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FIRST-ORDER AND SECOND-ORDER, CHAOS-FREE, FINITE DIFFERENCE SCHEMES FOR FISHER EQUATION[*1)]

Geng Sun
(*Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Hua-mo Wu
(*LSEC, ICMSEC, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Li-er Wang
(*Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

A new class of finite difference schemes is constructed for Fisher partial differential equation i.e. the reaction-diffusion equation with stiff source term: $\alpha u(1-u)$. These schemes have the properties that they reduce to high fidelity algorithms in the diffusion-free case namely in which the numerical solutions preserve the properties inherent in the exact solutions for arbitrary time step-size and reaction coefficient $\alpha > 0$, and all non-physical spurious solutions including bifurcations and chaos that normally appear in the standard discrete models of Fisher partial differential equation will not occur. The implicit schemes so developed obtain the numerical solutions by solving a single linear algebraic system at each step. The boundness and asymptotic behaviour of numerical solutions obtained by all these schemes are given. The approach constructing the above schemes can be extended to reaction-diffusion equations with other stiff source terms.

Key words: Reaction-diffusion equation, Fidelity algorithm.

1. Introduction

The Fisher partial differential equation (PDE.)[1,9]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \quad \alpha > 0 \tag{1.1}$$

is a mathematical model for the analysis of a number of natural phenomena: population growth with dispersal[2], flam propagation[7]and the neution population in a nuclear reaction[3], and has also been used as a test equation for the investigation of numerical integration schemes and the related issues of numerical chaos[8,4].

It is well known that the use of Euler finite difference scheme to solve the Fisher PDE. can produce bifurcations and chaos i.e. non physical spurious solutions which are contrived from the difference equation and are not a feature of the Fisher PDE. A major source of resulting

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in non physical spurious solutions is the existence of bifurcations and chaotie behaviour in numerical solutions to the corresponding ordinary differential equation (ODE.). Therefore, in numerical solution preceduces one must eliminate such non physical spurious solutions and faithfully approximates to the original initial-value problem (IVP.):

$$\begin{cases} \dfrac{du}{dt} = \alpha S(u), \alpha > 0 \\ u(0) = u^0 \end{cases} \tag{1.2}$$

where

$$S(u) = u(1-u) \tag{1.3}$$

i.e., so that the properties of particular interest that include asymptotic behaviour of solutions and the stability properties of fixed points should be all the same to IVP. (1.2)–(1.3). Consequently the corresponding numerical model is most important.

The present paper is a continuation of [11] where a new class of explicitly high fidelity algorithms for stiff IVP. (1.2) were presented, the major purpose is that the high fidelity algorithms so developed will be incorporated, in the appropriate, into finite-difference schemes for the Fisher PDE. (1.1) such that the numerical solutions arising will faithfully represent its genuine solutions, and non physical spurious solutions including bifurcations and chaos will not occur.

The paper is organized as following. In 2, the high fidelity algorithms for the diffusion-free case i.e. the IVP. (1.2)–(1.3) and their important properties are described. In §3 the full numerical schemes for the initial-boundary value problem (IBVP.) of the Fisher PDE. and the preliminaris are given. In §4 the boundness and asymptotic behaviour of numerical solutions obtained by all the schemes are discussed in detail.

2. Diffusion-Free Case

2.1. The properties of solutions for Logistie equation

In diffusion-free case the Fisher PDE.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u)$$

become the Logistic ODE.

$$\frac{du}{dt} = \alpha u(1-u), \quad t > 0, \tag{2.1}$$

where $\alpha \gg 0$ is a stiff parameter.

For the Logistic equation the exact solution is given by

$$u(t) = \frac{u^0}{u^0 + (1-u^0)e^{-\alpha t}} = \frac{u^0 e^{\alpha t}}{(1-u^0) + u^0 e^{\alpha t}}, \tag{2.2}$$

where $u^0 > 0$ is the initial value.

Two properties are important for the Logistic equation (2.1): 1) It has two fixed points $u^* = 1$ and $u^* = 0$ in which the fixed $u^* = 1$ is stable while $u^* = 0$ is unstable; 2) The global asymptotic solution behavious is that for every positive initial value u^0 and $\alpha > 0$ the solution is monotone, and eventually tends to stable fixed point $u^* = 1$ and that $u(t) \nearrow 1$ if $0 < u^0 < 1$ and $u(t) \searrow 1$ if $u^0 > 1$.

