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# SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHOD FOR NONSTATIONARY HYPERBOLIC EQUATION*1)

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**Abstract**

For the first order nonstationary hyperbolic equation taking the piecewise linear discontinuous Galerkin solver, we prove that under the uniform rectangular partition, such a discontinuous solver, after postprossesing, can have two and half approximative order which is half order higher than the optimal estimate by Lesaint and Raviart under the rectangular partition.

*Key words*: Discontinuous Galerkin method, Hyperbolic equation, Nonstationary, Superconvergence.

## 1. Introduction

Consider the first order hyperbolic equation in two-dimensional nonstationary case,

$$\begin{aligned} u_t + u_x + u_y + u &= f, && in\ \Omega \\ u(x,y,t) &= 0, && on\ \Gamma_- \\ u(x,y,0) &= u_0(x,y) \end{aligned} \tag{1}$$

where the domain $\Omega = [0,1] \times [0,1]$ for the sake of simplicity , and the boundary $\Gamma_- = \{(0,y) \times (x,0)\}$ and $\Gamma_+ = \{(1,y) \times (x,1)\}$. We assume throughout that the solution $u, u_t \in H^4$.

Earlier in 1973, Strang[1] has indicated that the continuous Galerkin method(CGM) with piecewise polynomials has two shortcomings for solving the above equation: first, that they resulted in an implicit scheme, rather than an explicit scheme; and second, that the convergence rate would be reduced by an order compared to that of the ordinary polynomial approximation.

However, if the discontinuous Galerkin method(DGM) was employed, the situation would be improved in two ways: on one hand, the scheme becomes explicit, and on the other, under the rectangular partition, there is proved to result without loss in the order of convergence rate; that is, it gains the same order as the ordinary polynomial approximation (Lesiant- Raviart's 1974 [2]). Even under the general triangulation of domains with no particular geometry, only half an order was lost in the convergence rate(Johnson et al. 1986 [3]).

Recently, we [4] have found that for piecewise bilinear elements, under uniform rectangular partition, there is a half order increase in the convergence rate as against Lesiant-Raviart's optimal estimate. This again confirms a conclusion previously reached by us [5] that a careful selection of the partition would effect the convergence rate in a considerable degree, whether CGM or DGM be employed.

The above results of Lesiant-Raviart[2], Johnson[3], and us[4] were done for stationary problem only, but this paper will consider the nonstationary case.

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## 2. Preliminaries

Decompose $\Omega$ into the elements

$$e = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e]$$

where $(x_e, y_e)$ is the center of $e$, $2h_e$ and $2k_e$ are the length and height, respectively, of $e$.

Assume that $W_h$ is a finite element space of discontinuous piecewise bilinear functions. The bilinear form correspondering to stationary equation(1) has been extended from $C^1$ to $W_h$ in [4] as follows: $\forall v \in W_h$,

$$\begin{aligned} B(w,v) \quad = \quad & \textstyle\sum_e\{\int_e w(v - v_x - v_y) + \int_{y_e-k_e}^{y_e+k_e}[w(x_e + h_e - 0, y) \\ & v(x_e + h_e - 0, y) - w(x_e - h_e - 0, y)v(x_e - h_e + 0, y)]dy \\ & + \textstyle\int_{x_e-h_e}^{x_e+h_e}[w(x, y_e + k_e - 0)v(x, y_e + k_e - 0) \\ & -w(x, y_e - k_e - 0)v(x, y_e - k_e + 0)]dx\}, \end{aligned} \tag{2}$$

which returns to $B(w,v) = \int_\Omega (w_x + w_y + w)v$ when $w \in C^1$. Notice that the above line integrals can be rewritten into:

$$\begin{aligned} & \textstyle\sum_e \int_{y_e-k_e}^{y_e+k_e} w(x_e + h_e - 0, y)v(x_e + h_e - 0, y)dy \\ = \quad & \textstyle\sum_e \int_{y_e-k_e}^{y_e+k_e} w(x_e - h_e - 0, y)v(x_e - h_e - 0, y)dy \\ & + \textstyle\int_0^1 w(1-0, y)v(1-0, y)dy - \int_0^1 w(-0, y)v(-0, y)dy. \end{aligned}$$

