi_2[x_0,x_1,x_2]} + \cdots + \frac{x-x_{n-1}|}{|\varphi_n[x_0,x_1,\ldots,x_n]},$$

then $R_n(x)$ is a rational function of type $([\frac{n+1}{2}]/[\frac{n}{2}])$ satisfying

$$R_n(x_i) = f(x_i), \;\; i = 0, 1, \ldots, n.$$

Theorem 1. *Let*

$$\begin{aligned} \bar{R}_n(x) &= a_0 + \frac{x-x_0|}{|a_1} + \frac{x-x_1|}{|a_2} + \cdots + \frac{x-x_{n-1}|}{|a_n} \\ &= \frac{N_n(x)}{D_n(x)} \end{aligned}$$

and denote by $L(N_n)$ the leading coefficient of polynomial $N_n(x)$, then

$$L(N_n) = \begin{cases} a_0 + a_2 + \cdots + a_n, & \textit{for even } n \\ 1, & \textit{for odd } n \end{cases}$$

and

$$L(D_n) = \begin{cases} 1, & \text{for even } n \\ a_1 + a_3 + \cdots + a_n, & \text{for odd } n \end{cases}$$

Proof. From

$$\bar{R}_1(x) = \frac{N_1(x)}{D_1(x)} = a_0 + \frac{x-x_0}{a_1} = \frac{a_0a_1 + x - x_0}{a_1}$$

and

$$\begin{aligned} \bar{R}_2(x) &= \frac{N_2(x)}{D_2(x)} = a_0 + \frac{x-x_0|}{|a_1} + \frac{x-x_1|}{|a_2} \\ &= \frac{a_0a_1a_2 + a_0(x-x_1) + a_2(x-x_0)}{a_1a_2 + x - x_1} \end{aligned}$$

one gets

$$\begin{array}{ll} L(N_1) = 1, & L(D_1) = a_1 \\ L(N_2) = a_0 + a_2, & L(D_2) = 1 \end{array}$$

Now assume

$$\begin{aligned} L(N_{2k}) &= a_0 + a_2 + \cdots + a_{2k} \\ L(D_{2k}) &= 1 \\ L(N_{2k+1}) &= 1 \\ L(D_{2k+1}) &= a_1 + a_3 + \cdots + a_{2k+1} \end{aligned}$$

then from the recursion formula

$$\begin{aligned} N_n(x) &= a_nN_{n-1}(x) + (x - x_{n-1})N_{n-2}(x) \\ D_n(x) &= a_nD_{n-1}(x) + (x - x_{n-1})D_{n-2}(x) \end{aligned}$$

and

$$\partial(N_{n-1}) = \left[\tfrac{n}{2}\right], \qquad \partial(N_{n-2}) = \left[\tfrac{n-1}{2}\right]$$

$$\partial(D_{n-1}) = \left[\tfrac{n-1}{2}\right], \quad \partial(D_{n-2}) = \left[\tfrac{n-2}{2}\right]$$

where $\partial(p)$ denotes the highest degree of the polynomial p, it follows

$$L(N_n) = \begin{cases} a_nL(N_{n-1}) + L(N_{n-2}), & \text{for even } n \\ L(N_{n-2}), & \text{for odd } n \end{cases}$$

and

$$L(D_n) = \begin{cases} L(D_{n-2}), & \text{for even } n \\ a_nL(D_{n-1}) + L(D_{n-2}), & \text{for odd } n \end{cases}$$

Therefore

$$\begin{aligned} L(N_{2k+2}) &= a_{2k+2}L(N_{2k+1}) + L(N_{2k}) \\ &= a_0 + a_2 + \cdots + a_{2k} + a_{2k+2} \\ L(D_{2k+2}) &= L(D_{2k}) = 1 \\ L(N_{2k+3}) &= L(N_{2k+1}) = 1 \\ L(D_{2k+3}) &= a_{2k+3}L(D_{2k+2}) + L(D_{2k+1}) \\ &= a_1 + a_3 + \cdots + a_{2k+1} + a_{2k+3} \end{aligned}$$

Theorem 1 is thus proved by induction.

