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THE MORTAR ELEMENT METHOD FOR ROTATED $Q1$ ELEMENT[*1)]

Jin-ru Chen
(*School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing 210097, China*)

Xue-jun Xu
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, a mortar element version for rotated $Q1$ element is proposed. The optimal error estimate is proven for the rotated $Q1$ mortar element method.

Key words: Mortar element method, Rotated $Q1$ element.

1. Introduction

Many authors have made significant contributions to the so-called mortar element method (see [4] [5] [7] [8] [10] [11], and references therein). The mortar element method is a nonconforming domain decomposition method with non-overlapping subdomains. The meshes on different subdomains need not align across subdomain interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. This offers the advantages of freely choosing highly varying mesh sizes on different subdomains and is very promising to approximate the problems with abruptly changing diffision coefficients or local anisotropies.

The rotated $Q1$ element is an important nonconforming element. It was first proposed and analysed in [12] for numerically solving the Stokes problem. The rotated $Q1$ element provides the simplest example of discretely divergence-free nonconforming element on quadrilaterals. Due to its simplicity, the rotated $Q1$ element is used to simulate the deformation of martensitic crystals with microstructure in [9]. Independently, it also was derived within the framework of mixed element method (see [2]). In [2] it was proven that Raviart-Thomas mixed rectangle element method is equivalent to rotated $Q1$ nonconforming element method.

The purpose of this paper is to study the rotated $Q1$ mortar element method. A mortar element version for rotated Q1 element is proposed. By constructing some relations between rotated $Q1$ mortar element and bilinear element, the optimal error estimate for rotated $Q1$ mortar element method is proven.

The remainder of this paper is organized as follows. In §2 we introduce model problem, the rotated $Q1$ mortar element method, and some notations. In §3 some technical Lemmas are given. In §4 the optimal error estimate is shown. For convenience, the symbols $\preceq$, $\succeq$, and $\asymp$ will be used in this paper, and $x_1 \preceq y_1$, $x_2 \succeq y_2$, and $x_3 \asymp y_3$ mean that $x_1 \le C_1 y_1$, $x_2 \ge c_2 y_2$, and $c_3 x_3 \le y_3 \le C_3 x_3$ for some constants C_1, c_2, c_3, and C_3 that are independent of mesh

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parameters. For any subdomain $D \subset \Omega$, we use usual L^2 inner product $(\cdot,\cdot)_D$, Sobolev space $H^s(D)$ with usual Sobolev norm $\|\cdot\|_{H^s(D)}$ and seminorm $|\cdot|_{H^s(D)}$. If $D=\Omega$, we denote the usual L^2 inner product by $(\cdot,\cdot)$, the Sobolev norm by $\|\cdot\|_s$ and seminorm by $|\cdot|_s$, where s may be fractional (for details see [1]).

2. Preliminaries

Consider the following model problem: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v), \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(u,v) = (\bigtriangledown u, \bigtriangledown v), \quad f(v) = (f,v),$$

$f \in L^2(\Omega)$, Ω is a rectangular or L-shape bounded domain.

Divide Ω into geometrically conforming rectangular substructures, i.e., $\bar\Omega = \bigcup_{k=1}^{N} \bar\Omega_k$ with $\bar\Omega_k \cap \bar\Omega_l$ being empty set or a vertex or an edge for $k \neq l$. With each Ω_k we associate a quasi-uniform triangulation $\mathcal{T}_h(\Omega_k)$ made of elements that are rectangles whose edges are parallel to x-axis or y-axis. The mesh parameter h_k is the diameter of the largest element in $\mathcal{T}_h(\Omega_k)$. Let Γ_{kl} denote the open edge that is common to Ω_k and Ω_l. Denote by Γ the set of all interfaces between the subdomains, i.e., $\Gamma = \bigcup \partial\Omega_k \backslash \partial\Omega$. Each edge inherits two triangulations made of segments that are edges of elements of the triangulations of Ω_k and Ω_l respectively. In this way each Γ_{kl} is provided with two independent and different one dimensional meshes, which are denoted by $\mathcal{T}_h^k(\Gamma_{kl})$ and $\mathcal{T}_h^l(\Gamma_{kl})$ respectively. Let $\Omega_{k,h}$ and $\partial\Omega_{k,h}$ be the sets of vertices of the triangulation $\mathcal{T}_h(\Omega_k)$ that are in $\overline{\Omega}_k$ and $\partial\Omega_k$ respectively.

For each triangulation $\mathcal{T}_h(\Omega_k)$, the rotated $Q1$ element space is defined by

$$\begin{aligned} X_h(\Omega_k) = \{v \in L^2(\Omega_k) \quad | \quad & v|_E = a_E^1 + a_E^2 x + a_E^3 y + a_E^4 (x^2 - y^2), \\ & a_E^i \in \mathcal{R}, \ \int_{\partial E \cap \partial\Omega} v|_{\partial\Omega} ds = 0, \ \forall E \in \mathcal{T}_h(\Omega_k); \\ & \text{for } \ E_1, E_2 \in \mathcal{T}_h(\Omega_k), \ \text{ if } \ \partial E_1 \cap \partial E_2 = e, \ \text{ then} \\ & \int_e v|_{\partial E_1} ds = \int_e v|_{\partial E_2} ds\}, \end{aligned}$$

with norm and seminorm

$$\|v\|_{H_h^1(\Omega_k)} = (\sum_{E\in\mathcal{T}_h(\Omega_k)} \|v\|_{H^1(E)}^2)^{1/2}, \quad |v|_{H_h^1(\Omega_k)} = (\sum_{E\in\mathcal{T}_h(\Omega_k)} |v|_{H^1(E)}^2)^{1/2}.$$