Hence we also have

$$\begin{aligned} B(w,v) \quad = \quad & \textstyle\sum_e\{\int_e w(v - v_x - v_y) + \int_{y_e-k_e}^{y_e+k_e} w(x_e - h_e - 0, y) \\ & [v(x_e - h_e - 0, y) - v(x_e - h_e + 0, y)]dy \\ & + \textstyle\int_{x_e-h_e}^{x_e+h_e} w(x, y_e - k_e - 0)[v(x, y_e - k_e - 0) - v(x, y_e - k_e + 0)]dx \\ & + \textstyle\int_{\Gamma_+} wvds - \int_{\Gamma_-} wvds\}. \end{aligned} \tag{3}$$

Notice that (3) includes a jump, of v, across the boundaries between adjacent elements. $B$ is positive definite when $v|_{\Gamma_-}=0$ and contains a $L^2$-norm and the jumps of element boundaries:

$$
\begin{aligned}
B(v,v) &= \textstyle\sum_e\{-\frac{1}{2}\int_{y_e-k_e}^{y_e+k_e}[v^2(x_e+h_e-0,y)-v^2(x_e-h_e+0,y)]dy \\
&\quad +\textstyle\int_{y_e-k_e}^{y_e+k_e}[v^2(x_e-h_e-0,y)-v(x_e-h_e-0,y)v(x_e-h_e+0,y)]dy \\
&\quad -\cdots+\textstyle\int_e v^2\} \\
&= \textstyle\sum_e\{\frac{1}{2}\int_{y_e-k_e}^{y_e+k_e}[v^2(x_e-h_e-0,y)+v^2(x_e-h_e+0,y)] \\
&\quad -2v(x_e-h_e-0,y)v(x_e-h_e+0,y)]dy \\
&\quad +\cdots+\textstyle\int_e v^2dy\}+\frac{1}{2}\int_{\Gamma_+}v^2(1-0,y)dy \\
&= \textstyle\sum_e\{\frac{1}{2}\int_{y_e-k_e}^{y_e+k_e}[v(x_e-h_e-0,y)-v(x_e-h_e+0,y)]^2dy \\
&\quad +\cdots+\textstyle\int_e v^2dy\}+\frac{1}{2}\int_0^1 v^2(1-0,y)dy.
\end{aligned} \tag{4}
$$

Now the DGM is taken to be $u^h\in W^h$ satisfying

$$B(u^h,v)=\textstyle\int_\Omega fv,\quad \forall v\in W_h,$$

$$u^h_-=0,\quad on\ \Gamma_-.$$

Hence

$$B(u^h,v)=B(u,v),\quad \forall v\in W_h.$$

In addition, in accordance with Lesaint-Raviart [2], the interpolation $u^I$ of u was defined to be a bilinear function over $e$, which assumed the same values as those of u at the four points

$$(x_e+h_e,y_e+k_e),\quad (x_e-\frac{h_e}{3},y_e+k_e),$$

$$(x_e-\frac{h_e}{3},y_e-\frac{k_e}{3}),\quad (x_e+h_e,y_e-\frac{k_e}{3}).$$

In particular,

$$B(u^h-u^I,v)=B(u-u^I,v),\quad \forall v\in W_h. \tag{5}$$

To have an expansion for $u^h-u^I$, from the identity (5) we have only to expand

$$B(u-u^I,v),\quad \forall v\in W_h.$$

For this purpose we first consider the expansion of the following block of terms in (2).