Suppose

$$\begin{aligned} R_n(x) &= \varphi_0[x_0] + \frac{x-x_0}{|\,\varphi_1[x_0,x_1]} + \frac{x-x_1}{|\,\varphi_2[x_0,x_1,x_2]} + \cdots + \frac{x-x_{n-1}}{|\,\varphi_n[x_0,x_1,\ldots,x_n]} \\ &= \frac{A_n(x)}{B_n(x)} \end{aligned} \tag{2.3}$$

and define

$$\rho_n[x_0,x_1,\ldots,x_n] = \begin{cases} \frac{L(A_n)}{L(B_n)}, & \text{for even } n \\ \frac{L(B_n)}{L(A_n)}, & \text{for odd } n \end{cases} \tag{2.4}$$

then we have by Theorem 1

$$\rho_n[x_0,x_1,\ldots,x_n] = \begin{cases} \varphi_0[x_0] + \varphi_2[x_0,x_1,x_2] + \cdots + \varphi_n[x_0,x_1,\ldots,x_n], & \text{for even } n \\ \varphi_1[x_0,x_1] + \varphi_3[x_0,x_1,x_2,x_3] + \cdots + \varphi_n[x_0,x_1,\ldots,x_n], & \text{for odd } n \end{cases} \tag{2.5}$$

and as a result we get

$$\begin{aligned} \rho_0[x_0] &= \varphi_0[x_0] \\ \rho_1[x_0,x_1] &= \varphi_1[x_0,x_1] \end{aligned} \tag{2.6}$$

and

$$\rho_n[x_0,x_1,\ldots,x_n] = \rho_{n-2}[x_0,x_1,\ldots,x_{n-2}] + \varphi_n[x_0,x_1,\ldots,x_n] \tag{2.7}$$

$\rho_n[x_0,x_1,\ldots,x_n]$ defined above is usually called reciprocal difference of the function $f(x)$ at points $x_0,x_1,\ldots,x_n$. Cuyt and Verdonk ([4]) gave the following determinant formula for reciprocal differences

$$\rho_{2k}[x_0,x_1,\ldots,x_{2k}] = \frac{\begin{vmatrix} 1 & f_0 & (x-x_0) & \cdots & (x-x_0)^{k-1} & f_0(x-x_0)^{k-1} & f_0(x-x_0)^k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & f_{2k} & (x-x_{2k}) & \cdots & & & f_{2k}(x-x_{2k})^k \end{vmatrix}}{\begin{vmatrix} 1 & f_0 & (x-x_0) & \cdots & (x-x_0)^{k-1} & f_0(x-x_0)^{k-1} & (x-x_0)^k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & f_{2k} & (x-x_{2k}) & \cdots & & & (x-x_{2k})^k \end{vmatrix}}$$

and

$$\rho_{2k+1}[x_0,x_1,\ldots,x_{2k+1}] = \frac{\begin{vmatrix} 1 & f_0 & (x-x_0) & f_0(x-x_0) & \cdots & (x-x_0)^k & (x-x_0)^{k+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & f_{2k+1} & \cdots & \cdots & & & (x-x_{2k+1})^{k+1} \end{vmatrix}}{\begin{vmatrix} 1 & f_0 & (x-x_0) & f_0(x-x_0) & \cdots & (x-x_0)^k & f_0(x-x_0)^k \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & f_{2k+1} & \cdots & \cdots & & & f_{2k+1}(x-x_{2k+1})^k \end{vmatrix}}.$$

Clearly the above formulae cannot be generalized to the case where the points $x_0, x_1, \ldots, x_n$ are coincident. In what follows, we give another determinant representation for reciprocal difference which allows the above mentioned coincidence of points. Let

$$D(m,n) = \begin{vmatrix} C_{0,m} & C_{1,m} & \cdots & C_{n,m} \\ C_{0,m+1} & C_{1,m+1} & \cdots & C_{n,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,m+n} & C_{1,m+n} & \cdots & C_{n,m+n} \end{vmatrix} \tag{2.8}$$

Theorem 2. *If $D(k+1,k-1)D(k+1,k) \neq 0$ for any $k \in \mathbb{N}$, then*

$$\rho_{2k}[x_0, x_1, \ldots, x_{2k}] = (-1)^k \frac{D(k,k)}{D(k+1,k-1)} \tag{2.9}$$

and

$$\rho_{2k+1}[x_0, x_1, \ldots, x_{2k+1}] = (-1)^k \frac{D(k+2,k-1)}{D(k+1,k)} \tag{2.10}$$

