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ABSOLUTE STABLE HOMOTOPY FINITE ELEMENT METHODS FOR CIRCULAR ARCH PROBLEM AND ASYMPTOTIC EXACTNESS POSTERIORI ERROR ESTIMATE*

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Abstract

In this paper, HFEM is proposed to investigate the circular arch problem. Optimal error estimates are derived, some superconvergence results are established, and an asymptotic exactness posteriori error estimator is presented. In contrast with the classical displacement variational method, the optimal convergence rate for displacement is uniform to the small parameter. In contrast with classical mixed finite element methods, our results are free of the strict restriction on h(the mesh size) which is preserved by all the previous papers. Furtheremore we introduce an asymptotic exactness posteriori error estimator based on a global superconvergence result which is discovered in this kind of problem for the first time.

Key words: HFEM, arch, superconvergence, asymptotic exactness, posteriori error estimator

1. Introduction

Homotopy Finite Element Method(HFEM) is a new finite element method, but it's idea can be traced back to 1983 with M.Fortin. R.Glowinski's pioneering work[30], thereafter, D.N.Arnold[1] extend this method to shell problem using mixed finite element method. Tianxiao Zhou[23] applying this method to beam and Reissner-Mindlin plate model has been attained success. This method has been used by us to overcome the locking phenomenon of arch beam models recently [16], furthermore we had used the same idea to difference approximation of a nonlinear fluid bed model in some a different form [14]. Now we introduce this method in an abstract framework.

Assuming L is a differential operator, We consider the following Dirichlet problem:

$$Lu = f \quad in\ \Omega \quad u = 0 \quad on\ \Gamma$$

Ω is a bounded open domain with Lipschitz-Continuous boundary Γ for this Dirichlet problem, we consider its variational equivalent form:
Find $u \in U$ such that

$$a(u,v) = (Lu,v) = (f,v) \quad \forall v \in V$$

$$a : U \times V \to R$$

where U, V are Banach spaces, It's evident one Dirichlet problem has not one variational-equivalent form [28]. Assuming this problem has another variational-equivalent form:
Find $u \in U$ such that

$$b(u,v) = (Lu,v) = (f,v) \quad \forall v \in V$$

$$b : U \times V \to R$$

* Received.

Now, we can consider the homotopy form of a (u, v) and $b(u, v)$.

$$H(t; u, v) : [0, 1] \times U \times V \to R$$
$$H(t; u, v) = (1 - t)a(u, v) + tb(u, v)$$

Thus we have a new variational problem:
Find $u \in U$ such that

$$H(t; u, v) = (f, v) \quad \forall v \in V$$

The homotopy variational form is a triliner form with a parameter t besides the two primal variables. There are many parameter-dependent models in beam, arch, shell problems, the parameters usually are the proportion ratio of thickness and the length of beam, or the thickness of the arch, or the mid-thickness of the shell. The small parameter is the main source of locking. This kind of locking phenomenon results from lacking of K-ellipticity or having difficulty in fulfilling the Babuska-Brezzi condition. Homotopy variational principle is to construct a homotopy between two variational forms. Tianxiao Zhou and D.N.Arnold have shown the new variational form could not only enhance the K-ellipticity but also help to fulfil the Babuska-Brezzi condition. In most mechanical problem enhancing K-ellipticity is of most momentous. The method presented in this paper is a panacea to problem lacking K-ellipticity in some sense. We must remark that in most cases, the form $H(t; u, v)$ is not the real homotopy-form as defined in [27], especially in the variational principle lacks symmetry, more precisly it is only a homotopy-form in essence but not in form, this will be presented in this paper.

The circular arch model presented in this paper based upon the Timoshenko-Mindlin-Reissner assumption. The Timoshenko-Mindlin-Reissner assumption is the basis of the governing equations for this model. The key feature of this model is that the shear strain is not neglected, this assumption imposed in this kind of problem is a two folds swore, the applicable of this theory to the problem in which the thickness is not small on the one hand, but on the other hand it becomes the source of locking. For the locking phenomenon of this type, some analyse have been performed in [16,17].

A series of papers have been contributed to analyse the finite element approximation of this problem. In [2] Arnold.D.N. investigated the beam model and derived the sharp estimates, furthermore he introduced one approach to analyse this kind of problem, namely he proposed a mixed method for little parameter problem. Thereafter Kikuchi.F[9,10] carried out a detailed analysis in arch model, in [9] he analyse a arch model without shear deformation, in [10] the asymptotic expansion in terms of d(the little parameter) is presented. Recently Zhimin Zhang[25], Loula.F.D [11], Reddy.B.D.[17] presented several mixed finite element methods for these problems, optimal error estimates uniformly to the little parameter are obtained under some restrictions. In all these restrictions the restriction on the h(the mesh size) is common and that in [25] is the weakest. In [16] based on HFEM, a new variational problem is introduced to analyse the model in [25] and the uniform error estimate is obtained without any restriction on h. However the model in this paper is not the same with [25], it is the model in [11] and like the model in [17], but it is more intricate in equation and variational principle, especially it's variational principle lacks symmetry. First we introduce a new variational principle instead of the non-symmetry form in [11], Secondly as in our previous paper [16] a new mixed finite element approximation is presented by using the idea of HFEM is presented. We point out that our methods can extend to the model in [17] without any diffculty.