2.2. Construction of high fidelity algorithms

For IVP of the Logistic equation a numerical method that generates numerical solutions satisfying such properties 1) and 2) for arbitrary $\alpha > 0$ and steplength τ, is called the high fidelity algorithm.

Let u^k be numerical solution for the Logistic equation at the point $t = k\tau$.

The standard discrete model for the Logistic equation, for example, the explicit Euler method

$$u^{k+1} = u^k + Au^k(1 - u^k), \quad k \geq 0, \quad A = \alpha\tau \tag{2.3}$$

gives stable asymptote $u^* = 1$ if and only if $0 < A < 2$ and $0 < u^0 < 1 + 1/A$. The numerical solution satisfies above the properties 1) and 2) if and only if $0 < A < 1$ and $0 < u^0 < 1$. In other case non physical solutions $u < 0$ or bifurcation or even chaos may take place (See [4] for details).

Applying the linearly implicit θ-method[10] to IVP. (1.2)-(1.3) obtains

$$u^{k+1} = u^k + AS(u^k) + \theta AS'_u(u^k)(u^{k+1} - u^k), \tag{2.4}$$

where the scheme (2.4) with $\theta = 1/2$ is of second order.

For IVP. of the Logistic equation LIM (2.4) with $\theta = 1/2$ leads to[6]

$$u^{k+1} = u^k + \frac{1}{2}A(u^k + u^{k+1}) - Au^k u^{k+1}. \tag{2.5}$$

It can be shown that its solution u^k tends to the fixed point $u^* = 1$ when $k \to \infty$ for all $A > 0$ and non physical solution $u^k < 0$ exists when $0 < u^0 < \frac{1}{2} - \frac{1}{A}$ and $A > 2$, but the bifurcation will not occur (See [12] for details). Obviouslly the second-order scheme LIM (2.5) is superior to the explicit Euler method (2.3).

On the right hand side in (2.5) if $\frac{1}{2}Au^k$ is used instead of $\frac{1}{2}Au^{k+1}$ and then we arrive at

$$u^{k+1} = u^k + Au^k(1 - u^{k+1}) \tag{2.6a}$$

i.e.

$$u^{k+1} = \frac{(1 + AP_1(u^k))}{(1 + AQ_1(u^k))}u^k \equiv R_1(u^k)u^k, \tag{2.6b}$$

where $P_1(u^k) = 1$ and $Q_1(u^k) = u^k$. Clearly the scheme (2.6) is only of first-order.

Inserting the right hand side of (2.6a) into the term in (2.5), that is modified just now leads to the second order scheme

$$u^{k+1} = \left(1 + A + \frac{1}{2}A^2\right)u^k - A\left(1 + \frac{1}{2}A\right)u^k u^{k+1} \tag{2.7a}$$

i.e.

the second order explicit scheme

$$u^{k+1} = \frac{(1 + AP_1(u^k) + \frac{1}{2}A^2 P_2(u^k))}{(1 + AQ_1(u^k) + \frac{1}{2}A^2 Q_2(u^k))}u^k = R_2(u^k)u^k \tag{2.7b}$$

where $P_i(u^k) \equiv 1$ and $Q_i(u^k) \equiv u^k (i = 1, 2)$.

In fact, in the case of Logistic equation we can obtain from the expression of the exact solution (2.2),

$$u^1 = \frac{(1 + \sum_{l=1}^{\infty} \frac{1}{l!} A^l)}{(1 + \sum_{l=1}^{\infty} \frac{1}{l!} A^l u^0)} u^0. \tag{2.8}$$

Truncating first m terms of the sums we have the following m-th order method

$$u^{k+1} = \frac{(1 + \sum_{l=1}^{m} \frac{1}{l!} A^l P_l(u^k))}{(1 + \sum_{l=1}^{m} \frac{1}{l!} A^l Q_l(u^k))} u^k \equiv R_m(u^k) u^k \tag{2.9}$$

where $P_l(u^k) \equiv 1$ and $Q_l(u^k) \equiv u^k, l = 1, 2, \cdots, m$.

For the scheme (2.9) there is the following result:

Theorem 2.1. *For the IVP. of Logistic equation the scheme (2.9) is the high fidelity algorithm.*

The proof see [11].

Remark 1. A simple way constructing first order high fidelity algorithm for the IVP. (1.2) is that we may start from any first-order method and then, on condition that order of the method has no change, appropriately modify the scheme such that $P_1(u^0) \geq 0$ and $Q_1(u^0) \geq 0$.

