Journal of Computational Mathematics, Vol.19, No.6, 2001, 601–606.

D-CONVERGENCE OF ONE-LEG METHODS FOR STIFF DELAY DIFFERENTIAL EQUATIONS*1)

Cheng-ming Huang
(*Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, China*)

Shou-fu Li
(*Department of Mathematics, Xiangtan University, Xiangtan 411105, China*)

Hong-yuan Fu   Guang-nan Chen
(*Institute of Applied Physics and Computational Mathematics, Beijing 100088, China*)

Abstract

This paper is concerned with the numerical solution of delay differential equations(DDEs). We focus on the error analysis of one-leg methods applied nonlinear stiff DDEs. It is proved that an A-stable one-leg method with a simple linear interpolation is D-convergent of order p, if it is consistent of order p in the classical sense.

Key words: Nonlinear delay differential equations, One-leg methods, D-convergence.

1. Introduction

In recent years, many papers discussed numerical methods for the solution of delay differential equation (DDE)

$$y'(t) = f(t, y(t), y(t-\tau)). \tag{1.1}$$

For linear stability of numerical methods , a significant number of results have already been found for both Runge-Kutta methods and linear multistep methods (cf.[4] [7] [8]).Recently, we further established the relationship between G-stability and nonlinear stability (cf.[3]). Error analysis of DDE solvers is another important issue. In fact, many papers investigated the local and global error behaviour of DDE solvers (cf.[1] [10]). However, error analysis of numerical methods for DDEs is mostly based on the fact that the function $f(t, y, z)$ satisfies Lipschitz conditions in both the last two variables. They are suitable for nonstiff DDEs because the Lipschitz constants are moderate. However, they can not be applied to stiff DDEs. For example, consider semi-discrete Hutchinson's equation (cf.[2]) with

$$f(t,y(t),y(t-\tau)) = \frac{a}{\Delta x^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ \vdots \\ y_N(t) \end{bmatrix} + \begin{bmatrix} y_1(t)(1-y_1(t-\tau)) \\ y_2(t)(1-y_2(t-\tau)) \\ \vdots \\ \vdots \\ y_N(t)(1-y_N(t-\tau)) \end{bmatrix}, \tag{1.2}$$

where $a > 0$ is the diffusion coefficient, $\Delta x = 1/(N+1)$ is a constant stepsize in space. In this case, the Lipschitz constant L of the function $f(t, y, z)$ with respect to y will contain negative powers of the meshwidth Δx in space. As a consequence, L will be very large for fine space grids, and the error estimates based on L are not realistic. On the other hand, the one-sided

* Received January 13, 1998.
1)Project supported by NSF of China.

Lipschitz constant α is only moderate. Hence estimates based on α are often considerably more realistic than that based on L. Recently, the concept of D-convergence [11] for DDEs, which is a generalization of the concept of B-convergence (cf.[5] [6]) for ODEs, was introduced. In [3], we discussed D-convergence of A-stable one-leg methods with a complex interpolation procedure . In this paper, we further discuss D-convergence of A-stable one-leg methods with a more simple interpolation procedure.

2. Preliminaries

Let $\langle\cdot,\cdot\rangle$ be an inner product on C^N and $\|\cdot\|$ the corresponding norm. Consider the following nonlinear equation

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau)), \quad t \geq 0, \\ y(t) = \phi_1(t), \quad t \leq 0, \end{cases} \tag{2.1}$$

where τ is a positive delay term , ϕ_1 is a continuous function, and $f : [0, +\infty) \times C^N \times C^N \to C^N$, is a given mapping which satisfies the following conditions:

$$\mathrm{Re}\langle u_1 - u_2, f(t, u_1, v) - f(t, u_2, v)\rangle \leq \alpha\|u_1 - u_2\|^2, \quad t \geq 0, u_1, u_2, v \in C^N, \tag{2.2}$$

$$\|f(t, u, v_1) - f(t, u, v_2)\| \leq \beta\|v_1 - v_2\|, \quad t \geq 0, u, v_1, v_2 \in C^N, \tag{2.3}$$

where α and β are real constants. In order to make the error analysis feasible, we always assume that the problem (2.1) has a unique solution $y(t)$ which is sufficiently differentiable and satisfies

$$\|\frac{d^i y(t)}{dt^i}\| \leq M_i.$$

Remark 2.1. When $\beta = 0$, the above problem class has been used widely in stiff ODEs field (cf.[6]).