$$
\begin{aligned}
F(u,v) &= -\textstyle\int_e(u-u^I)v_x+\int_{y_e-k_e}^{y_e+k_e}[(u-u^I)(x_e+h_e-0,y)v(x_e+h_e-0,y) \\
&\quad -(u-u^I)(x_e-h_e-0,y)v(x_e-h_e+0,y)]dy.
\end{aligned}
$$

In order to use Bramble-Hilbert lemma, let us consider a bilinear functional over $e$

$$Z(u,v) = F(u,v) - \frac{h_e^3}{27}\int_e u_{xxx}v_x + \frac{k_e^3}{9}\int_e u_{xyy}v_y.$$

We need to transform the definition of $Z(u,v)$ over to a reference element $\widehat{e} = [-1,1]\times[-1,1]$. To this end, let $G : e \to \widehat{e}$ be a map defined as

$$G : (x,y) \mapsto (\hat{x},\hat{y}),$$

$$\hat{x} = \frac{x - x_e}{h_e}, \quad \hat{y} = \frac{y - y_e}{k_e}.$$

Let

$$(\hat{u} - \hat{u}^I)(\hat{x},\hat{y}) = (u - u^I)(x,y), \quad \hat{v}(\hat{x},\hat{y}) = v(x,y).$$

Consider the bilinear functional over $\hat{e}$

$$\begin{aligned} \hat{Z}(\hat{u},\hat{v}) \quad = \quad & -\textstyle\int_{-1}^1\int_{-1}^1(\hat{u}-\hat{u}^I)\hat{v}_{\hat{x}} + \int_{-1}^1[(\hat{u}-\hat{u}^I)(1-0,\hat{y})\hat{v}(1-0,\hat{y}) \\ & -(\hat{u}-\hat{u}^I)(-1-0,\hat{y})\hat{v}(-1+0,\hat{y})]d\hat{y} - \frac{1}{27}\int_{-1}^1\int_{-1}^1 \\ & \hat{u}_{\hat{x}\hat{x}\hat{x}}\hat{v}_{\hat{x}} + \frac{1}{9}\int_{-1}^1\int_{-1}^1 \hat{u}_{\hat{x}\hat{y}\hat{y}}\hat{v}_{\hat{y}}. \end{aligned}$$

By imbedding theorem and inverse inequality, we have

$$|\hat{Z}(\hat{v},\hat{v})| \leq c||\hat{v}||_{4,\hat{e}}||\hat{v}||_{0,\hat{e}}.$$

If $\hat{u}$ takes the following forms, respectively

$$\hat{x}^2, \quad \hat{y}^2, \quad \hat{x}^3, \quad \hat{x}^2\hat{y}, \quad \hat{x}\hat{y}^2, \quad \hat{y}^3,$$

then $\hat{u}^I$ will have the respective forms, correspondingly

$$\frac{2}{3}\hat{x} + \frac{1}{3}, \quad \frac{2}{3}\hat{y} + \frac{1}{3}, \quad \frac{7}{9}\hat{x} + \frac{2}{9},$$

$$(\frac{2}{3}\hat{x} + \frac{1}{3})\hat{y}, \quad \hat{x}(\frac{2}{3}\hat{y} + \frac{1}{3}), \quad \frac{7}{9}\hat{y} + \frac{2}{9}.$$

Through direct computing, we know that for any bilinear functional $\hat{v}$ on $\hat{e}$

$$\hat{Z}(\hat{u},\hat{v}) = 0.$$

Applying Bramble-Hilbert lemma we obtain

$$|\hat{Z}(\hat{u},\hat{v})| \leq c|\hat{u}|_{4,\hat{e}}|\hat{v}|_{0,\hat{e}}$$

and consequently,

$$\begin{aligned} |Z(u,v)| &= k_e|\hat{z}(\hat{u},\hat{v})| \\ &\leq ck_e|\hat{u}|_{4,\hat{e}}|\hat{v}|_{0,\hat{e}} \\ &\leq ch^3||u||_{4,e}||v||_{0,e}. \end{aligned}$$