Introduce the global discrete space

$$X_h(\Omega) = \prod_{k=1}^{N} X_h(\Omega_k),$$

with norm $\|v\|_{1,h} = (\sum_{k=1}^{N} \|v\|_{H_h^1(\Omega_k)}^2)^{1/2}$ and seminorm $|v|_{1,h} = (\sum_{k=1}^{N} |v|_{H_h^1(\Omega_k)}^2)^{1/2}$.

Define one of the sides of Γ_{kl} as mortar denoted by $\gamma_{m(k)}$ and the other as nonmortar denoted by $\delta_{m(l)}$. Assume that the mortar for $\gamma_{m(k)} = \delta_{m(l)} = \Gamma_{kl}$ is chosen by the condition $h_k \leq h_l$, i.e., the fine side is chosen as mortar. Based on this assumption, the two elements of the slave triangulation $\mathcal{T}_h^l(\delta_{m(l)})$ that touch the ends of $\delta_{m(l)}$ are longer than the respective elements of the mortar triangulation $\mathcal{T}_h^k(\gamma_{m(k)})$. Define an auxiliary test space $M^{h_l}(\delta_{m(l)})$ to be a subspace of the space $L^2(\Gamma_{kl})$ such that its functions are piecewise constants on $\mathcal{T}_h^l(\delta_{m(l)})$. The dimension of $M^{h_l}(\delta_{m(l)})$ is equal to the number of elements on the $\delta_{m(l)}$. For each nonmortar $\delta_{m(l)} = \Gamma_{kl}$, we define an L^2-orthogonal projection $Q_m : L^2(\Gamma_{kl}) \to M^{h_l}(\delta_{m(l)})$ by

$$(Q_m v, w)_{L^2(\delta_{m(l)})} = (v, w)_{L^2(\delta_{m(l)})}, \quad \forall w \in M^{h_l}(\delta_{m(l)}). \tag{2.2}$$

Now we define rotated $Q1$ mortar element space

$$V_h = \{v \in X_h(\Omega) \quad | \quad Q_m v_l = Q_m v_k, \quad \forall \delta_{m(l)} = \gamma_{m(k)} \subset \Gamma\},$$

where $v_k = v|_{\gamma_{m(k)}}$ and $v_l = v|_{\delta_{m(l)}}$. The condition of the equality of the L^2-orthogonal projection of traces onto the test space for each interface is called the mortar condition. The rotated $Q1$ mortar element approximation of problem (2.1) is: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{2.3}$$

where

$$a_h(u_h, v_h) = \sum_{k=1}^{N} a_{h,k}(u_h, v_h), \quad a_{h,k}(u_h, v_h) = \sum_{E \in \mathcal{T}_h(\Omega_k)} (\nabla u_h, \nabla v_h)_E.$$

3. Some Technical Lemmas

In this section we present some auxiliary technical lemmas that are necessary to prove our results.

Let $\mathcal{T}_{h/2}(\Omega_k)$ be the partition which is constructed by connecting midpoints of the opposite edges of elements of $\mathcal{T}_h(\Omega_k)$, $\tilde{V}^{h/2}(\Omega_k)$ be piecewise bilinear conforming element space defined on $\mathcal{T}_{h/2}(\Omega_k)$, and $\tilde{V}_0^{h/2}(\Omega_k)$ be the subspace of $\tilde{V}^{h/2}(\Omega_k)$ consisting of functions with zero traces on $\partial\Omega_k$. Define operator $\mathcal{M}_k : X_h(\Omega_k) \to \tilde{V}^{h/2}(\Omega_k)$ as follows:

Definition 3.1. *Given $v \in X_h(\Omega_k)$, we define $\mathcal{M}_k v \in \tilde{V}^{h/2}(\Omega_k)$ by the values of $\mathcal{M}_k v$ at the vertices of the partition $\mathcal{T}_{h/2}(\Omega_k)$. The vertices are divided into four sets of points:*

- *If P is a central point of E, $E \in \mathcal{T}_h(\Omega_k)$, then*

$$(\mathcal{M}_k v)(P) = \frac{1}{4} \sum_{e_i \in \partial E} \frac{1}{|e_i|} \int_{e_i} v ds;$$

- *If P is a midpoint of one dege $e \in \partial E$, $E \in \mathcal{T}_h(\Omega_k)$, then*

$$(\mathcal{M}_k v)(P) = \frac{1}{|e|} \int_e v ds;$$

- *If* $P \in \Omega_{k,h} \backslash \partial\Omega_{k,h}$, *then*

$$(\mathcal{M}_k v)(P) = \frac{1}{4} \sum_{e_i} \frac{1}{|e_i|} \int_{e_i} v ds,$$

where the sum is taken over all edges e_i *with the common vertex* P, $e_i \in \partial E_i$, $E_i \in \mathcal{T}_h(\Omega_k)$*;*

- *If* $P \in \partial\Omega_{k,h}$, *then*

$$(M_k v)(P) = \frac{|e_l|}{|e_l| + |e_r|} (\frac{1}{|e_l|} \int_{e_l} v ds) + \frac{|e_r|}{|e_l| + |e_r|} (\frac{1}{|e_r|} \int_{e_r} v ds),$$

where $e_l \in \partial E_1 \cap \partial\Omega_k$ *and* $e_r \in \partial E_2 \cap \partial\Omega_k$ *are the left and right neighbor edges of* P, E_1, $E_2 \in \mathcal{T}_h(\Omega_k)$*. If* P *is a vertex of* Ω_k*, then* $E_1 = E_2$.