Proof. From the condition $D(k+1,k-1)D(k+1,k) \neq 0$, we know both the matrices

$$\begin{bmatrix} C_{0,k+1} & C_{1,k+1} & \cdots & C_{k,k+1} \\ C_{0,k+2} & C_{1,k+2} & \cdots & C_{k,k+2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k} & C_{1,2k} & \cdots & C_{k,2k} \end{bmatrix}$$

and

$$\begin{bmatrix} C_{0,k+2} & C_{1,k+2} & \cdots & C_{k,k+2} \\ C_{0,k+3} & C_{1,k+3} & \cdots & C_{k,k+3} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k+1} & C_{1,2k+1} & \cdots & C_{k,2k+1} \end{bmatrix}$$

are of maximal rank and therefore there exist the Newton-Padé approximants of order (k,k) and $(k+1,k)$ for $f(x)$. In the light of (1.8) and (1.9), by setting $m = n = k$ and $C = 1/(-1)^k D(k+1,k-1)$, we get

$$P_{k,k}(x) = (-1)^k \begin{vmatrix} F_{0,k}(x) & F_{1,k}(x) & \cdots & F_{k,k}(x) \\ C_{0,k+1} & C_{1,k+1} & \cdots & C_{k,k+1} \\ C_{0,k+2} & C_{1,k+2} & \cdots & C_{k,k+2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k} & C_{1,2k} & \cdots & C_{k,2k} \end{vmatrix} \Bigg/ D(k+1,k-1) \tag{2.11}$$

and

$$Q_{k,k}(x) = (-1)^k \begin{vmatrix} \omega_0(x) & \omega_1(x) & \cdots & \omega_k(x) \\ C_{0,k+1} & C_{1,k+1} & \cdots & C_{k,k+1} \\ C_{0,k+2} & C_{1,k+2} & \cdots & C_{k,k+2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k} & C_{1,2k} & \cdots & C_{k,2k} \end{vmatrix} \Bigg/ D(k+1,k-1). \tag{2.12}$$

By setting $m = k+1, n = k$ and $C = 1/D(k+1,k)$, we get

$$P_{k+1,k}(x) = \begin{vmatrix} F_{0,k+1}(x) & F_{1,k+1}(x) & \cdots & F_{k,k+1}(x) \\ C_{0,k+2} & C_{1,k+2} & \cdots & C_{k,k+2} \\ C_{0,k+3} & C_{1,k+3} & \cdots & C_{k,k+3} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k+1} & C_{1,2k+1} & \cdots & C_{k,2k+1} \end{vmatrix} \Bigg/ D(k+1,k) \tag{2.13}$$

and

$$Q_{k+1,k}(x) = \begin{vmatrix} \omega_0(x) & \omega_1(x) & \cdots & \omega_k(x) \\ C_{0,k+2} & C_{1,k+2} & \cdots & C_{k,k+2} \\ C_{0,k+3} & C_{1,k+3} & \cdots & C_{k,k+3} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k+1} & C_{1,2k+1} & \cdots & C_{k,2k+1} \end{vmatrix} \Bigg/ D(k+1,k). \tag{2.14}$$

From

$$\begin{array}{ll} \partial(P_{k,k}) = k, & \partial(Q_{k,k}) = k, \\ \partial(P_{k+1,k}) = k+1, & \partial(Q_{k+1,k}) = k, \end{array}$$

and

$$L(Q_{k,k}) = L(P_{k+1,k}) = 1,$$

it follows

$$\begin{array}{ll} P_{k,k}(x) = A_{2k}(x), & Q_{k,k}(x) = B_{2k}(x) \\ P_{k+1,k}(x) = A_{2k+1}(x), & Q_{k+1,k}(x) = B_{2k+1}(x) \end{array}$$

which results in

$$\begin{aligned} \rho_{2k}[x_0, x_1, \ldots, x_{2k}] &= \frac{L(A_{2k})}{L(B_{2k})} \\ &= \frac{L(P_{k,k})}{L(Q_{k,k})} \\ &= (-1)^k \begin{vmatrix} C_{0,k} & C_{1,k} & \cdots & C_{k,k} \\ C_{0,k+1} & C_{1,k+1} & \cdots & C_{k,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k} & C_{1,2k} & \cdots & C_{k,2k} \end{vmatrix} \Big/ D(k+1, k-1) \\ &= (-1)^k D(k,k)/D(k+1,k-1) \end{aligned}$$

and

$$\begin{aligned} \rho_{2k+1}[x_0, x_1, \ldots, x_{2k+1}] &= \frac{L(B_{2k+1})}{L(A_{2k+1})} \\ &= \frac{L(Q_{k+1,k})}{L(P_{k+1,k})} \\ &= (-1)^k \begin{vmatrix} C_{0,k+2} & C_{1,k+2} & \cdots & C_{k-1,k+2} \\ C_{0,k+3} & C_{1,k+3} & \cdots & C_{k-1,k+3} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k+1} & C_{1,2k+1} & \cdots & C_{k-1,2k+1} \end{vmatrix} \Big/ D(k+1, k) \\ &= (-1)^k D(k+2,k-1)/D(k+1,k) \end{aligned}$$

as asserted.