Before stating stability results, we introduce another system, defined by the same function $f(t, u, v)$, but with another initial condition:

$$\begin{cases} z'(t) = f(t, z(t), z(t-\tau)), \quad t \geq 0, \\ z(t) = \phi_2(t), \quad t \leq 0. \end{cases} \tag{2.4}$$

Proposition 2.2. *Suppose $\beta \leq -\alpha$.Then the following is true:*

$$\|y(t) - z(t)\| \leq \max_{x \leq 0} \|\phi_1(x) - \phi_2(x)\|, \quad t \geq 0. \tag{2.5}$$

The proof of this proposition can be found in [9]. Similarily, we can easily obtain the following result.

Proposition 2.3. *Suppose $\beta < -\alpha$. Then the following holds:*

$$\lim_{t \to +\infty} \|y(t) - z(t)\| = 0. \tag{2.6}$$

Now we consider the adaptation of one-leg methods to (2.1). We briefly recall the form of a one-leg method for the numerical solution of the ordinary differential equation

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \geq 0, \\ y(0) = y_0. \end{cases} \tag{2.7}$$

The one-leg k step method is the following

$$\rho(E)y_n = hf(\sigma(E)t_n, \sigma(E)y_n), \tag{2.8}$$

where $h > 0$ is the stepsize, E is the translation operator:$Ey_n = y_{n+1}$, each y_n is an approximation to the exact solution $y(t_n)$ with $t_n = nh$, and $\rho(x) = \sum_{j=0}^{k} \alpha_j x^j$ and $\sigma(x) = \sum_{j=0}^{k} \beta_j x^j$ are generating polynomials, which are assumed to have real coefficients, no common divisor .

We also assume $\rho(1) = 0, \rho'(1) = \sigma(1) = 1$.

Apply the one-leg k-step method (ρ, σ) to DDE (2.1)

$$\rho(E)y_n = hf(\sigma(E)t_n, \sigma(E)y_n, \bar{y}_n), \quad n = 0, 1, 2, \cdots, \tag{2.9}$$

where the argument $\bar{y}_n$ denotes an approximation to $y(\sigma(E)t_n - \tau)$ that is obtained by a specific interpolation at the point $t = \sigma(E)t_n - \tau$

using $\{y_i\}_{i<n+k}$.

Process (2.9) is defined completely by the one-leg method (ρ, σ) and the interpolation procedure for $\bar{y}_n$.

It is well known that any A-stable one-leg method for ODEs has order at most 2. So we can use the linear interpolation procedure for $\bar{y}_n$. Let $\tau = (m - \delta)h$ with integer $m \geq 1$ and $\delta \in [0, 1)$.

In [3], We define

$$\bar{y}_n = \delta\sigma(E)y_{n-m+1} + (1 - \delta)\sigma(E)y_{n-m}, \tag{2.10}$$

where $y_l = \phi_1(lh)$ for $l \leq 0$. For this kind of interpolation procedure, it is proved that an A-stable one-leg method (ρ, σ) is D-convergent of order p if it is consistent of p.

In this paper, we consider another more simple interpolation procedure for $\bar{y}_n$. Let $\sigma'(1) - m + \delta = \mu + x$ with integer μ and $x \in [0, 1)$. Define

$$\bar{y}_n = \begin{cases} \phi_1(t_{n+\mu} + xh), & t_{n+\mu} + xh \leq 0, \\ xy_{n+\mu+1} + (1 - x)y_{n+\mu}, & t_{n+\mu} + xh > 0, m \geq \sigma'(1) + 2 - k, \end{cases} \tag{2.11}$$

where we assume $m \geq \sigma'(1) + 2 - k$ so as to guarantee that, in the interpolation procedure for $\bar{y}_n$, no unknown values y_i with $i > n + k - 1$ are used.

Definition 2.4. (cf.[3] [11]) *The one-leg method (2.9) with interpolation procedure (2.11) is said to be D-convergent of order p if this method when applied to any given problem (2.1) with initial values $y_0, y_1, \cdots, y_{k-1}$, produces an*

approximation sequence $\{y_n\}$, and the global error satisfies a bound of the form

$$\|y(t_n) - y_n\| \leq C(t_n)(h^p + \max_{0 \leq i < k} \|y(t_i) - y_i\|), \quad n \geq k, h \in (0, h_0], \tag{2.12}$$

where the function $C(t)$ and the maximum stepsize h_0 depend only on the method, some of the bounds M_i, the parameters α, β and τ.