The above operator $\mathcal{M}_k$ has the following properties.

Lemma 3.1. *For any* $v \in X_h(\Omega_k)$*, we have*

$$|\mathcal{M}_k v|_{H^1(\Omega_k)} \asymp |v|_{H_h^1(\Omega_k)}, \tag{3.1}$$

$$\|\mathcal{M}_k v\|_{L^2(\Omega_k)} \asymp \|v\|_{L^2(\Omega_k)}, \tag{3.2}$$

$$\int_{\partial\Omega_k} \mathcal{M}_k v ds = \int_{\partial\Omega_k} v ds, \tag{3.3}$$

$$\|\mathcal{M}_k v - v\|_{L^2(\Omega_k)} \preceq h_k |v|_{H_h^1(\Omega_k)}, \tag{3.4}$$

$$\|\mathcal{M}_k v - v\|_{L^2(\varepsilon)} \preceq h_k^{1/2} |v|_{H_h^1(\Omega_k)}, \tag{3.5}$$

where ε *is an edge of* Ω_k.

Proof. For any $K \in \mathcal{T}_{h/2}(\Omega_k)$, there exists an element $E \in \mathcal{T}_h(\Omega_k)$, such that $K \subset E$. Assume that P_1 is the common vertex of K and E, P_2 and P_3 are other two vertices of K which lie on the edges of E, and P_4 is the fourth vertex of K which lies in E. Then

$$\begin{aligned}
|\mathcal{M}_k v|^2_{H^1(K)} &\asymp \sum_{i,j=1}^{4} |(\mathcal{M}_k v)(P_i) - (\mathcal{M}_k v)(P_j)|^2 \\
&\preceq \sum_{\tau \in \mathcal{T}_h(\Omega_k), P_1 \in \bar{\tau}} \sum_{e_i, e_j \in \partial\tau} (\frac{1}{|e_i|} \int_{e_i} v ds - \frac{1}{|e_j|} \int_{e_j} v ds)^2 \\
&\preceq \sum_{\tau \in \mathcal{T}_h(\Omega_k), P_1 \in \bar{\tau}} |v|^2_{H^1(\tau)}.
\end{aligned}$$

Summing over all $K \in \mathcal{T}_{h/2}(\Omega_k)$ yields

$$|\mathcal{M}_k v|_{H^1(\Omega_k)} \preceq |v|_{H_h^1(\Omega_k)}. \tag{3.6}$$

For any $E \in \mathcal{T}_h(\Omega_k)$, let $e_i (i = 1, 2, 3, 4)$ be the edges of E numbered clockwisely, we have

$$\begin{aligned}
|v|^2_{H^1(E)} &\asymp \sum_{i,j=1}^{4} (\frac{1}{|e_i|} \int_{e_i} v ds - \frac{1}{|e_j|} \int_{e_j} v ds)^2 \\
&\preceq \sum_{|i-j| \neq 2} (\frac{1}{|e_i|} \int_{e_i} v ds - \frac{1}{|e_j|} \int_{e_j} v ds)^2 \\
&\preceq \sum_{K \subset E, K \in \mathcal{T}_{h/2}(\Omega_k)} |\mathcal{M}_k v|^2_{H^1(K)}.
\end{aligned}$$

Summing over all $E \in \mathcal{T}_h(\Omega_k)$ gives

$$|v|_{H_h^1(\Omega_k)} \preceq |\mathcal{M}_k v|_{H^1(\Omega_k)}. \tag{3.7}$$

(3.1) follows from (3.6) and (3.7).

Similarly, we can show (3.2) holds.

Next we prove (3.3). For any $E \in \mathcal{T}_h(\Omega_k)$, $\partial E \cap \partial\Omega_k \neq \emptyset$, let e be an edge of E which lies on $\partial\Omega_k$. Assume e_1 and e_2 are two neighbor edges of e which lie in $\partial\Omega_k$. Since $\mathcal{M}_k v|_e$ is piecewise linear, according to the definition 3.1, we deduce that

$$\begin{aligned}
\int_e \mathcal{M}_k v ds &= \frac{|e|}{4}\{\frac{1}{|e_1|+|e|}(\int_{e_1} vds + \int_e vds) + \frac{2}{|e|}\int_e vds + \\
&\quad + \frac{1}{|e|+|e_2|}(\int_e vds + \int_{e_2} vds)\} \\
&= \frac{|e|}{4(|e_1|+|e|)}\int_{e_1} vds + \frac{1}{4}(\frac{|e|}{|e_1|+|e|} + 2 + \\
&\quad + \frac{|e|}{|e|+|e_2|})\int_e vds + \frac{|e|}{4(|e|+|e_2|)}\int_{e_2} vds.
\end{aligned}$$

Summing over all $e \in \partial\Omega_k$ yields

$$\int_e \mathcal{M}_k v ds = \int_e vds,$$

i.e., (3.3) holds.