We conclude that

$$\begin{aligned} F(u,v) &= -\int_e(u-u^I)v_x + \int_{y_e-k_e}^{y_e+k_e}[(u-u^I)(x_e+h_e-0,y) \\ &\quad v(x_e+h_e-0,y) - (u-u^I)(x_e-h_e-0,y) \\ &\quad v(x_e-h_e+0,y)]dy \\ &= \frac{h_e^3}{27}\int_e u_{xxx}v_x - \frac{k_e^3}{9}\int_e u_{xyy}v_y + O(h^3)||u||_{4,e}||v||_{0,e}. \end{aligned} \tag{6}$$

The other terms of (2) can be expanded in a similar way,

$$\begin{aligned} &-\int_e(u-u^I)v_y + \int_{x_e-h_e}^{x_e+h_e}[(u-u^I)(x,y_e+k_e-)v(x,y_e+k_e-0) \\ &-(u-u^I)(x,y_e-k_e-0)v(x,y_e-k_e+0)]dx \\ &= \frac{k_e^3}{27}\int_e u_{yyy}v_y - \frac{h_e^3}{9}\int_e u_{xxy}v_x + O(h^3)||u||_{4,e}||v||_{0,e}, \end{aligned} \tag{7}$$

$$\begin{aligned} \int_e(u-u^I)v\,dxdy &= -\frac{h_e^3}{9}\int_e u_{xx}v_x - \frac{k_e^3}{9}\int_e u_{yy}v_y \\ &\quad + O(h^3)||u||_{4,e}||v||_{0,e}. \end{aligned} \tag{8}$$

In the end, we get the following expansion of $B$:

$$\begin{aligned} B(u^h-u^I,v) &= B(u-u^I,v) \\ &= \sum_e\int_e(\frac{h_e^3}{27}u_{xxx}v_x - \frac{k_e^3}{9}u_{xyy}v_y + \frac{k_e^3}{27}u_{yyy}v_y - \frac{h_e^3}{9}u_{xxy}v_x \\ &\quad -\frac{h_e^3}{9}u_{xx}v_x - \frac{k_e^3}{9}u_{yy}v_y) + O(h^3)||u||_4||v||_0. \end{aligned} \tag{9}$$

So far we consider only the stationary case, which has been done in [4]. We now turn to the main part of this paper.

## 3. Superclose for Nonstationary Problem

Consider the nonstationary problem(1). The associated DGM:
Find

$$u^h \in W_h$$

satisfying

$$(u_t^h, v) + B(u^h, v) = (f, v), \quad \forall v \in W_h.$$

Particularly, we have

$$(u_t^h, v) + B(u^h, v) = (u_t, v) + B(u, v), \quad \forall v \in W_h.$$

Hence, $\forall v \in W_h$,

$$((u^h-u^I)_t, v) + B(u^h-u^I, v) = ((u_t-u_t^I), v) + B(u-u^I, v).$$

Similar to expansion (8), we have

$$\int_e (u_t - u_t^I)v = -\frac{h_e^3}{9}\int_e u_{txx}v_x - \frac{k_e^3}{9}\int_e u_{tyy}v_y$$
$$+O(h^3)||u_t||_{4,e}||v||_{0,e}.$$

We then obtain

$$((u^h - u^I)_t, v) + B(u^h - u^I, v)$$
$$= \sum_e \int_e \{\frac{h_e^3}{27}u_{xxx}v_x - \frac{k_e^3}{9}u_{xyy}v_y + \frac{k_e^3}{27}u_{yyy}v_y - \frac{h_e^3}{9}u_{xxy}v_x$$
$$-\frac{h_e^3}{9}(u_{xx} + u_{txx})v_x - \frac{k_e^3}{9}(u_{yy} + u_{tyy})v_y\} \tag{10}$$
$$+O(h^3)(||u_t||_{4,e} + ||u||_{4,e})|v|_{0,e}.$$

(10) can be further processed with intergration by parts.

