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A NEW APPROACH TO SOLVE SYSTEMS OF LINEAR EQUATIONS*[1)]

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Abstract

We propose a new iterative approach to solve systems of linear equations. The new strategy integrates the algebraic basis of the problem with elements from classical mechanics and the finite difference method. The approach defines two families of convergent iterative methods. Each family is characterized by a linear differential equation, and every method is obtained from a suitable finite difference scheme to integrate the associated differential equation. The methods are general and depend on neither the matrix dimension nor the matrix structure. In this preliminary work, we present the basic features of the method with a simple application to a low dimensional system.

Key words: Iterative method, Linear systems, Classical dynamics.

1. Introduction

The new approach is based on the analysis of the motion of a damped harmonic oscillator in the gravitational field [1]. The associated equation of motion is

$$mX_{tt} + \alpha X_t + aX = b \tag{1}$$

where $X = X(t)$, is the one dimensional displacement of of a mass m under a dissipation ($\alpha > 0$), a harmonic potential ($a > 0$) and a constant acceleration (b, gravitational field). The total energy variation is given by the equation

$$\frac{dE}{dt} = -\alpha(\frac{dX}{dt})^2 \tag{2}$$

where

$$E = \frac{1}{2}m(\frac{dX}{dt})^2 + \frac{1}{2}aX^2 - bX \tag{3}$$

The solution of the motion equation (1) is given by the sum of two contributions: the homogeneous part and a particular solution:

$$X(t) = X_{hom.} + X_p \tag{4}$$

where

$$X_{hom.} = Me^{\lambda_1 t} + Ne^{\lambda_2 t} \tag{5}$$

being M and N constants.

$$X_p = \frac{b}{a} \tag{6}$$

that is

$$X(t) = Me^{\lambda_1 t} + Ne^{\lambda_2 t} + \frac{b}{a} \tag{7}$$

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The homogeneous component of the motion depends on the initial conditions $X(0), X_t(0)$, which are related to the two constants M, N, and the parameters λ_1 and λ_2 are either real negative or complex with a negative real part:

$$\lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4am}}{2m} \tag{8}$$

For the critical case $\alpha^2 = 4am$, the two parameters λ are real and equal, and the solution is

$$X(t) = (M + Nt)e^{\frac{-\alpha t}{2m}} + \frac{b}{a} \tag{9}$$

In the three cases of possible values of λ we have that

$$lim_{t\to\infty} X_{hom.}(t) = 0 \tag{10}$$

Thus we get

$$lim_{t\to\infty} X(t) = \frac{b}{a} \tag{11}$$

This limit is the solution of the linear equation

$$aX = b \tag{12}$$

which, also, can be understood as the minimum of the potential $V(X) = \frac{1}{2}aX^2 - bX$, associated to the particle motion of mass m. In the above context, let us consider the following two remarks:

Remark. If $b = 0$, the solution of (12) is $X = 0$, which is the limit of the solution (7) and (9) when $t \to \infty$.

Remark. If $b = 0$ and $a = 0$, the limit of (7), when $t \to \infty$, is an arbitrary constant, depending on the initial conditions. This will be related to the undetermined systems.

Let us consider again the mechanical model (1) in the overdamped limit: the dominant effect is the dissipative one. Thus, by taking $\alpha = 1$ we get the equation:

$$X_t + aX = b \tag{13}$$

The equilibrium point of the dynamical system (13) is once more the solution of the linear equation (12). Such equilibrium point is stable.

The above mechanical considerations are the basis for the two new families of iterative methods to solve a system of linear equations. Each iterative method will be defined by a finite difference method to solve the linear equations (1) and (13). In the next two sections, we will extend the previous one dimensional mechanical considerations to the case of a system of equations.

An important feature of this approach is that we relate the problem of solving a system of linear equations to integrate the equation of motion of one particle, that tends asymptotically to a position which is identified with the solution of the above linear system. Thus, we can expect to have general iterative methods which need many iterations, but the convergence is satisfied while the discretization of the differential equation of motion satisfies conditions related to the conservation and variation of the energy of the system.

2. The Damped Methods

Let us consider the system of linear equations

$$AX = B \tag{14}$$

where A is a real $n \times n$ matrix, X and B real n-dimensional vectors. Also, we can interpret (14) as the extremum of the potential $V(X) = \frac{1}{2}X^T AX - X^T B$. We have three possibilities: one unique extremum, infinite extremums and no extremum, which correspond to one, infinite or no solutions for the system (14). In this context, we can consider the vectorial dynamical equation, similar to (1):

$$X_{tt} + \alpha X_t + AX = B \tag{15}$$

Key Assumption. Let A be a matrix with positive real spectrum. Then, we can consider a similarity transformation in the equation (15), and to reduce it to a system of n equations related to the type of (1), such that the positive eigenvalues of the matrix A guarantee that the asymptotic behaviour of the solution of (15) gives the solution of the system (14)