3. Numerical Methods and Preliminaries

The IBVP. of the Fisher PDE. to be discussed reads

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{L^2} \dfrac{\partial^2 u}{\partial x^2} + \alpha u(1-u), 0 < x < 1, t > 0, \alpha > 0, & (3.1) \\ u(0,t) = b_1 \quad \text{and} \quad u(1,t) = b_2, \quad 0 \leq b_i \leq 1, i = 1, 2, t \geq 0 & (3.2) \\ u(x,0) = U_0(x), 0 \leq U_0(x) \leq 1, 0 < x < 1, & (3.3) \end{cases}$$

Where α is a positive stiff parameter, $U_0(x)$is a continuous function for $0 < x < 1$ and the parameter L, the critical length L^* at which bifurcation occurs, is contained in [5]. The IBVP. (3.1)–(3.3) has applications in a number of fields ([2], [3], [7]).

3.1. Numerical Methods

In order to solve numerically this problem, first of all, the region $[0 \leq x \leq 1] \times [t \geq 0]$ is divided by a rectangular grid with steplengths h (space interval $0 \leq x \leq 1$) and τ (time variable) such that $Nh = 1$, N is an integer and $t_k = k\tau (k = 0, 1, 2, \cdots)$.

We define

$$I_h = \{x | x = h, 2h, \cdots, (N-1)h\} \quad \text{and} \quad \overline{I}_h = I_h + \{0\} + \{1\}.$$

Let $\eta^k(x)$ be the value of the mesh function η at mesh point $x \in \overline{I}_h, t = k\tau (k \geq 0)$. We use the following notations

$$\eta_x^k(x) = \frac{1}{h}[\eta^k(x+h) - \eta^k(x)], \quad \eta_{\overline{x}}^k(x) = \eta_x^k(x-h)$$

$$\eta_{x\overline{x}}^k(x) = \frac{1}{h^2}[\eta^k(x+h) - 2\eta^k(x) + \eta^k(x-h)]$$

and

$$\eta_t^k(x) = \frac{1}{\tau}[\eta^{k+1}(x) - \eta^k(x)].$$

Let $u^k(x)$ be the solution of numerical method for IBVP. (3.1)–(3.3). Approximating the space derivative in (3.1) to

$$\frac{\partial^2 u}{\partial x^2}|_{x\in I_h} = u^k_{x\overline{x}}(x) + O(h^2)$$

leads to the ODEs.

$$\frac{du^k(x)}{dt} = \frac{1}{L^2}u^k_{x\overline{x}}(x) + \alpha u^k(x)(1-u^k(x)), \tag{3.4}$$

and then applying the high fidelity scheme (2.6a) given in §2 to the ODEs. (3.4) obtains the linearly implicit scheme with the $(h^2+\tau)$ accuracy

$$u^{k+1}(x) = u^k(x) + \frac{\tau}{L^2}(\theta u^{k+1}_{x\overline{x}}(x) + (1-\theta)u^k_{x\overline{x}}(x)) + Au^k(x)(1-u^{k+1}(x)). \tag{3.5}$$

where $A = \alpha\tau$.

Still further applyng the linearly implicit θ-method (2.4) with $\theta = 1/2$ given in §2 to the ODEs. (3.4) leads to the linearly implicit scheme with the $(h^2+\tau^2)$ accuracy

$$u^{k+1}(x) = u^k(x) + \frac{\tau}{2L^2}(u^{k+1}_{x\overline{x}}(x) + u^k_{x\overline{x}}(x)) + \frac{1}{2}A(u^k(x)+u^{k+1}(x)) - Au^k(x)u^{k+1}(x). \tag{3.6}$$

and then inserting the right hand side of the fornula (3.5) with $\theta = 1/2$ into the place of the term $\frac{1}{2}Au^{k+1}(x)$ in (3.6) can obtain the linearly implicit scheme with the $(h^2+\tau^2)$ accuracy

$$u^{k+1}(x) = u^k(x) + \frac{\tau r}{2L^2}(u^{k+1}_{x\overline{x}}(x) + u^k_{x\overline{x}}(x)) + rAu^k(x)(1-u^{k+1}(x)), \tag{3.7}$$

where $r = \left(1+\frac{1}{2}A\right)$.

Finally the schemes (3.5) and (3.7) can be written as the unified form

$$u^{k+1}(x) = u^k(x) + \frac{\tau r}{L^2}\left(\theta u^{k+1}_{x\overline{x}}(x) + (1-\theta)u^k_{x\overline{x}}(x)\right) + rAu^k(x)(1-u^{k+1}(x)). \tag{3.8}$$

We will discuss the following three schemes.
ME: (the first-order explicit scheme) $r = 1$ and $\theta = 0$,
MI: (the first-order implicit scheme) $r = 1$ and $\theta = 1$, and
MS: (the second-order implicit scheme) $r = \left(1+\frac{1}{2}A\right)$ and $\theta = 1/2$ in the formula (3.8).