To see (3.4), we introduce a local interpolation operator I_K. For any $v \in X_h(\Omega_k)$, define $I_K v$ to be a bilinear function on $K \in \mathcal{T}_{h/2}(\Omega_k)$ by $(I_K v)(P_i) = v(P_i)$, where P_i $(i = 1,2,3,4)$ are four vertices of K. Using standard interpolation estimate [6] and discrete norm, we obtain

$$\begin{aligned}
\|v - \mathcal{M}_k v\|_{L^2(K)}^2 &\preceq \|v - I_K v\|_{L^2(K)}^2 + \|I_K v - \mathcal{M}_k v\|_{L^2(K)}^2 \\
&\preceq h_k^2 |v|_{H^1(K)}^2 + \sum_{i=1}^4 h_k^2 (v(P_i) - (\mathcal{M}_k v)(P_i))^2.
\end{aligned} \tag{3.8}$$

If P_i is a midpoint of one edge e of $E \in \mathcal{T}_h(\Omega_k)$, $K \subset E$, then there exists a $\xi \in e$ such that

$$v(\xi) = \frac{1}{|e|}\int_e vds.$$

Using above equality and inverse inequality yields

$$\begin{aligned}
(v(P_i) - (\mathcal{M}_k v)(P_i))^2 &= (v(P_i) - \frac{1}{|e|}\int_e vds)^2 \\
&= (v(P_i) - v(\xi))^2 \\
&\preceq h_k^2 |v|_{W_\infty^1(E)}^2 \\
&\preceq |v|_{H^1(E)}^2.
\end{aligned} \tag{3.9}$$

If P_i is a central point of E, $E \in \mathcal{T}_h(\Omega_k)$, arguing as in (3.9) follows

$$\begin{aligned}(v(P_i)-(\mathcal{M}_k v)(P_i))^2 &= (v(P_i)-\frac{1}{4}\sum_{e_j\in\partial E}\frac{1}{|e_j|}\int_{e_j} vds)^2\\ &\preceq \sum_{e_j\in\partial E}(v(p_i)-\frac{1}{|e_j|}\int_{e_j} vds)^2\\ &\preceq |v|^2_{H^1(E)}. \end{aligned}\tag{3.10}$$

If $P_i \in \Omega_{k,h}\backslash\partial\Omega_{k,h}$, assume P_i is a common vertex of $E_j \in \mathcal{T}_h(\Omega_k)$ $(j=1,2,3,4)$, $K \subset E_1$, $\partial E_1 \cap \partial E_2 = e_1$, $\partial E_2 \cap \partial E_3 = e_2$, $\partial E_3 \cap \partial E_4 = e_3$, $\partial E_4 \cap \partial E_1 = e_4$. Arguing as in (3.9) we deduce that

$$\begin{aligned}(v(P_i)-(\mathcal{M}_k v)(P_i))^2 &= (v(P_i)-\frac{1}{4}\sum_{j=1}^{4}\frac{1}{|e_j|}\int_{e_j} vds)^2\\ &\preceq (v(P_i)-\frac{1}{|e_1|}\int_{e_1} vds)^2+(v(P_i)-\frac{1}{|e_4|}\int_{e_4} vds)^2\\ &\quad +(\frac{1}{|e_1|}\int_{e_1} vds-\frac{1}{|e_2|}\int_{e_2} vds)^2\\ &\quad +(\frac{1}{|e_3|}\int_{e_3} vds-\frac{1}{|e_4|}\int_{e_4} vds)^2\\ &\preceq \sum_{j=1}^{4}|v|^2_{H^1(E_j)}. \end{aligned}\tag{3.11}$$

Similarly, if $P_i \in \partial\Omega_{k,h}$, $P_i \in \bar{E}_1 \cap \bar{E}_2$, E_1 and $E_2 \in \mathcal{T}_h(\Omega_k)$, then

$$(v(P_i)-(\mathcal{M}_k v)(P_i))^2 \preceq |v|^2_{H^1(E_1)}+|v|^2_{H^1(E_2)},\tag{3.12}$$

where $E_1 = E_2$ if P_i is a vertex of Ω_k.

Combining (3.8)-(3.12), summing over all $K \in \mathcal{T}_{h/2}(\Omega_k)$, we obtain (3.4).

Finally we prove (3.5). Using trace theorem, (3.1), and (3.4), we derive that

$$\begin{aligned}\|\mathcal{M}_k v - v\|^2_{L^2(\varepsilon)} &\preceq h_k^{-1}\|\mathcal{M}_k v - v\|^2_{L^2(\Omega_k)} + h_k|\mathcal{M}_k v - v|^2_{H^1_h(\Omega_k)}\\ &\preceq h_k|v|^2_{H^1_h(\Omega_k)}.\end{aligned}$$

From this we know (3.5) holds and the proof is completed.