Theorem 3. *If $D(k+1,k-1)D(k,k-2)D(k+1,k)D(k,k-1) \neq 0$, then for $k \geq 1$*

$$\varphi_{2k}[x_0, x_1, \cdots, x_{2k}] = (-1)^k \left[\frac{D(k,k)}{D(k+1,k-1)} + \frac{D(k-1,k-1)}{D(k,k-2)} \right] \tag{2.15}$$

$$\varphi_{2k+1}[x_0, x_1, \cdots, x_{2k+1}] = (-1)^k \left[\frac{D(k+2,k-1)}{D(k+1,k)} + \frac{D(k+1,k-2)}{D(k,k-1)} \right] \tag{2.16}$$

Proof. By using Theorem 2 and the relation between reciprocal differences and inverse differences (2.7), one immediately comes to the conclusion in Theorem 3.

Besides the above expressions there exist another determinant representations for inverse differences which look more compact.

Theorem 4. *If $D(k+1,k-1) \cdot D(k+1,k) \neq 0$ and $Q_{k,k-1}(x_{2k-1}) \cdot Q_{k,k}(x_{2k}) \neq 0$, then*

$$\varphi_{2k}[x_0, x_1, \ldots, x_{2k}] = (-1)^k \frac{D(k,k-1)}{D(k+1,k-1)} \cdot \frac{\begin{vmatrix} \omega_0(x_{2k-1}) & \omega_1(x_{2k-1}) & \cdots & \omega_k(x_{2k-1}) \\ C_{0,k+1} & C_{1,k+1} & \cdots & C_{k,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k} & C_{1,2k} & \cdots & C_{k,2k} \end{vmatrix}}{\begin{vmatrix} \omega_0(x_{2k-1}) & \omega_1(x_{2k-1}) & \cdots & \omega_{k-1}(x_{2k-1}) \\ C_{0,k+1} & C_{1,k+1} & \cdots & C_{k-1,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,2k-1} & C_{1,2k-1} & \cdots & C_{k-1,2k-1} \end{vmatrix}} \tag{2.17}$$

and

$$\varphi_{2k+1}[x_0,x_1,\ldots,x_{2k+1}] = (-1)^k\frac{D(k+1,k-1)}{D(k+1,k)}\cdot\frac{\begin{vmatrix}\omega_0(x_{2k}) & \omega_1(x_{2k}) & \cdots & \omega_k(x_{2k})\\ C_{0,k+2} & C_{1,k+2} & \cdots & C_{k,k+2}\\ \vdots & \vdots & \ddots & \vdots\\ C_{0,2k+1} & C_{1,2k+1} & \cdots & C_{k,2k+1}\end{vmatrix}}{\begin{vmatrix}\omega_0(x_{2k}) & \omega_1(x_{2k}) & \cdots & \omega_k(x_{2k})\\ C_{0,k+1} & C_{1,k+1} & \cdots & C_{k,k+1}\\ \vdots & \vdots & \ddots & \vdots\\ C_{0,2k} & C_{1,2k} & \cdots & C_{k,2k}\end{vmatrix}} \tag{2.18}$$

Proof. Applying recursion formula to (2.3), we have

$$\begin{aligned} A_n(x) &= \varphi_n[x_0,x_1,\ldots,x_n]A_{n-1}(x)+(x-x_{n-1})A_{n-2}(x)\\ B_n(x) &= \varphi_n[x_0,x_1,\ldots,x_n]B_{n-1}(x)+(x-x_{n-1})B_{n-2}(x)\end{aligned}$$

from which we get

$$\varphi_n[x_0,x_1,\ldots,x_n]=\frac{A_n(x_{n-1})}{A_{n-1}(x_{n-1})}=\frac{B_n(x_{n-1})}{B_{n-1}(x_{n-1})}$$

provided $B_n(x_{n-1})\neq 0$ or $A_{n-1}(x_{n-1})\neq 0$. Therefore

$$\varphi_{2k}[x_0,x_1,\ldots,x_{2k}]=\frac{B_{2k}(x_{2k-1})}{B_{2k-1}(x_{2k-1})}=\frac{Q_{k,k}(x_{2k-1})}{Q_{k,k-1}(x_{2k-1})}$$

$$\varphi_{2k+1}[x_0,x_1,\ldots,x_{2k+1}]=\frac{B_{2k+1}(x_{2k})}{B_{2k}(x_{2k})}=\frac{Q_{k+1,k}(x_{2k})}{Q_{k,k}(x_{2k})}$$

Substituting (2.12) and (2.14) into the above expressions, we immediately obtain the results stated in Theorem 4.