3.2. Preliminaries

We need to make provision for discussion of the properties of schemes ME, MI and MS.
First define

$$D(h,\tau,L,\alpha)\eta^k(x) = \eta^k_t(x) - \frac{r}{L^2}(\theta\eta^{k+1}_{x\overline{x}}(x) + (1-\theta)\eta^k_{x\overline{x}}(x)) - \alpha r\eta^k(x)(1-\eta^{k+1}(x)).$$

Thus we have the following result.

Lemma A1. *Assume* $0 \le \eta^k(x), \max_{x\in\overline{I}_h,k\ge 0} \xi^k(x) \le 1$ *and*

$$\begin{cases} D(h,\tau,L,\alpha)\eta^k(x) \le D(h,\tau,L,\alpha)\xi^k(x), & x\in I_h, k\ge 0,\\ \eta^k(x) \le \xi^k(x), & x = 0,1, k\ge 0\\ \eta^0(x) \le \xi^k(x), & x\in I_h \end{cases} \tag{3.9}$$

then for $\theta = 0$ *and* $r = 1$ *with* $\left(1 - \dfrac{2\tau}{L^2h^2}\right) \ge 0$ *or* $\theta = \dfrac{1}{2}$ *with* $\left(1 - \dfrac{\tau r}{L^2h^2}\right) \ge 0$ *there is all*

$$\eta^k(x) \le \xi^k(x), \qquad x \in \overline{I}_h, k \ge 0.$$

Proof. Put $\eta^k(x) = \xi^k(x) + \tilde{\eta}^k(x)$. We can obtain from the inequality (3.9)

$$\begin{aligned} &\left(1 + \frac{2\theta\tau r}{L^2h^2} + rA\eta^k(x)\right)\tilde{\eta}^{k+1}(x) - \frac{\theta r\tau}{L^2h^2}(\tilde{\eta}^{k+1}(x+h) + \tilde{\eta}^{k+1}(x-h)) \\ &\quad \le \left(1 - \frac{2(1-\theta)\tau r}{L^2h^2} + rA(1 - \xi^{k+1}(x))\right)\tilde{\eta}^k(x) \\ &\qquad + \frac{(1-\theta)\tau r}{L^2h^2}(\tilde{\eta}^k(x+h) + \tilde{\eta}^k(x-h)). \end{aligned} \tag{3.10}$$

Obviously, $\tilde{\eta}^0(x) \le 0$ for all $x \in \overline{I}_h$ and suppose that for all $x \in \overline{I}_h, \tilde{\eta}^i(x) \le 0, i \le k$.

Let $\tilde{\eta}^{k+1}(x^{(0)}) = \max\limits_{x\in\overline{I}_h} \tilde{\eta}^{k+1}(x)$ and Assume $x^{(0)} \ne 0$ or 1 (otherwise one of both $\tilde{\eta}^{k+1}(0)$ and $\tilde{\eta}^{k+1}(1) = \max\limits_{x\in\overline{I}_h} \tilde{\eta}^{k+1}(x) \le 0$ from which $\tilde{\eta}^{k+1}(x) \le 0, x \in \overline{I}_h$, thus the induction is completed).

Hence, when $\theta = 0$ and $r = 1$ with $\left(1 - \dfrac{2\tau}{L^2h^2}\right) \ge 0$ or $\theta = 1/2$ with $\left(1 - \dfrac{\tau r}{L^2h^2}\right) \ge 0$ in the inequality (3.10) there is

$$(1 + rA\eta^k(x^{(0)}))\tilde{\eta}^{k+1}(x^{(0)}) \le (1 + rA(1 - \xi^{k+1}(x^{(0)}))) \max_{x\in\overline{I}_h} \tilde{\eta}^k(x) \le 0,$$

from which $\tilde{\eta}^{k+1}(x^{(0)}) \le 0$ and so $\tilde{\eta}^{k+1}(x) \le 0$ for all $x \in \overline{I}_h$. Thus the induction is completed.