A suitable finite difference scheme to solve (15) is the following [2, 3]:

$$\frac{X^{n+2}-2X^{n+1}+X^n}{\tau^2}+\alpha\frac{X^{n+2}-X^n}{2\tau}+AX^{n+1}=B \tag{16}$$

where τ is the mesh size of the time variable and X^n denotes the position at the time $t=n\tau$. If we multiply (16) by $((X^T)^{n+2}-(X^T)^n)/2\tau$, and rearrange the terms, we get

$$\frac{E^{n+1}-E^n}{\tau}=-\alpha\left(\frac{X^{n+2}-X^n}{2\tau}\right)^2 \tag{17}$$

where E^n is the discrete energy defined by

$$E^n=\frac{1}{2}\left(\frac{X^{n+1}-X^n}{\tau}\right)^2+\frac{1}{2}(X^T)^nAX^{n+1}-\frac{1}{2}((X^T)^{n+1}B+(X^T)^nB) \tag{18}$$

which is a discrete energy similar to that of (3). As we can see the variation of the discrete energy associated with the difference equation (16) is the same as in (2) depending only on the sign of α and not on the solution. This property guarantees the convergence of the numerical solution to that of the system (14).

The scheme (16) defines a possible iterative convergent method for the linear system of equations (14). If the spectrum of A is not real positive, the solution of (15) and the numerical one (16) do not converge to the desired limit. When we check the divergence, we have the following procedure to get the solution of the problem, by studying the numerical solution of the following equivalent systems:

1. When the spectrum of A is real negative:
$$X_{tt}+\alpha X_t-AX=-B \tag{19}$$
2. When A has positive and negative eigenvalues:
$$X_{tt}+\alpha X_t+AAX=AB \tag{20}$$
3. When A has complex eigenvalues:
$$X_{tt}+\alpha X_t+A^TAX=A^TB \tag{21}$$

Remark. Given a matrix A, the matrix A^TA has non-negative eigenvalues. Thus, we can start the study of the associated system (21) to get a convergent iterative method.

The parameter τ is related to the value of α and must allow to the finite difference equation (16) to have solutions with the homogeneous part exponentially decaying.

With this approach we obtain information about the spectrum of the matrix A at the same time that we get the appropriate iterative process to solve the original linear system.

If $B=0$, the system (14) has either the solution $X=0$, or infinite number of solutions if at least one eigenvalue of A is zero. This situation is mechanically well detected as it was indicated in the previous section.

When $B\neq 0$, and the system (14) has no solution, this means that at least one eigenvalue of A is zero and then the solution of the equation (15) has a linear component growing linearly in the time.

3. The Overdamped Methods

As in the previous section, we can consider the equations (14) defining the equilibrium configurations of the system

$$X_t+AX=B \tag{22}$$

As in the previous section we have to consider the possibilities of the different systems according the spectrum of A. By using the simple Euler method, we can approach the equation (22) as follows

$$\frac{X^{n+1}-X^n}{\tau}+AX^n=B \tag{23}$$

from this we get the iterative method for solving (14):

$$X^{n+1}=(I-\tau A)X^n+\tau B \tag{24}$$

This iteration always converges to the desired solution, while τ is sufficiently small and the spectrum of A is real positive. As we can see, with this method we also obtain information about the eigenvalues of A in the process of selecting the right equivalent linear system in order to get the convergence to the solution. The three possibilities of solutions for (14) are well reflected by (22)-(24) as in the previous damped methods.

4. Numerical Simulations

In the computations with the damped and overdamped methods, we have to fix the initial conditions, τ, and the value of α in the damped ones. The convergence is independent of the initial conditions, and we selected the values $X(0) = 0, X_t(0) = 0$, for the damped ones and $X(0) = 0$, for the overdamped ones. An open question is to find the optimal initial conditions in order to get a fast convergence. In the damped methods we used $\alpha = 1$.

The correct values of τ are related to the norm of the matrix A and to the value of the constant α in the damped methods. The behaviour of the numerical solution is monitored against different values of τ with the above reference values, and the numerical results from the equivalent systems described in (19)-(21).

As a basic illustration of the method, let us consider the system:

$$\begin{pmatrix} s & 1 \\ s^2 - 5 & s + 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{25}$$

where s is a real parameter, according to its values the eigenvalues of the matrix A are the following:

- If $-2 < s < 2$ the two eigenvalues are complex.
- If $s^2 - 4 > 0$ the two eigenvalues are real and with opposite sign if $s < -\frac{5}{2}$ and the same sign in the other cases.

We tried the iterative methods defined by (16) and (24), with the zero initial conditions and $\alpha = 1$. We found the convergence to the exact solution any time that the system was transformed such that the associated matrix has positive eigenvalues, while τ is sufficiently small. For instance in the case $s = 3$, the exact solution is $x = \frac{3}{11} = 0.2727, y = \frac{2}{11} = 0.1818$, while the iterative one ($\tau = 0.6$) obtained with (16) is $x = 0.2711, y = 0.1806$ and 15 steps. This corresponds to follow the trajectory of the particle during 9 times the characteristic time associated with the damping. The iterative solution with (24) and $\tau = 0.2$ is $x = 0.2707, y = 0.1843$ and 10 steps.

5. Conclusions

We described the main features of a new approach to solve systems of linear equations. This approach allows us to define a general convergent iterative method to solve any linear system, independently of the structure of the associated matrix ($n \times n$). Each method is related to a finite difference scheme to solve a linear differential equation. Further studies [4] related to limiting cases, optimal parameters as well as initial conditions, and comparison with other methods and different finite difference schemes in the case of large systems are in progress.

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