3. The Limiting Case of the Thiele's Interpolant

Let us now consider the limiting case of the Thiele's interpolating continued fraction expansion (2.1), by which we mean the corresponding continued fraction yielded from (2.1) when all the points in set $X=\{x_i|i\in\mathbb{N}\}$ are coincident with certain point, say, x_0. Naturally the limits of inverse differences play key roles in this case. For simplicity, we introduce the following notations

$$\begin{aligned}\varphi_n[x_0] &= \varphi_n[\overbrace{x_0,x_0,\ldots,x_0}^{n+1}]\\ &= \lim_{\substack{x_i\to x_0\\ i=1,\cdots,n}}\varphi_n[x_0,x_1,\ldots,x_n],\end{aligned} \tag{3.1}$$

$$C_i=\frac{f^{(i)}(x_0)}{i!} \tag{3.2}$$

and

$$C(L/M)=\begin{vmatrix} C_{L-M+1} & C_{L-M+2} & \cdots & C_L\\ C_{L-M+2} & C_{L-M+3} & \cdots & C_{L+1}\\ \vdots & \vdots & \ddots & \vdots\\ C_L & C_{L+1} & \cdots & C_{L+M-1}\end{vmatrix}, \tag{3.3}$$

then

$$\begin{aligned}\rho_0[x_0] &= \varphi_0[x_0],\\ \rho_1[x_0] &= \varphi_1[x_0],\\ \rho_n[x_0] &= \rho_{n-2}[x_0]+\varphi_n[x_0], n\geq 2\end{aligned}$$

and (2.1) turns out to be

$$\varphi_0[x_0] + \frac{x - x_0|}{|\varphi_1[x_0]} + \frac{x - x_0|}{|\varphi_2[x_0]} + \frac{x - x_0|}{|\varphi_3[x_0]} + \cdots \tag{3.4}$$

Two methods known as the Thiele's algorithm and the Viscovatov's algorithm are available for the computations of the limits of inverse differences, which we formulate as follows.

Thiele's algorithm. (see [3] or [5])

$$\begin{aligned}
\varphi_0[x_0] &= \rho_0[x_0] = f(x_0) \\
\varphi_1[x_0] &= \rho_1[x_0] = \frac{1}{\frac{\mathrm{d}f(x_0)}{\mathrm{d}x}} \\
\varphi_l[x_0] &= \frac{l}{\frac{\mathrm{d}\rho_{l-1}[x_0]}{\mathrm{d}x}},\ l \geq 2 \\
\rho_l[x_0] &= \rho_{l-2}[x_0] + \varphi_l[x_0],\ \ l \geq 2.
\end{aligned}$$

Viscovatov's algorithm. (see [5])

$$\begin{aligned}
C_i^{(0)} &= f^{(i)}(x_0)/i!,\ i = 0, 1, 2, \ldots \\
\varphi_0[x_0] &= C_0^{(0)} \\
\varphi_1[x_0] &= 1/C_1^{(0)} \\
C_i^{(1)} &= -C_{i+1}^{(0)}/C_1^{(0)},\ i \geq 1 \\
\varphi_l[x_0] &= C_1^{(l-2)}/C_1^{(l-1)},\ l \geq 2 \\
C_i^{(l)} &= C_{i+1}^{(l-2)} - \varphi_l[x_0]C_{i+1}^{(l-1)},\ i \geq 1, l \geq 2.
\end{aligned}$$

Obviously, when using the Thiele's algorithm or the Viscovatov's algorithm, one has to carry out computations step by step with great patience for it is hardly possible to derive a general expression for $\varphi_n[x_0]$ from these algorithms. The following theorem presents a general determinantal expression for $\varphi_n[x_0]$, which may be regarded as a make-up for both the Thiele's algorithm and Viscovatov's algorithm.

Theorem 5. *If* $C(k/k-1)C(k+1/k)C(k/k)C(k+1/k+1) \neq 0$, *then*

$$\varphi_{2k}[x_0] = -\frac{[C(k/k)]^2}{C(k/k-1)C(k+1/k)}$$

and

$$\varphi_{2k+1}[x_0] = \frac{[C(k+1/k)]^2}{C(k/k)C(k+1/k+1)}$$

Proof. Noticing

$$\begin{aligned}
\lim_{\substack{x_l \to x_0 \\ l = i, \cdots, j}} C_{i,j} &= \lim_{\substack{x_l \to x_0 \\ l = i, \cdots, j}} f[x_i, \cdots, x_j] \\
&= \frac{f^{(j-i)}(x_0)}{(j-i)!} = C_{j-i},
\end{aligned}$$