Lemma A2. *Let* $\varphi(y) = \dfrac{(1+\beta)y}{(1+\beta y)}, \beta > 0, y \ge 0$. *Then* $\varphi(y)$ *has the following properties:*

(i) $\varphi'(y) : \begin{cases} > 0 & \text{as } 0 \le y < 1 \\ \le 0 & \text{as } y \ge 1 \end{cases}$

and

(ii) $\varphi(0) = 0, \varphi(1) = 1$ *and* $1 \le \varphi(c) < c$ *as* $c \ge 1$.

4. The Properties of Schemes

In this section the boundness and asymptotic behaviour of numerical solution for schemes ME, MI and MS will be discussed in detail.

4.1. The Scheme ME

For first-order explicit scheme ME the difference equation reads

$$(ME) : \begin{cases} u^{k+1}(x) = u^k(x) + \dfrac{\tau}{L^2}u^k_{x\overline{x}}(x) + Au^k(x)(1 - u^{k+1}(x)), \quad k \ge 0 \\ u^k(0) = b_1 \quad \text{and} \quad u^k(1) = b_2, \quad 0 \le b_i \le 1, i = 1, 2, k \ge 0 \\ u(x, 0) = U_0(x) \equiv u^0(x), x \in I_h, A = \alpha\tau. \end{cases} \tag{4.1}$$

Let $u^k(x)$ be the solution of the difference equation (ME). We have the following results.

Theorem ME1. *Suppose that the conditions*

(1) $0 \le U_0(x) \le 1, x \in I_h$

and

(2) $\left(1-\frac{2\tau}{L^2h^2}\right)\geq 0$

hold. Then for arbitrary $\alpha \geq 0$

$$0 \leq u^k(x) \leq 1, \quad x \in \overline{I}_h, k \geq 0.$$

Proof. Solving Eq. (4.1) for $u^{k+1}(x)$ gives

$$u^{k+1}(x) = \frac{(1+A-\frac{2\tau}{L^2h^2})u^k(x)+\frac{\tau}{L^2h^2}(u^k(x+h)+u^k(x-h))}{(1+Au^k(x))}, \quad k \geq 0. \tag{4.1'}$$

By $u^0(x)\geq 0, x\in \overline{I}_h$ and $\left(1-\frac{2\tau}{L^2h^2}\right)\geq 0$ we obtain from (4.1)'

$$u^k(x)\geq 0, \quad x\in \overline{I}_h, k\geq 0.$$

On the other hand taking $\theta = 0, r = 1, \eta^k(x) = u^k(x) \geq 0$ and $\xi^k(x)\equiv 1$ in the Lemma A1. leads to

$$0 \leq u^k(x) \leq 1, \quad x \in \overline{I}_h, k \geq 0.$$

Thus the Theorem is completely proved.

Theorem ME2. *Suppose that the conditions (1) and (2) for Theorem ME1 and*

(3) $-\frac{1}{L^2}u^0_{x\overline{x}}(x) - \alpha u^0(x)(1-u^0(x)) \leq 0, \quad x \in I_h$

hold. Then $u^k(x)$ *is a nondecreasing function of* k *for all* $x \in \overline{I}_h$ *and* $\alpha \geq 0$*. In particular, if the condition (3) is a strict inequality then* $u^k(x)$ *is monotone increasing and*

$$\lim_{k\to\infty} u^k\left(\left[\frac{N}{2}\right]\right) = 1.$$

Proof. Applying Theorem ME1 leads to

$$0 \leq u^k(x) \leq 1, x \in \overline{I}_h, k \geq 0.$$

Using the equation (4.1) can obtain

$$\begin{aligned}(1+Au^k(x))(u^{k+1}(x)-u^k(x)) &= \frac{\tau}{L^2}u^k_{x\overline{x}}(x)+Au^k(x)(1-u^k(x))\\ &\geq 0, k=0 \quad \text{(by the condition (3))}\end{aligned}$$

and

$$\begin{aligned}&-\frac{\tau}{L^2}u^{k+1}_{x\overline{x}}(x) - Au^{k+1}(x)(1-u^{k+1}(x))\\ &= \frac{\tau}{L^2}(u^k_{x\overline{x}}(x)-u^{k+1}_{x\overline{x}}(x)) + [1+A(1-u^{k+1}(x))](u^k(x)-u^{k+1}(x))\\ &= \frac{\tau}{L^2h^2}[(u^k(x+h)-u^{k+1}(x+h)) + (u^k(x-h)-u^{k+1}(x-h))]\\ &\quad + \left[1-\frac{2\tau}{L^2h^2}+A(1-u^{k+1}(x))\right](u^k(x)-u^{k+1}(x))\\ &\leq (1+A(1-u^{k+1}(x)))\max_{x\in\overline{I}_h}(u^k(x)-u^{k+1}(x))\\ &\leq 0, k=0 \quad \text{(by the above inequality).}\end{aligned}$$