Proposition 2.5. *D-convergence implies B-convergence.*

In this paper, for a real symmetric positive definite $k \times k$ matrix $G = [g_{ij}]$, the norm $\|\cdot\|_G$ is defined by

$$\|U\|_G = (\sum_{i,j=1}^{k} g_{ij}\langle u_i, u_j\rangle)^{\frac{1}{2}}, \quad U = (u_1^T, u_2^T, \cdots, u_k^T)^T \in C^{kN}.$$

3. Error Analysis

Consider the scheme (2.9)-(2.11) and the following scheme

$$\rho(E)\hat{y}_n + \alpha_k e_n = hf(\sigma(E)t_n, \sigma(E)\hat{y}_n + \beta_k e_n, \bar{Y}_n), \quad n = 0, 1, 2, \cdots. \tag{3.1}$$

Theorem 3.1. *Assume that the one-leg (ρ, σ) is G-stable with respect to positive definite matrix G. Then*

$$\|\epsilon_{n+1}\|_G^2 \leq \|\epsilon_0\|_G^2 + h\sum_{i=0}^{n}[\|\epsilon_i\|_G^2 + 2c_2(1 + h)\|\sigma(E)(y_i - \hat{y}_i)\|^2$$

$$+\beta(1 + h)\|\bar{y}_i - \bar{Y}_i\|^2 + (1 + h)(2c_2\beta_k^2 + \lambda_1 h^{-2})\|e_i\|^2], n = 0, 1, 2, \cdots, \tag{3.2}$$

where

$$\epsilon_n = ((y_n - \hat{y}_n)^T, (y_{n+1} - \hat{y}_{n+1})^T, \cdots, (y_{n+k-1} - \hat{y}_{n+k-1})^T)^T,$$

$$c_2 = \max(0, 2\alpha + \beta),$$

and λ_1 *denotes the maximum eigenvalue of the matrix* G.

Proof. Let $z_n = \hat{y}_n, \cdots, z_{n+k-1} = \hat{y}_{n+k-1}, z_{n+k} = \hat{y}_{n+k} + e_n$,

$$w_1 = \begin{bmatrix} y_{n+1} - z_{n+1} \\ y_{n+2} - z_{n+2} \\ \vdots \\ y_{n+k} - z_{n+k} \end{bmatrix}. \tag{3.3}$$

Then

$$\rho(E)z_n = hf(\sigma(E)t_n, \sigma(E)z_n, \bar{Y}_n). \tag{3.4}$$

In view of G-stability of the method, there exists positive definite G such that for all real $a_0, a_1, \cdots, a_k$,

$$A_1^T G A_1 - A_0^T G A_0 \le 2\sigma(E)a_0\rho(E)a_0,$$

where $A_i = (a_i, a_{i+1}, \cdots, a_{i+k-1})^T, i = 0, 1$. Therefore, we can easily obtain(cf.[5] [6])

$$\|w_1\|_G^2 - \|\epsilon_n\|_G^2 \le 2\mathrm{Re}\langle\sigma(E)(y_n - z_n), \rho(E)(y_n - z_n)\rangle. \tag{3.5}$$

Hence

$$\begin{aligned}\|w_1\|_G^2 - \|\epsilon_n\|_G^2 &\le 2\mathrm{Re}\langle\sigma(E)(y_n - z_n), h(f(\sigma(E)t_n, \sigma(E)y_n, \bar{y}_n) \\ &\quad -f(\sigma(E)t_n, \sigma(E)z_n, \bar{Y}_n))\rangle. \\ &\le 2\mathrm{Re}\langle\sigma(E)(y_n - z_n), h(f(\sigma(E)t_n, \sigma(E)y_n, \bar{y}_n) - f(\sigma(E)t_n, \sigma(E)z_n, \bar{y}_n))\rangle \\ &\quad +2\mathrm{Re}\langle\sigma(E)(y_n - z_n), h(f(\sigma(E)t_n, \sigma(E)z_n, \bar{y}_n) - f(\sigma(E)t_n, \sigma(E)z_n, \bar{Y}_n))\rangle \\ &\le 2h[\alpha\|\sigma(E)(y_n - z_n)\|^2 + \beta\|\sigma(E)(y_n - z_n)\| \cdot \|\bar{y}_n - \bar{Y}_n\|] \\ &\le (2\alpha + \beta)h\|\sigma(E)(y_n - z_n)\|^2 + \beta h\|\bar{y}_n - \bar{Y}_n\|^2.\end{aligned} \tag{3.6}$$