we have

$$\begin{aligned}
\lim_{\substack{x_i \to x_0 \\ i = 0, 1, \cdots, m+n}} D(m,n) &= \begin{vmatrix} C_m & C_{m-1} & \cdots & C_{m-n} \\ C_{m+1} & C_m & \cdots & C_{m-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m+n} & C_{m+n-1} & \cdots & C_m \end{vmatrix} \\
&= (-1)^{\frac{n(n+1)}{2}} \begin{vmatrix} C_{m-n} & C_{m-n+1} & \cdots & C_m \\ C_{m-n+1} & C_{m-n+2} & \cdots & C_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_m & C_{m+1} & \cdots & C_{m+n} \end{vmatrix} \\
&= (-1)^{\frac{n(n+1)}{2}} C(m/n+1).
\end{aligned}$$

By Theorem 4, we get

$$\begin{aligned}
\varphi_{2k}[x_0] &= (-1)^k \frac{(-1)^{\frac{k(k-1)}{2}} C(k/k)}{(-1)^{\frac{k(k-1)}{2}} C(k+1/k)} \cdot \frac{\begin{vmatrix} C_k & C_{k-1} & \cdots & C_1 \\ C_{k+1} & C_k & \cdots & C_2 \\ \vdots & \vdots & \ddots & \vdots \\ C_{2k-1} & C_{2k-2} & \cdots & C_k \end{vmatrix}}{\begin{vmatrix} C_k & C_{k-1} & \cdots & C_2 \\ C_{k+1} & C_k & \cdots & C_3 \\ \vdots & \vdots & \ddots & \vdots \\ C_{2k-2} & C_{2k-3} & \cdots & C_k \end{vmatrix}} \\
&= (-1)^k \frac{C(k/k)}{C(k+1/k)} \cdot (-1)^{k-1} \frac{C(k/k)}{C(k/k-1)} \\
&= -\frac{[C(k/k)]^2}{C(k+1/k)C(k/k-1)}
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{2k+1}[x_0] &= (-1)^k \frac{(-1)^{\frac{k(k-1)}{2}} C(k+1/k)}{(-1)^{\frac{k(k+1)}{2}} C(k+1/k+1)} \cdot \frac{\begin{vmatrix} C_{k+1} & C_k & \cdots & C_2 \\ C_{k+2} & C_{k+1} & \cdots & C_3 \\ \vdots & \vdots & \ddots & \vdots \\ C_{2k} & C_{2k-1} & \cdots & C_{k+1} \end{vmatrix}}{\begin{vmatrix} C_k & C_{k-1} & \cdots & C_1 \\ C_{k+1} & C_k & \cdots & C_2 \\ \vdots & \vdots & \ddots & \vdots \\ C_{2k-1} & C_{2k-2} & \cdots & C_k \end{vmatrix}} \\
&= \frac{C(k+1/k)}{C(k+1/k+1)} \cdot \frac{C(k+1/k)}{C(k/k)} \\
&= \frac{[C(k+1/k)]^2}{C(k/k)C(k+1/k+1)}
\end{aligned}$$

The proof of Theorem 5 is completed.

Remark. One may also prove Theorem 5 by means of Theorem 3. As a matter of fact, from (2.15) and (2.16) it follows

$$\begin{aligned}
\varphi_{2k}[x_0] &= (-1)^k \left[\frac{(-1)^{\frac{k(k+1)}{2}} C(k/k+1)}{(-1)^{\frac{k(k-1)}{2}} C(k+1/k)} + \frac{(-1)^{\frac{k(k-1)}{2}} C(k-1/k)}{(-1)^{\frac{(k-2)(k-1)}{2}} C(k/k-1)} \right] \\
&= \frac{C(k/k+1)}{C(k+1/k)} - \frac{C(k-1/k)}{C(k/k-1)} \\
&= \frac{C(k/k+1)C(k/k-1) - C(k-1/k)C(k+1/k)}{C(k+1/k)C(k/k-1)}
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{2k+1}[x_0] &= (-1)^k \left[\frac{(-1)^{\frac{k(k-1)}{2}} C(k+2/k)}{(-1)^{\frac{k(k+1)}{2}} C(k+1/k+1)} + \frac{(-1)^{\frac{(k-2)(k-1)}{2}} C(k+1/k-1)}{(-1)^{\frac{k(k-1)}{2}} C(k/k)} \right] \\
&= \frac{C(k+2/k)}{C(k+1/k+1)} - \frac{C(k+1/k-1)}{C(k/k)} \\
&= \frac{C(k+2/k)C(k/k) - C(k+1/k-1)C(k+1/k+1)}{C(k/k)C(k+1/k+1)}.
\end{aligned}$$