We now introduce a subspace $X_h^\varepsilon(\Omega_k)$ of $X_h(\Omega_k)$ for each open edge ε of Ω_k as follows:

$$X_h^\varepsilon(\Omega_k) = \{v \in X_h(\Omega_k) \ | \ \int_e vds = 0, \quad \forall e \in \partial\Omega_k\backslash\varepsilon\}.$$

Define an operator $\mathcal{M}_k^\varepsilon : X_h^\varepsilon(\Omega_k) \to \tilde{V}^{h/2}(\Omega_k)$ by

Definition 3.2. *Given $v \in X_h^\varepsilon(\Omega_k)$, we define $\mathcal{M}_k^\varepsilon v \in \tilde{V}^{h/2}(\Omega_k)$ by the values of $\mathcal{M}_k^\varepsilon v$ at the vertices of the partition $\mathcal{T}_{h/2}(\Omega_k)$.*

- *If P is a central point of E or a midpoint of one edge of E, $E \in \mathcal{T}_h(\Omega_k)$, or $P \in \Omega_{k,h}\backslash\partial\Omega_{k,h}$, then $(\mathcal{M}_k^\varepsilon v)(P) = (\mathcal{M}_k v)(P)$;*

- *If* $P \in \partial\Omega_{k,h} \backslash \varepsilon$, *then* $(\mathcal{M}_k^{\varepsilon} v)(P) = 0$;
- *If* $P \in \partial\Omega_{k,h} \cap \varepsilon$, *then*

$$(\mathcal{M}_k^{\varepsilon} v)(P) = \frac{|e_r|}{|e_l| + |e_r|}\left(\frac{1}{|e_l|}\int_{e_l} vds\right) + \frac{|e_l|}{|e_l| + |e_r|}\left(\frac{1}{|e_r|}\int_{e_r} vds\right),$$

where $e_l \in \partial E_1 \cap \partial\Omega_k$ *and* $e_r \in \partial E_2 \cap \partial\Omega_k$ *are the left and right neighbor edges of* P, $E_1, E_2 \in \mathcal{T}_h(\Omega_k)$. *If* P *is a vertex of* Ω_k, $E_1 = E_2$.

Define the pseudo-inverse map $(\mathcal{M}_k)^+ : \tilde{V}^{h/2}(\Omega_k) \to X_h(\Omega_k)$ by

$$\frac{1}{|e|}\int_e (\mathcal{M}_k)^+ vds = v(P), \quad \forall v \in \tilde{V}^{h/2}(\Omega_k),$$

where $e \in \partial E$, $E \in \mathcal{T}_h(\Omega_k)$, P is the midpoint of e. Obviously, we have

$$(\mathcal{M}_k)^+ \mathcal{M}_k v = v, \quad (\mathcal{M}_k)^+ \mathcal{M}_k^{\varepsilon} w = w, \quad \forall v \in X_h(\Omega_k), \ \forall w \in X_h^{\varepsilon}(\Omega_k).$$

Using the discrete norms, we can prove the following Lemma holds.

Lemma 3.2. *For any* $v \in \tilde{V}^{h/2}(\Omega_k)$, *we have*

$$|(\mathcal{M}_k)^+ v|_{H_h^1(\Omega_k)} \preceq |v|_{H_h^1(\Omega_k)}, \quad \|(\mathcal{M}_k)^+ v\|_{L^2(\Omega_k)} \preceq \|v\|_{L^2(\Omega_k)}.$$

Let $\mathcal{A}_k$ be a special set of edges which belong to $\partial\Omega_k$ or are the edges of rectangles which have one side on a mortar $\gamma_{m(k)}$. We introduce a special subspace $X_h^k(\Omega_k) \subset X_h(\Omega_k)$ as follows:

$$X_h^k(\Omega_k) = \{v \in X_h(\Omega_k) \ \mid \ \int_e vds = 0, \quad \forall e \in \mathcal{A}_k\}.$$

Define a discrete harmonic part $H_k v$ of $v \in X_h(\Omega_k)$ by

$$a_{h,k}(H_k v, w) = 0, \quad \forall w \in X_h^k(\Omega_k),$$
$$\int_e H_k vds = \int_e vds, \quad \forall e \in \mathcal{A}_k.$$

Also we define a projection operator $P_k : X_h(\Omega_k) \to X_h^k(\Omega_k)$ by

$$a_{h,k}(P_k v, w) = a_{h,k}(v, w), \quad \forall w \in X_h^k(\Omega_k).$$

Lemma 3.3. *Let* $\varepsilon = \delta_{m(k)}$ *be a nonmortar edge of* Ω_k, *and* v *be discrete harmonic in* Ω_k *with* $\int_e vds = 0$ *for any* $e \in \mathcal{A}_k \backslash \delta_{m(k)}$. *Then*

$$|v|_{H_h^1(\Omega_k)} \preceq \|\mathcal{M}_k^{\varepsilon} v\|_{H_{00}^{1/2}(\delta_{m(k)})}.$$

Proof. Let $\mathcal{H}_k : \tilde{V}^{h/2}(\partial\Omega_k) \to \tilde{V}^{h/2}(\Omega_k)$ be conforming discrete harmonic defined by

$$(\nabla \mathcal{H}_k w, \nabla \psi)_{L^2(\Omega_k)} = 0, \quad \forall \psi \in \tilde{V}_0^{h/2}(\Omega_k),$$
$$\mathcal{H}_k w = w, \qquad \text{on} \quad \partial\Omega_k.$$