Letting $h_e = k_e = h$(uniform rectangular partition) and $v|_{\Gamma_-} = 0$, the first sum on the right of (10) is

$$h^3\sum_e \int_e u_{xxx}v_x = -h^3\sum_e \{\int_e u_{xxxx}v + \int_{y_e-h}^{y_e+h}[u_{xxx}(x_e+h,y)v(x_e+h-0,y)$$
$$-u_{xxx}(x_e-h,y)v(x_e-h+0,y)]dy\}$$
$$= O(h^3)|u|_4|v|_0 - h^3\sum_e \int_{y_e-h}^{y_e+h} u_{xxx}(x_e-h,y)[v(x_e-h-0,y)$$
$$-v(x_e-h+0,y)]dy + h^3\int_0^1 u_{xxx}(1,y)v(1-0,y)dy.$$

Here we have used

$$\sum_e \int_{y_e-h}^{y_e+h} u_{xxx}(x_e+h,y)v(x_e+h-0,y)dy$$
$$= \sum \int_{y_e-h}^{y_e+h} u_{xxx}(x_e-h,y)v(x_e-h-0,y)dy + \int_0^1 u_{xxx}(1,y)v(1-0,y)dy.$$

By inverse theorem, we note that

$$h^3\int_{y_e-h}^{y_e+h} u_{xxx}(x_e-h,y)[v(x_e-h-0,y) - v(x_e-h+0,y)]dy$$
$$= O(h^{2.5})||u||_{4,e}(\int_{y_e-h}^{y_e+h}[v(x_e-h-0,y) - v(x_e-h+0,y)]^2dy)^{0.5},$$

and that (4) contains a similar jump, of v, across the boundary of adjacent elements. We have

$$h^3\sum_e \int_e u_{xxx}v_x = O(h^{2.5})||u||_4 B(v,v)^{0.5}.$$

Upon a similar consideration in other terms of (10), we conclude that

$$((u^h - u^I)_t, v) + B(u^h - u^I, v) = O(h^{2.5})(||u||_4 + ||u_t||_4)B(v,v)^{0.5}.$$

Letting

$$v = u^h - u^I$$

we obtain

$$\frac{1}{2}\frac{d}{dt}||v||_0^2 + B(v,v) \le ch^5(||u||_4 + ||u_t||_4)^2 + \frac{1}{2}B(v,v).$$

Thus,

$$\frac{d}{dt}||v||_0^2 + B(v,v) \le ch^5(||u||_4 + ||u_t||_4)^2.$$

Intergrating on $t$ over $[0,T]$, note that $v(0)=0$, we have

$$||v||_0^2 + \int_0^T B(v,v)dt \le ch^5 \int_0^T (||u||_4 + ||u_t||_4)^2 dt.$$

In particular, we obtain the following superclose result:

$$||u^h - u^I||_0 \le ch^{2.5} \int_0^T (||u||_4 + ||u_t||_4)dt. \tag{11}$$

In addition, a discrete $L_2-$estimate of the jump, of $u^h - u^I$, across the boundary can also be obtained, which reflects a superconvergence for second derivatives. We omitte the details here to avoid technical trouble.

## 4. Postprocessing

Formula (11) gives a superclose to $u^I$ rather than to $u$, so we have to use postprocessing for $u^h$ by the technique developed in our previous work [5]. Let each four small elements combine to be a big element. Take a biquadratic interpolation $I_{2h}$ on the big element acting on $u^h$. $I_{2h}u^h$, which has same value as $u^h$ on the nine interpolation points from four small elements inside the big element, then

$$I_{2h}u^I = I_{2h}u$$

and hence

$$I_{2h}u^h - u = I_{2h}(u^h - u^I) + I_{2h}u - u.$$

Thus

$$||I_{2h}u^h - u||_0 = O(h^{2.5})||u||_4.$$

Therefore, although $u^h$ does not superclose to $u$, postprocessing of $u^h$ does superclose to $u$.