By sylvester's determinant identity (see [1] or [9])

$$C(L-1/M)C(L+1/M) - C(L/M-1)C(L/M+1) = [C(L/M)]^2,$$

we see

$$\begin{aligned}
C(k/k+1)C(k/k-1) - C(k-1/k)C(k+1/k) &= -[C(k/k)]^2, \\
C(k+2/k)C(k/k) - C(k+1/k-1)C(k+1/k+1) &= [C(k+1/k)]^2
\end{aligned}$$

and herewith we obtain

$$\varphi_{2k}[x_0] = -\frac{[C(k/k)]^2}{C(k+1/k)C(k/k-1)}$$

and

$$\varphi_{2k+1}[x_0] = \frac{[C(k+1/k)]^2}{C(k/k)C(k+1/k+1)}$$

4. Numerical Example

Let us take $f(x) = e^x$ as an example to show the construction of limiting Thiele's interpolating continued fraction expansion at $x_0 = 0$. In this case, $f(x)$ may formally be expanded as

$$f(x) = \varphi_0[0] + \frac{x}{\lceil \varphi_1[0]} + \frac{x}{\lceil \varphi_2[0]} + \cdots$$

In order to compute $\varphi_i[0],\ i = 0, 1, 2, \ldots$, we need the following lemma.

Lemma 2.

$$\begin{vmatrix} 1 & \binom{p}{1} & \binom{p}{2} & \cdots & \binom{p}{n} \\ 1 & \binom{p+1}{1} & \binom{p+1}{2} & \cdots & \binom{p+1}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \binom{p+n}{1} & \binom{p+n}{2} & \cdots & \binom{p+n}{n} \end{vmatrix} = 1. \tag{4.1}$$

Proof. Subtract the i^{th} row from the $(i+1)^{th}$ row for $i = n, n-1, \ldots, 2, 1$ and keep using the combinatorial identity

$$\binom{m+1}{n} - \binom{m}{n} = \binom{m}{n-1},$$

one at once gets (4.1).

Now

$$C(L/M) = \begin{vmatrix} \frac{1}{(L-M+1)!} & \frac{1}{(L-M+2)!} & \cdots & \frac{1}{L!} \\ \frac{1}{(L-M+2)!} & \frac{1}{(L-M+3)!} & \cdots & \frac{1}{(L+1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L!} & \frac{1}{(L+1)!} & \cdots & \frac{1}{(L+M-1)!} \end{vmatrix}$$

$$= \frac{1!2!\cdots(M-1)!}{L!(L+1)!\cdots(L+M-1)!} \cdot \begin{vmatrix} \binom{L}{M-1} & \cdots & \binom{L}{1} & 1 \\ \binom{L+1}{M-1} & \cdots & \binom{L+1}{1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \binom{L+M-1}{M-1} & \cdots & \binom{L+M-1}{1} & 1 \end{vmatrix}.$$

It follows from Lemma 2

$$C(L/M) = (-1)^{\frac{M(M-1)}{2}} \frac{1!2!\cdots(M-1)!}{L!(L+1)!\cdots(L+M-1)!}. \tag{4.2}$$

By Theorem 5 and (4.2), we derive for $k \geq 1$

$$\begin{aligned}\varphi_{2k}[0] &= (-1)^k \frac{\left[\frac{1!2!\cdots(k-1)!}{k!(k+1)!\cdots(2k-1)!}\right]^2}{\frac{1!2!\cdots(k-2)!}{k!(k+1)!\cdots(2k-2)!}\cdot\frac{1!2!\cdots(k-1)!}{(k+1)!(k+2)!\cdots(2k)!}} \\ &= (-1)^k \frac{(k-1)!}{(2k-1)!}\cdot\frac{(2k)!}{k!} \\ &= (-1)^k 2\end{aligned}$$

and

$$\begin{aligned}\varphi_{2k+1}[0] &= (-1)^k \frac{\left[\frac{1!2!\cdots(k-1)!}{(k+1)!(k+2)!\cdots(2k)!}\right]^2}{\frac{1!2!\cdots(k-1)!}{k!(k+1)!\cdots(2k-1)!}\cdot\frac{1!2!\cdots k!}{(k+1)!(k+2)!\cdots(2k+1)!}} \\ &= (-1)^k \frac{(2k+1)!}{k!}\cdot\frac{k!}{(2k)!} \\ &= (-1)^k (2k+1)\end{aligned}$$