By Lemma 3.2 and the property of conforming discrete harmonic (see (4.13) in [13]), we obtain

$$|(\mathcal{M}_k)^+ \mathcal{H}_k \mathcal{M}_k^{\epsilon} v|_{H_h^1(\Omega_k)} \preceq |\mathcal{H}_k \mathcal{M}_k^{\epsilon} v|_{H_h^1(\Omega_k)} \preceq \|\mathcal{M}_k^{\epsilon} v\|_{H_{00}^{1/2}(\delta_{m(k)})}. \tag{3.13}$$

Define $\tilde{v} \in X_h(\Omega_k)$ by

$$\int_e \tilde{v} ds = \begin{cases} 0, & e \in \mathcal{A}_k \backslash \partial\Omega_k, \\ \int_e (\mathcal{M}_k)^+ \mathcal{H}_k \mathcal{M}_k^\epsilon v ds, & \text{otherwise.} \end{cases}$$

Obviously we have

$$\int_e \tilde{v} ds = \int_e v ds, \quad \forall e \in \mathcal{A}_k.$$

According to the property of discrete harmonic it follows that

$$|v|_{H_h^1(\Omega_k)} \preceq |\tilde{v}|_{H_h^1(\Omega_k)}. \tag{3.14}$$

Using H^1 and L^2 discrete norms and Poincare inequality in each element yields

$$\begin{aligned} |\tilde{v}|_{H_h^1(\Omega_k)} &\preceq |(\mathcal{M}_k)^+ \mathcal{H}_k \mathcal{M}_k^\epsilon v|_{H_h^1(\Omega_k)} + |\tilde{v} - (\mathcal{M}_k)^+ \mathcal{H}_k \mathcal{M}_k^\epsilon v|_{H_h^1(\Omega_k)} \\ &\preceq \sum_{E \in \mathcal{T}_h(\Omega_k)} \sum_{e_i, e_j \subset \partial E} \Big(\frac{1}{|e_i|} \int_{e_i} (\mathcal{M}_k)^+ \mathcal{H}_k \mathcal{M}_k^\epsilon v ds - \\ &\quad - \frac{1}{|e_j|} \int_{e_j} (\mathcal{M}_k)^+ \mathcal{H}_k \mathcal{M}_k^\epsilon v ds \Big)^2 \\ &\preceq |(\mathcal{M}_k)^+ \mathcal{H}_k \mathcal{M}_k^\epsilon v|_{H_h^1(\Omega_k)}. \end{aligned} \tag{3.15}$$

The desired result follows from (3.13)-(3.15).

Let $\delta_{m(l)}$ be a nonmortar edge of Ω_l, $W_0^{h_l}(\delta_{m(l)})$ be the continuous function space whose elements are piecewise linear over all segments that have the midpoints of edges belonging to $\delta_{m(l)}$ as their nodals and equal zero at the ends of $\delta_{m(l)}$. Let $\delta_{m(l)}^m$ be the set of midpoints of edges in $\mathcal{T}_h^l(\delta_{m(l)})$. Define an auxiliary operator $\Pi_m : L^2(\delta_{m(l)}) \to W_0^{h_l}(\delta_{m(l)})$ as follows:

$$(\Pi_m v)(P) = (Q_m v)(P), \quad \forall P \in \delta_{m(l)}^m.$$

Lemma 3.4. $\|\Pi_m v\|_{L^2(\delta_{m(l)})} \preceq \|v\|_{L^2(\delta_{m(l)})}$, $\forall v \in L^2(\delta_{m(l)})$.

Proof. Using the discrete norm and L^2-stability of operator Q_m yields

$$\begin{aligned} \|\Pi_m v\|^2_{L^2(\delta_{m(l)})} &\preceq h_l \sum_{P \in \delta_{m(l)}^m} |(\Pi_m v)(P)|^2 = h_l \sum_{P \in \delta_{m(l)}^m} |(Q_m v)(P)|^2 \\ &\preceq \|Q_m v\|^2_{L^2(\delta_{m(l)})} \preceq \|v\|^2_{L^2(\delta_{m(l)})}. \end{aligned}$$

Thus the proof is completed.

By interpolation estimate [6] and interpolation space theory [3], we can derive the following result.

Lemma 3.5. $\|v - Q_m v\|_{L^2(\delta_{m(l)})} \preceq h_l^{1/2} |v|_{H^{1/2}(\delta_{m(l)})}$, $\forall v \in H^{1/2}(\delta_{m(l)})$.

4. Error Estimate

The following result is the well-known second Strang Lemma.

Lemma 4.1. *Let u and u_h be the solutions of (2.1) and (2.3) respectively, then*

$$|u-u_h|_{H_h^1(\Omega)} \preceq \inf_{v\in V_h} |u-v|_{H_h^1(\Omega)} + \sup_{w\in V_h} |\sum_{k=1}^{N} \sum_{E\in\mathcal{T}_h(\Omega_k)} \frac{\int_{\partial E} \frac{\partial u}{\partial n} w ds}{|w|_{H_h^1(\Omega)}}|. \tag{4.1}$$

The first term in (4.1) is known as the approximation error, while the second term is called the consistency error.