By the above inequalities there is $u^0(x) \leq u^1(x) \leq 1$ and the condition (3) for $k = 1$ holds and that the induction can be carried out. In particular, if (3) is a strict inequality then

$$u^k(x) < u^{k+1}(x) \leq 1, \quad x \in \overline{I}_h, \quad k \geq 0$$

and

$$\lim_{k\to\infty} u^k\left(\left[\frac{N}{2}\right]\right) = 1.$$

Thus the Theorem is completely proved.

Similarly we can prove the following result.

Theorem ME3. *Suppose that the conditions (1) and (2) for Theorem ME1 and*

(3)' $-\frac{1}{L^2}u^0_{x\overline{x}}(x) - \alpha u^0(x)(1-u^0(x)) \geq 0, \quad x \in I_h$

hold. Then $u^k(x)$ is a nonincreasing function of k for all $x \in \overline{I}_h$ and $\alpha \geq 0$. In particular, if (3)' is a strict inequality then $u^k(x)$ is monotone decreasing and

$$\lim_{k\to\infty} u^k\left(\left[\frac{N}{2}\right]\right) = 0.$$

Remark 2. Let $k \to \infty$ in (4.1) and

$$v(x) = \lim_{k\to\infty} u^k(x), \quad x \in I_h.$$

Then $v(x)$ is a solution of the following steady problem

$$\begin{cases} -\frac{1}{L^2}v_{x\overline{x}}(x) - \alpha v(x)(1-v(x)) = 0 \\ v(0) = b_1 \quad \text{and} \quad v(1) = b_2, 0 \leq b_i \leq 1, i = 1, 2 \end{cases}$$

and the limit value $v\left(\left[\frac{N}{2}\right]\right)$ is determined by the initial-boundary value, the second-order difference quotient of $U_0(x)$ and L, the eritical length of (ME).

Remark 3. The condition $\left(1 - \frac{2\tau}{L^2h^2}\right) \geq 0$ for (ME) which is indepentent of α, is not improvable because it is the stability conditiion for the corresponding heat equation ($\alpha = 0$).

4.2. The scheme MI

For first-order implicit scheme MI the difference equation reads

$$(MI): \begin{cases} u^{k+1}(x) = u^k(x) + \frac{\tau}{L^2}u^{k+1}_{x\overline{x}}(x) + Au^k(x)(1-u^{k+1}(x)), k \geq 0, \\ u^k(0) = b_1 \quad \text{and} \quad u^k(1) = b_2, 0 \leq b_i \leq 1, i = 1, 2, k \geq 0 \\ u(x,0) = U_0(x) \equiv u^0(x), x \in I_h, A = \alpha\tau. \end{cases} \tag{4.2}$$

Let $u^k(x)$ be the solution of (MI). We have the following results.

Theorem MI1. *Suppose $0 \leq U_0(x) \leq 1, x \in I_h$. Then for all positive parameters h, τ and α,*

$$0 \leq u^k(x) \leq 1, \quad x \in \overline{I}_h, \quad k \geq 0.$$

Proof. The equation (4.2) can be rewritten as

$$(1 + Au^k(x))u^{k+1}(x) = \frac{\tau}{L^2}u^{k+1}_{x\overline{x}}(x) + (1+A)u^k(x), \quad k \geq 0. \tag{4.2'}$$

We may suppose that no boundary value arrives at $\max_{x\in\overline{I}_h} u^{k+1}(x)$ or $\min_{x\in\overline{I}_h} u^{k+1}(x)$ otherwise, at least, one side of the inequality $0 \le u^k(x) \le 1$ is proved.

Now suppose that for all $x \in \overline{I}_h$ and $j \le k, 0 \le u^j(x) \le 1$. Assume $u^{k+1}(x^{(0)}) = \max_{x\in I_h} u^{k+1}(x)$ and $u^{k+1}(x^{(1)}) = \min_{x\in I_h} u^{k+1}(x)$. Then there are $u^{k+1}_{x\overline{x}}(x^{(0)}) \le 0$ and $u^{k+1}_{x\overline{x}}(x^{(1)}) \ge 0$ respectively. By the formula (4.2)' and Lemma A2, we can obtain

$$u^{k+1}(x^{(0)}) \le \frac{(1+A)u^k(x^{(0)})}{(1+Au^k(x^{(0)}))} \le 1$$

and

$$u^{k+1}(x^{(1)}) \ge \frac{(1+A)u^k(x^{(1)})}{(1+Au^k(x^{(1)}))} \ge 0$$

from which $0 \le u^{k+1}(x) \le 1$, $x \in \overline{I}_h, k \ge 0$. Thus the induction is completed.