## 5. Numerical Results

Consider the above equation $u_t + u_x + u_y + u = f$, we assume $f(x,y,t) = 0$, and the accurate solution is $u = e^{x+y-3t}$.

The following is our numerical results. we get the $L_2$-error and $L_\infty$-error of $u - u^h$, the $L_2$-error of $u^h - u^I$, and its convergence order, where we set $t = 0.5$.

| n x m | 10x10 | 20x20 | 40x40 | 80x80 | 160x160 |
|---|---|---|---|---|---|
| $L_\infty$-error of $u - u^h$ | 0.1258E-2 | 0.3284E-3 | 0.8396E-4 | 0.2123E-4 | 0.5337E-5 |
| $L_\infty$-order of $u - u^h$ | | 1.94 | 1.97 | 1.98 | 1.99 |
| $L_2$-error of $u - u^h$ | 0.6249E-3 | 0.1547E-3 | 0.3851E-4 | 0.9607E-5 | 0.2386E-5 |
| $L_2$-order of $u - u^h$ | | 2.01 | 2.01 | 2.00 | 2.01 |
| $L_2$-error of $u^h - u^I$ | 0.4158E-4 | 0.6784E-5 | 0.1133E-5 | 0.1891E-6 | 0.3200E-7 |
| $L_2$-order of $u^h - u^I$ | | 2.62 | 2.58 | 2.58 | 2.56 |

$Fig1$, $Fig2$ are the images of numerical results and exact results repectively.

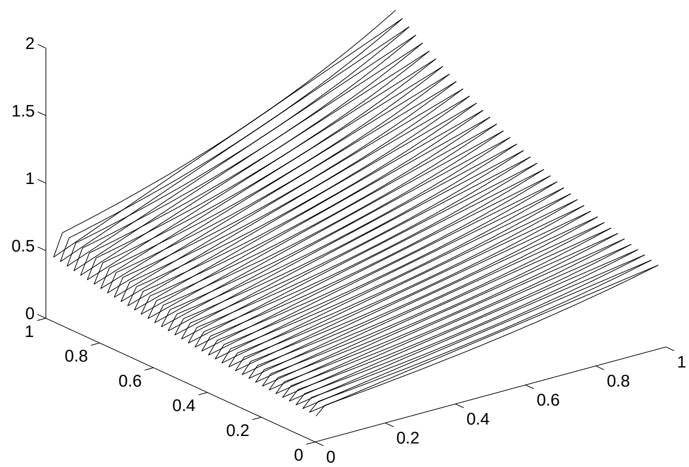


Figure 1. The numerical results.

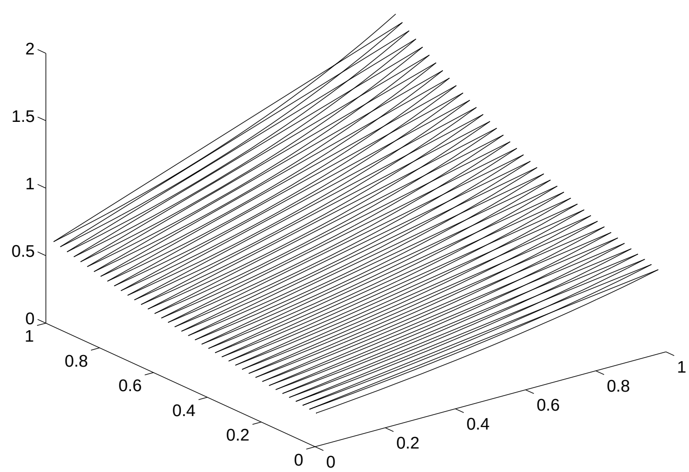


Figure 2. The exact results.

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