Lemma 4.2. *Let u and u_h be the solution of (2.1) and (2.3) respectively. Assume $u|_{\Omega_k} \in H^2(\Omega_k)$, then we have*

$$|\sum_{k=1}^{N} \sum_{E\in\mathcal{T}_h(\Omega_k)} \int_{\partial E} \frac{\partial u}{\partial n} w ds| \preceq (\sum_{k=1}^{N} h_k^2 |u|_{H^2(\Omega_k)}^2)^{1/2} |w|_{H_h^1(\Omega)}, \quad \forall w \in V_h.$$

Proof. Note that

$$\begin{aligned}\sum_{k=1}^{N} \sum_{E\in\mathcal{T}_h(\Omega_k)} \int_{\partial E} \frac{\partial u}{\partial n} w ds &= \sum_{k=1}^{N} \sum_{E\in\mathcal{T}_h(\Omega_k)} \sum_{e\in\partial E\setminus\Gamma} \int_e \frac{\partial u}{\partial n} w ds \\ &+ \sum_{\Gamma_{kl}\in\Gamma} \int_{\Gamma_{kl}} \frac{\partial u}{\partial n} [w] ds, \end{aligned} \tag{4.2}$$

where e is an edge of E, $[w]$ denotes the jump of w across Γ_{kl}.

The first term in the right hand of (4.2) can be estimated by standard argument as follows (cf. [9] for details):

$$|\sum_{k=1}^{N} \sum_{E\in\mathcal{T}_h(\Omega_k)} \sum_{e\in E\setminus\Gamma} \int_e \frac{\partial u}{\partial n} w ds| \preceq (h_k^2 \sum_{k=1}^{N} |u|_{H^2(\Omega_k)}^2)^{1/2} |w|_{H_h^1(\Omega)}. \tag{4.3}$$

For any $\gamma_{m(k)} = \delta_{m(l)} = \Gamma_{kl} \subset \Gamma$, by the mortar condition,

$$\begin{aligned}|\int_{\Gamma_{kl}} \frac{\partial u}{\partial n}[w] ds| &= |\int_{\delta_{m(l)}} (\frac{\partial u}{\partial n} - Q_m \frac{\partial u}{\partial n})[w] ds| \\ &= |\int_{\delta_{m(l)}} (\frac{\partial u}{\partial n} - Q_m \frac{\partial u}{\partial n})\{(w_k - \mathcal{M}_k w_k) + \\ &\quad +(\mathcal{M}_k w_k - Q_m \mathcal{M}_k w_k) - (w_l - Q_m w_l)\} ds| \\ &\leq \|\frac{\partial u}{\partial n} - Q_m \frac{\partial u}{\partial n}\|_{L^2(\delta_{m(l)})} \{\|\mathcal{M}_k w_k - Q_m \mathcal{M}_k w_k\|_{L^2(\delta_{m(l)})} \\ &\quad + \|w_k - \mathcal{M}_k w_k\|_{L^2(\delta_{m(l)})} + \|w_l - Q_m w_l\|_{L^2(\delta_{m(l)})}\}, \end{aligned} \tag{4.4}$$

where $w_k = w|_{\gamma_{m(k)}}$ and $w_l = w|_{\delta_{m(l)}}$.

By Lemma 3.1, Lemma 3.5, and trace theorem, we have

$$\begin{aligned}
&\|\frac{\partial u}{\partial n} - Q_m\frac{\partial u}{\partial n}\|_{L^2(\delta_{m(l)})} \preceq h_l^{1/2}|\frac{\partial u}{\partial n}|_{H^{1/2}(\delta_{m(l)})} \preceq h_l^{1/2}|u|_{H^2(\Omega_l)},\\
&\|w_k - \mathcal{M}_k w_k\|_{L^2(\delta_{m(l)})} \preceq h_k^{1/2}|w|_{H_h^1(\Omega_k)} \le h_l^{1/2}|w|_{H_h^1(\Omega_k)},\\
&\|\mathcal{M}_k w_k - Q_m\mathcal{M}_k w_k\|_{L^2(\delta_{m(l)})} \preceq h_l^{1/2}|\mathcal{M}_k w_k|_{H^{1/2}(\delta_{m(l)})}\\
&\qquad \preceq h_l^{1/2}|\mathcal{M}_k w|_{H^1(\Omega_k)} \preceq h_l^{1/2}|w|_{H_h^1(\Omega_k)},\\
&\|w_l - Q_m w_l\|_{L^2(\delta_{m(l)})} \preceq h_l^{1/2}|w_l|_{H^{1/2}(\delta_{m(l)})} \preceq h_l^{1/2}|w|_{H_h^1(\Omega_l)},
\end{aligned}$$

where the assumption $h_k \le h_l$ is used.

Combining above four inequalities with (4.4) yields

$$|\int_{\Gamma_{kl}} \frac{\partial u}{\partial n}[w]ds| \preceq h_l|u|_{H^2(\Omega_l)}(|w|_{H_h^1(\Omega_k)} + |w|_{H_h^1(\Omega_l)}).$$

Summing over all $\Gamma_{kl} \in \Gamma$, using (4.2) and (4.3), the desired result follows.