Theorem MI2. *Suppose that*

(1) $0 \le U_0(x) \le 1, x \in I_h, k \ge 0$

and

(3) $-\frac{1}{L^2}u^0_{x\overline{x}}(x) - \alpha u^0(x)(1-u^0(x)) \le 0, x \in I_h$

hold. Then $u^k(x), x \in \overline{I}_h$, is a nondecreasing function of k for all positive parameters h, τ and α. In particular, if the condition (3) is a strict inequality then $u^k(x), x \in I_h$, is monotone increasing and $\lim_{k\to\infty} u^k\left(\left[\frac{N}{2}\right]\right) = 1$.

Proof. Applying Theorem MI1 leads to

$$0 \le u^k(x) \le 1, \quad x \in \overline{I}_h, \quad k \ge 0.$$

The equation (4.2) can be rewritten as

$$\begin{aligned}&(1+Au^k(x))(u^{k+1}(x) - u^k(x)) - \frac{\tau}{L^2}(u^{k+1}_{x\overline{x}}(x) - u^k_{x\overline{x}}(x))\\ &= \frac{\tau}{L^2}u^k_{x\overline{x}}(x) + Au^k(x)(1-u^k(x)).\end{aligned}$$

Let $\psi^k(x) = u^{k+1}(x) - u^k(x), u \in \overline{I}_h$. Clearly, $\psi^k(x)|_{x=0,1} = 0$. Assume $\psi^k(x^{(1)}) = \min_{x\in\overline{I}_h} \psi^k(x), k \ge 0$ and $x^{(1)} \ne 0$ or 1. Then

$$(1+Au^k(x^{(1)}))\psi^k(x^{(1)}) \ge \frac{\tau}{L^2}u^k_{x\overline{x}}(x^{(1)}) + Au^k(x^{(1)})(1-u^k(x^{(1)})),$$

from which $(u^1(x) - u^0(x)) \ge 0, x \in \overline{I}_h$, moreover, by the equation (4.2) there is

$$\begin{aligned}&-\frac{\tau}{L^2}u^{k+1}_{x\overline{x}}(x) - Au^{k+1}(x)(1-u^{k+1}(x))\\ &= -(1+A(1-u^{k+1}(x)))(u^{k+1}(x) - u^k(x)),\end{aligned}$$

from which the condition (3) for $k = 1$ holds and that the induction can be carried out. In particular, if (3) is a strict inequality then

$$u^k(x) < u^{k+1}(x) \le 1, \quad x \in I_h, k \ge 0$$

and $\lim_{k\to\infty} u^k\left(\left[\frac{N}{2}\right]\right) = 1$. Thus the Theorem is completed proved.

Similarly, we can prove the following result.

Theorem MI3. *Suppose that the condition (1) and*

(3)' $-\frac{1}{L^2}u^0_{xx}(x) - \alpha u^0(x)(1-u^0(x)) \geq 0, x \in I_h$

hold. Then $u^k(x)$ *is a nonincreasing function of* k *for all positive parameters* h, τ *and* α*. In particular, if (3)' is a strict inequality then* $u^k(x)$ *is monotone decreasing and* $\lim_{k\to\infty} u^k\left(\left[\frac{N}{2}\right]\right) = 0$.

4.3. The Scheme MS

For second-order scheme MS the difference equation reads

$$(MS): \begin{cases} u^{k+1}(x) = u^k(x) + \dfrac{\tau r}{2L^2h^2}(u^{k+1}_{x\overline{x}}(x) + u^k_{x\overline{x}}(x)) + rAu^k(x)(1-u^{k+1}(x)), \\ \qquad\qquad x \in I_h, k \geq 0 \\ u^k(0) = b_1 \quad \text{and} \quad u^k(1) = b_2, 0 \leq b_i \leq 1, i = 1,2, k \geq 0 \\ u^0(x,0) = U_0(x) \equiv u^0(x), x \in I_h, A = \alpha\tau, \end{cases} \tag{4.3}$$

where $r = \left(1 + \frac{1}{2}A\right)$.

Applying the techniques used in §4.1 and §4.2 to the difference equation (MS) can obtain the similar results. Only the key steps to the proof of these results are given in the following.