On the other hand, it follows from Cauchy inequality that

$$\|\epsilon_{n+1}\|_G^2 \le (1 + h)\|w_1\|_G^2 + (1 + h^{-1})\lambda_1\|e_n\|^2, \tag{3.7}$$

and

$$\|\sigma(E)(y_n - z_n)\|^2 \le 2\|\sigma(E)(y_n - \hat{y}_n)\|^2 + 2\beta_k^2\|e_n\|^2. \tag{3.8}$$

A combination of (3.6),(3.7) and (3.8) leads to

$$\|\epsilon_{n+1}\|_G^2 \le (1+h)[\|\epsilon_n\|_G^2 + 2c_2h\|\sigma(E)(y_n - \hat{y}_n)\|^2 + \beta h\|\bar{y}_n - \bar{Y}_n\|^2 + (2c_2h\beta_k^2 + h^{-1}\lambda_1)\|e_n\|^2]. \tag{3.9}$$

By induction we can easily obtain that (3.2) holds, which completes the proof of Theorem 3.1.

In the following, we further assume

$$c_1 = -\frac{1}{2}\sum_{j=0}^{k-1}(\beta_j - \frac{\beta_k}{\alpha_k}\alpha_j)j^2 - \frac{\beta_k}{\alpha_k}\sum_{j=0}^{k} j\beta_j + \frac{1}{2}(\sum_{j=0}^{k} j\beta_j)^2, \tag{3.10}$$

$$\hat{y}_n = y(t_n) + c_1h^2y''(t_n), \tag{3.11}$$

$$\bar{Y}_n = y(\sigma(E)t_n - \tau). \tag{3.12}$$

Theorem 3.2. *Assume that the one-leg method* (ρ, σ) *is consistent of order* $s \le 2$ *in the classical sense and that* $\frac{\beta_k}{\alpha_k} > 0$, *then there exist constants* d_1 *and* $h_1 \le 1$, *which depend only on the method, some of the bounds* M_i, *the parameters* α, β *and* τ, *such that*

$$\|e_n\| \le d_1h^{s+1}, h \in (0, h_1], n = 0, 1, 2, \cdots. \tag{3.13}$$

The proof of Theorem 3.2 can be found in [3].

Theorem 3.3. *If the A-stable one-leg k step method (ρ,σ) is consistent of order p in the classical sense for ODEs, then it is D-convergent of order p, where $k \geq 1, p = 1$ or 2.*

Proof. Suppose the method (ρ,σ) is A-stable. Then $\frac{\beta_k}{\alpha_k} > 0$ and the method is G-stable . From Theorem 3.1 and Theorem 3.2, we have

$$\|\epsilon_{n+1}\|_G^2 \leq \|\epsilon_0\|_G^2 + h\sum_{i=0}^{n}[\|\epsilon_i\|_G^2 + 4c_2\|\sigma(E)(y_i - \hat{y}_i)\|^2$$
$$+2\beta\|\bar{y}_i - \bar{Y}_i\|^2 + 2(2c_2\beta_k^2 + h^{-2}\lambda_1)d_1^2h^{2(p+1)}], h \in (0, h_1]. \quad (3.14)$$

On the one hand,

$$\|\bar{y}_i - \bar{Y}_i\| = \|\bar{y}_i - y(\sigma(E)t_i - \tau)\| = \|\bar{y}_i - y(t_{i+\mu} + xh)\|. \quad (3.15)$$

Hence $\|\bar{y}_i - \bar{Y}_i\| = 0$ when $i + \mu < 0$, and when $i + \mu \geq 0$ we have

$$\|\bar{y}_i - \bar{Y}_i\| = \|xy_{i+\mu+1} + (1-x)y_{i+\mu} - y(t_{i+\mu} + xh)\|$$
$$\leq x\|y_{i+\mu+1} - \hat{y}_{i+\mu+1}\| + (1-x)\|y_{i+\mu} - \hat{y}_{i+\mu}\| + \|x\hat{y}_{i+\mu+1} + (1-x)\hat{y}_{i+\mu} - y(t_{i+\mu} + xh)\|$$
$$\leq \|y_{i+\mu+1} - \hat{y}_{i+\mu+1}\| + \|y_{i+\mu} - \hat{y}_{i+\mu}\| + (1 + |c_1|)M_2h^2. \quad (3.16)$$

Therefore

$$\|\bar{y}_i - \bar{Y}_i\|^2 \leq 3[\|y_{i+\mu+1} - \hat{y}_{i+\mu+1}\|^2 + \|y_{i+\mu} - \hat{y}_{i+\mu}\|^2 + (1 + |c_1|)^2M_2^2h^4]. \quad (3.17)$$