Lemma 4.3. *For any $u \in H_0^1(\Omega)$ with $u|_{\Omega_k} \in H^2(\Omega_k)$, we have*

$$\inf_{v\in V_h} |u-v|_{H_h^1(\Omega)} \preceq (\sum_{k=1}^N h_k^2|u|^2_{H^2(\Omega_k)})^{1/2}.$$

Proof. Let $\Pi_h^k u \in X_h(\Omega_k)$ be the interpolation of u in $X_h(\Omega_k)$. We have interpolation estimate (cf. [9])

$$\|u - \Pi_h^k u\|_{L^2(\Omega_k)} + h_k|u - \Pi_h^k u|_{H_h^1(\Omega_k)} \preceq h_k^2|u|_{H^2(\Omega_k)}. \tag{4.5}$$

Define $\tilde{v}|_{\Omega_k} = \Pi_h^k u$, $k = 1,\cdots,N$. The function $\tilde{v} \in X_h(\Omega)$ may not satisfy the mortar condition across the interfaces. So we need to define a function $w \in X_h(\Omega)$ such that $v = w + \tilde{v}$ satisfies the mortar condition. To this end we first determine $\int_e wds$ for all $e \in \mathcal{A}_k$, $k = 1,\cdots,N$. Let $\int_e wds$ be zero for all e associated with mortars, i.e., $e \in \mathcal{A}_k\backslash \sum \delta_{m(k)}$, where the sum is taken over all nonmortars of Ω_k. On the slave side of an interface Γ_{kl} it is defined by

$$\int_{\delta_{m(l)}} w\psi ds = \int_{\delta_{m(l)}} (\Pi_h^k u - \Pi_h^l u)\psi ds, \quad \forall \psi \in M^{h_l}(\delta_{m(l)}). \tag{4.6}$$

Next we define w as discrete harmonic in all subdomains. It is obvious that $v = w + \tilde{v}$ satisfies the mortar condition. From (4.5) we know

$$\begin{aligned}
|u-v|_{H_h^1(\Omega)} &\le |u-\tilde{v}|_{H_h^1(\Omega)} + |w|_{H_h^1(\Omega)}\\
&\le (\sum_{k=1}^N h_k^2|u|^2_{H^2(\Omega_k)})^{1/2} + |w|_{H_h^1(\Omega)}.
\end{aligned}$$

To estimate $|w|_{H_h^1(\Omega)}$, we decompose $w|_{\Omega_l}$ as

$$w|_{\Omega_l} = \sum_{\delta_{m(l)}\subset\partial\Omega_l} w_{m(l)},$$

where the sum is taken over all nonmortars of Ω_l, $w_{m(l)}$ is discrete harmonic in Ω_l, and

$$\int_{\delta_{m(l)}} w_{m(l)} ds = \int_{\delta_{m(l)}} w ds, \quad \int_e w_{m(l)} ds = 0, \quad \forall e \in \mathcal{A}_l \backslash \delta_{m(l)}.$$

We further extend $w_{m(l)}$ as zero onto other subdomains and have

$$w = \sum w_{m(l)},$$

where the sum is taken over all nonmortars in $\mathcal{T}_h(\Omega)$.

Based on the above decomposition of w, we deduce that

$$|w|_{H_h^1(\Omega)} \leq \sum |w_{m(l)}|_{H_h^1(\Omega)} = \sum |w_{m(l)}|_{H_h^1(\Omega_l)}.$$

By (4.6) and the definitions of $\mathcal{M}_j^\varepsilon$ and Π_m,

$$\mathcal{M}_l^\varepsilon w_{m(l)} = \Pi_m(\Pi_h^k u - \Pi_h^l u) \quad \text{on} \quad \delta_{m(l)}.$$

From Lemmas 3.3-3.4, above equality, inverse inequality, trace theorem, and (4.5), it follows that

$$\begin{aligned} |w_{m(l)}|_{H_h^1(\Omega_l)} &\preceq \|\mathcal{M}_l^\varepsilon w_{m(l)}\|_{H_{00}^{1/2}(\Gamma_{kl})} = \|\Pi_m(\Pi_h^k u - \Pi_h^l u)\|_{H_{00}^{1/2}(\Gamma_{kl})} \\ &\preceq h_l^{-1/2}(\|\Pi_h^k u - u\|_{L^2(\Gamma_{kl})} + \|\Pi_h^l u - u\|_{L^2(\Gamma_{kl})}) \\ &\preceq h_l^{-1/2}\{(h_k^{-1}\|u - \Pi_h^k u\|_{L^2(\Omega_k)}^2 + h_k|u - \Pi_h^k u|_{H_h^1(\Omega_k)}^2)^{1/2} \\ &\quad +(h_l^{-1}\|u - \Pi_h^l u\|_{L^2(\Omega_l)}^2 + h_l|u - \Pi_h^l u|_{H_h^1(\Omega_l)}^2)^{1/2}\} \\ &\preceq h_l^{-1/2}(h_k^{3/2}|u|_{H_h^2(\Omega_k)} + h_l^{3/2}|u|_{H_h^2(\Omega_l)}) \\ &\preceq h_k|u|_{H_h^2(\Omega_k)} + h_l|u|_{H_h^2(\Omega_l)}. \end{aligned}$$

Here the assumption $h_k \leq h_l$ is used again. Summing over all nonmortars $\delta_{m(l)} \subset \partial\Omega_l$ and afterwards over the subdomains, we complete the proof.

From Lemmas 4.1-4.3 we obtain the following optimal error estimate.

Theorem 4.4. *Let u and u_h be the solution of (2.1) and (2.3) respectively, $u|_{\Omega_k} \in H^2(\Omega_k)$, then*

$$|u - u_h|_{H_h^1(\Omega)} \preceq (\sum_{k=1}^N h_k^2 |u|_{H^1(\Omega_k)}^2)^{1/2}.$$

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