Theorem MS1. *Suppose that the conditions*

(1) $0 \leq U_0(x) \leq 1, x \in I_h$

and

(2) $\left(1 - \frac{\tau r}{L^2h^2}\right) \geq 0$

hold. Then for all $x \in \overline{I}_h$,

$$0 \leq u^k(x) \leq 1, k \geq 0.$$

Proof. The equation (4.3) is rewritten as

$$(1 + rAu^k(x))u^{k+1}(x) = \frac{\tau r}{2L^2}(u^{k+1}_{x\overline{x}}(x) + u^k_{x\overline{x}}(x)) + (1+rA)u^k(x), \quad x \in I_h.$$

Assume $u^{k+1}(x^{(1)}) = \min_{x\in\overline{I}_h} u^{k+1}(x)$ and $x^{(1)} \neq 0$ or 1. Then

$$\begin{aligned} (1 + rAu^k(x^{(1)}))u^{k+1}(x^{(1)}) &\geq \left[1 - \frac{\tau r}{L^2h^2} + Ar\right]u^k(x^{(1)}) \\ &\quad + \frac{\tau r}{2L^2h^2}(u^k(x^{(1)}+h) + u^k(x^{(1)}-h)) \\ &\geq (1+rA)\min_{x\in\overline{I}_h} u^k(x) \geq 0. \end{aligned}$$

Moreover, in the Lemma A1 taking $\theta = 1/2, \eta^k(x) = u^k(x) \geq 0$ and $\xi^k(x) \equiv 1$ leads to

$$0 \leq u^k(x) \leq 1, \quad x \in \overline{I}_h, k \geq 0.$$

Theorem MS2. *Suppose that the conditions (1) and (2) for Theorem MS1 and*

(3) $-\frac{1}{L^2}u^0_{x\overline{x}}(x) - \alpha u^0(x)(1-u^0(x)) \leq 0, \quad x \in I_h$

hold. Then $u^k(x)$ is a nondecreasing function of k for all $x \in I_h$. In particular, if the condition (3) is a strict inequality then $u^k(x)$ is monotone increasing and $\lim_{k\to\infty} u\left(\left[\frac{N}{2}\right]\right) = 1$.

Proof.

i) Applying the Theorem MS1 leads to

$$0 \le u^k(x) \le 1, \quad x \in \overline{I}_h, k \ge 0.$$

ii) The equation (4.3) is rewritten as

$$\begin{aligned}&(1 + rAu^k(x))(u^{k+1}(x) - u^k(x)) - \frac{\tau}{2L^2}(u^{k+1}_{x\overline{x}}(x) - u^k_{x\overline{x}}(x))\\ &= \frac{\tau r}{L^2}u^k_{x\overline{x}}(x) + rAu^k(x)(1 - u^k(x))(\ge 0).\end{aligned}$$

iii) Using the equation (4.3) can obtain

$$\begin{aligned}&-\frac{\tau r}{L^2}u^{k+1}_{x\overline{x}}(x) - rAu^{k+1}(x)(1 - u^{k+1}(x))\\ &= \frac{\tau r}{2L^2}(u^k_{x\overline{x}}(x) - u^{k+1}_{x\overline{x}}(x)) + (1 + rA(1 - u^{k+1}(x)))(u^k(x) - u^{k+1}(x))\\ &\le (1 + rA(1 - u^{k+1}(x)))\max_{x\in\overline{I}_h}(u^k(x) - u^{k+1}(x)).\end{aligned}$$

On the basis of three inequalities above it is easy to prove the results of the Theorem.

Similarly, we can prove the following result.

Theorem MS3. *Suppose that the conditions (1) and (2) for Theorem MS1 and*

(3)' $-\frac{1}{L^2}u^0_{x\overline{x}}(x) - \alpha u^0(x)(1 - u^0(x)) \ge 0, \quad x \in I_h$

hold. Then $u^k(x)$ is a nonincreasing function of k for all $x \in \overline{I}_h$. In particular, if (3)' is a strict inequality then $u^k(x)$ is monotone decreasing and

$$\lim_{k\to\infty} u^k\left(\left[\frac{N}{2}\right]\right) = 0.$$

Remark 4. The approach constructing the schemes in this paper can be extended to reaction-diffusion equations with other stiff scource terms. It remains to be researched whether the condition (2) $\left((1 - \frac{r\tau}{L^2h^2}) \ge 0\right)$, in Theorem MS1, can be cut out.

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