with

$$\begin{aligned}\varphi_0[0] &= f(0) = 1 \\ \varphi_1[0] &= \lim_{x\to 0}\frac{x-0}{f(x)-f(0)} = \frac{1}{f'(0)} = 1\end{aligned}$$

and consequently we obtain the limiting Thiele's interpolating continued fraction expansion for e^x as follows

$$\begin{aligned}e^x &= 1 + \frac{x|}{|1} + \frac{x|}{|-2} + \frac{x|}{|-3} + \frac{x|}{|2} + \frac{x|}{|5} \\ &+ \frac{x|}{|-2} + \frac{x|}{|-7} + \frac{x|}{|2} + \frac{x|}{|9} + \cdots + \frac{x|}{|(-1)^k 2} + \frac{x|}{|(-1)^k(2k+1)} + \cdots\end{aligned}$$

Its first four convergents are

$$\begin{aligned}&1 + \frac{x|}{|1} = 1 + x \\ &1 + \frac{x|}{|1} + \frac{x|}{|-2} = \frac{2+x}{2-x} \\ &1 + \frac{x|}{|1} + \frac{x|}{|-2} + \frac{x|}{|-3} = \frac{6+4x+x^2}{6-2x} \\ &1 + \frac{x|}{|1} + \frac{x|}{|-2} + \frac{x|}{|-3} + \frac{x|}{|2} = \frac{12+6x+x^2}{12-6x+x^2}\end{aligned}$$

which are respectively the entries $r_{1,0}, r_{1,1}, r_{2,1}$ and $r_{2,2}$ in the Padé table of the function e^x (we refer to [1] or [9])

If one calculates $\varphi_n[x_0]$ by means of the Thiele's method, then the computing process proceeds as follows

$$\varphi_0[x] = e^x,\ \rho_0[x] = e^x,\ \varphi_0[0] = 1,$$

$$\varphi_1[x] = e^{-x},\ \rho_1[x] = e^{-x},\ \varphi_1[0] = 1,$$

$$\varphi_2[x] = \frac{2}{\frac{\mathrm{d}\rho_1[x]}{\mathrm{d}x}} = -2e^x,\ \rho_2[x] = \rho_0[x] + \varphi_2[x] = -e^x,\ \varphi_2[0] = -2,$$

$$\varphi_3[x] = \frac{3}{\frac{\mathrm{d}\rho_2[x]}{\mathrm{d}x}} = -3e^{-x},\ \rho_3[x] = \rho_1[x] + \varphi_3[x] = -2e^{-x},\ \varphi_3[0] = -3,$$

$$\varphi_4[x] = \frac{4}{\frac{\mathrm{d}\rho_3[x]}{\mathrm{d}x}} = 2e^x,\ \rho_4[x] = \rho_2[x] + \varphi_4[x] = e^x,\ \varphi_4[0] = 2,$$

$$\varphi_5[x] = \frac{5}{\frac{\mathrm{d}\rho_4[x]}{\mathrm{d}x}} = 5e^{-x},\ \rho_5[x] = \rho_3[x] + \varphi_5[x] = 3e^{-x},\ \varphi_4[0] = 5.$$

If one calculates $\varphi_n[x_0]$ by means of the Viscovatov's method, then the computing process proceeds as follows

$$\begin{array}{llllll}
C_0^{(0)} = 1 & C_1^{(0)} = 1 & C_2^{(0)} = 1/2 & C_3^{(0)} = 1/6 & C_4^{(0)} = 1/24 & C_5^{(0)} = 1/120 \\
 & C_1^{(1)} = -1/2 & C_2^{(1)} = -1/6 & C_3^{(1)} = -1/24 & C_4^{(1)} = -1/120 & \\
 & C_1^{(2)} = 1/6 & C_2^{(2)} = 1/12 & C_3^{(2)} = 1/40 & & \\
 & C_1^{(3)} = 1/12 & C_2^{(3)} = 1/30 & & & \\
 & C_1^{(4)} = 1/60 & & & &
\end{array}$$

$$\begin{array}{lll}
\varphi_0[0] = 1, & \varphi_1[0] = 1, & \varphi_2[0] = C_1^{(0)}/C_1^{(1)} = -2, \\
\varphi_3[0] = C_1^{(1)}/C_1^{(2)} = -3, & \varphi_4[0] = C_1^{(2)}/C_1^{(3)} = 2, & \varphi_5[0] = C_1^{(3)}/C_1^{(4)} = 5.
\end{array}$$

The above example shows that the method provided in our paper can serve to solve the limiting Thiele's interpolating continued fraction expansion problem once for all and seems more effective in some cases than the so called Thiele's method and Viscovatov's method (see [3], [5] and [6]).

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