It follows from (2.11) that $\mu \leq k - 2$ which implies $n + \mu + 1 \leq n + k - 1$. Hence

$$\sum_{i=0}^{n}\|\bar{y}_i - \bar{Y}_i\|^2 \leq \sum_{i=-\mu}^{n}\|\bar{y}_i - \bar{Y}_i\|^2 \leq 6\sum_{i=0}^{n+k-1}\|y_i - \hat{y}_i\|^2 + 3(1 + |c_1|)^2(n + \mu + 1)M_2^2h^4. \quad (3.18)$$

On the other hand,it follows from Cauchy inequality that

$$\|\sigma(E)(y_i - \hat{y}_i)\|^2 = \|\sum_{j=0}^{k}\beta_j(y_{i+j} - \hat{y}_{i+j})\|^2 \leq (k+1)\sum_{j=0}^{k}\beta_j^2\|y_{i+j} - \hat{y}_{i+j}\|^2. \quad (3.19)$$

Then

$$\sum_{i=0}^{n}\|\sigma(E)(y_i - \hat{y}_i)\|^2 \leq (k+1)\sum_{j=0}^{k}\beta_j^2\sum_{i=0}^{n}\|y_{i+j} - \hat{y}_{i+j}\|^2$$
$$\leq (k+1)\sum_{j=0}^{k}\beta_j^2\sum_{i=0}^{n+k-1}\|y_i - \hat{y}_i\|^2 + (k+1)\beta_k^2\|y_{n+k} - \hat{y}_{n+k}\|^2. \quad (3.20)$$

Let λ_1 and λ_2 denote the maximum and minimum eigenvalues of the matrix G respectively. It is easily seen that

$$\sum_{i=0}^{n}\|\epsilon_i\|_G^2 \leq \lambda_1\sum_{i=0}^{n}(\sum_{j=0}^{k-1}\|y_{i+j} - \hat{y}_{i+j}\|^2) \leq k\lambda_1\sum_{i=0}^{n+k-1}\|y_i - \hat{y}_i\|^2, \quad (3.21)$$

and

$$\|\epsilon_{n+1}\|^2 \geq \lambda_2\|y_{n+k} - \hat{y}_{n+k}\|^2. \quad (3.22)$$

Substituting (3.14) with (3.18) ,(3.20),(3.21) and (3.22) leads to

$$\|y_{n+k} - \hat{y}_{n+k}\|^2 \leq d_2h^{2p}(n+k)h + d_3\sum_{i=0}^{k-1}\|y_i - \hat{y}_i\|^2 + d_4h\sum_{i=0}^{n+k-1}\|y_i - \hat{y}_i\|^2, \quad h \in (0, h_0], \quad (3.23)$$

where

$$\begin{aligned} h_0 &= \begin{cases} h_1 & c_2 = 0, \\ \min(h_1, \lambda_2/(8c_2(k+1)\beta_k^2)), & c_2 \neq 0, \end{cases} \\ d_2 &= 2[2(2c_2\beta_k^2 + \lambda_1)d_1^2 + 6(1+|c_1|)^2 M_2^2]/\lambda_2, \\ d_3 &= 2\lambda_1/\lambda_2, \\ d_4 &= 2(k\lambda_1 + 4c_2(k+1)\sum_{j=0}^{k}\beta_j^2 + 12\beta)/\lambda_2. \end{aligned} \tag{3.24}$$

By the discrete Bellman inequality, we have

$$\|y_{n+k} - \hat{y}_{n+k}\|^2 \leq (d_2 t_{n+k} h^{2p} + k(d_3+d_4)\max_{0\leq i\leq k-1}\|y_i - \hat{y}_i\|^2)\exp(d_4 t_{n+k}), \quad h \in (0, h_0]. \tag{3.25}$$

In view of (3.11), we have

$$\|y_{n+k} - y(t_{n+k})\| \leq |c_1|M_2h^2 + [\sqrt{d_2 t_{n+k}}h^p + \sqrt{k(d_3+d_4)}(\max_{0\leq i\leq k-1}\|y_i - y(t_i)\| + |c_1|M_2h^2)]$$
$$\exp(\frac{1}{2}d_4 t_{n+k}), \quad h \in (0, h_0], \tag{3.26}$$

which shows the method is D-convergent of order $p, p = 1$ or 2.

Remark 3.1. For solvability of the equation (2.9) with (2.11), we refer to [6].

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