In this paper, HFEM is proposed to investigate the circular arch problem. Optimal error estimates are derived, some superconvergence results are establishd, and an asymptotic exactness posteriori error estimator is presented. In contrast with the classical displacement variational method, the optimal convergence rate for displacement is uniform to the small parameter. In contrast with classical mixed finite element methods, our results are free of the strict restriction on h(the mesh size) which is preserved by all the previous papers. Furtheremore we introduce an asymptotic exactness posteriori error estimator based on a global superconvergence result which is discovered in this kind of problem for the first time.

The structure of the paper follows. In section 2 we introduce the circular arch model and all kinds of variational problems including primal variational problem, mixed variational problem and homotopy variational problem. In section 3 we consider the finite element approximation of the homotopy variational problem. In section 4 two type superconvergence results are derived, In section 5 a global superconvergence is presented and an asymptotic exactness posteriori error estimator is proved via the global superconvergence results.

2. Circular Arch Model and Variational Problem

1. Circular arch model

We conside a clamped arch model. The unilaterial case has been considered and alayzed in [5]. This uniform arch of length L, radius R, cross-section A, moment of inertial I, Young's modulus E, and shear modulus G, are subjected to a distributed load $F = (f_u, f_w, f_\phi)$. We denote by $U_s = (u, v, w)^t, s \in [0, L]$, the vector generalized displacements, where u, v, w are tangential displacement, the transverse displacement, the rotation of the cross-section respectively. The stress resultant field is $N(s) = (N, Q, M)$, where N, Q, M are the axial force, the shear force, the bending moment respectively.

The arch problem considered here is described by the two differential equations.

-equalibrium equations.

$$-\frac{dN}{ds} - \frac{Q}{R} = f_u \tag{1}$$

$$-\frac{dQ}{ds} + \frac{N}{R} = f_w \tag{2}$$

$$-\frac{dM}{ds} - Q = f_\phi \tag{3}$$

constitutive equations

$$-\frac{N}{EA} + \frac{du}{ds} + \frac{w}{R} = 0 \tag{4}$$

$$-\frac{Q}{kGA} + \frac{dw}{ds} - \frac{u}{R} - \phi = 0 \tag{5}$$

$$-\frac{M}{EI} + \frac{d\phi}{ds} = 0 \tag{6}$$

Where k is the shear correction factor. We shall consider the clamped problem. So these equations have homogeneous boundary conditions.

$$u(0) = 0 = u(L) \tag{7}$$

$$w(0) = 0 = w(L) \tag{8}$$

$$\Phi(0) = 0 = \Phi(L) \tag{9}$$

To explicate the dependence of this problem on a small parameter, we set $\epsilon^2 = \frac{1}{AL^2}$. From physical meaning we have $\epsilon^2 \ll 1$.

We introduce some new variables to nondimensionalize this problem.

$$u_1 = \frac{u}{L} \quad u_2 = \frac{w}{L} \quad u_3 = \phi \tag{10}$$

$$w_1 = \frac{NL^2}{EI} \quad w_2 = \frac{QL^2}{EI} \quad w_3 = \frac{ML^2}{EI} \tag{11}$$

$$f_1 = \frac{f_u L^3}{EI} \quad f_2 = \frac{f_w L^3}{EI} \quad f_3 = \frac{f_\phi L^2}{EI} \tag{12}$$

We have the following non-dimensional form.

Find $u(x) = (u_1(x), u_2(x), u_3(x))^T, w(x) = (w_1(x), w_2(x), w_3(x))^t, x \in (0,1)$ satisfying,
-equilibrium equation

$$-w_1' - \lambda w_2 = f_1 \tag{13}$$

$$-w_2' + \lambda w_1 = f_2 \tag{14}$$

$$-w_3' - w_2 = f_3 \tag{15}$$

-constitutive equations

$$-\epsilon^2 w_1 + u_1' + \lambda u_2 = 0 \tag{16}$$

$$-\mu\epsilon^2 w_2 + u_2' - \lambda u_1 - u_3 = 0 \tag{17}$$

$$-w_3 + u_3' = 0 \tag{18}$$

with boundary conditoins:

$$u_1(0) = 0 = u_1(1) \tag{19}$$

$$u_2(0) = 0 = u_2(1) \tag{20}$$

$$u_3(0) = 0 = u_3(1) \tag{21}$$

where $x = \frac{s}{L}, \mu = \frac{E}{kG}, \lambda = \frac{L}{R}$. we assume $\lambda > 0$ to exclude the degenerate case $\lambda = 0$ corresponding a straight beam. For convience we assume $\mu = 1$.

We are now in a position to introduce the variational form of this problem, for the simplicity, we introduce some matrices.

$$E = \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad L(\frac{d}{dx}) = \begin{pmatrix} \frac{d}{dx} & \lambda & 0 \\ -\lambda & \frac{d}{dx} & -1 \\ 0 & 0 & \frac{d}{dx} \end{pmatrix}$$

$$L^*(\frac{d}{dx}) = \begin{pmatrix} -\frac{d}{dx} & -\lambda & 0 \\ \lambda & -\frac{d}{dx} & 0 \\ 0 & -1 & -\frac{d}{dx} \end{pmatrix}$$

L^2 is the adjoint of L.

Now we can rewrite (13)-(15), (16)-(18) in a more compact form.

$$L^*(\frac{d}{dx})w = f \tag{22}$$

$$Ew = L(\frac{d}{dx})u \tag{23}$$

$$u(0) = 0 = u(1)$$

2. Prelimilary

Let us denote $\bar{J} = [0,1]$ and define

$$U = H_0^1(0,1)^3, \quad W = L_2(0,1)^3,$$

$$(u,v) = \int_0^1 uv dx$$

where $u \in U$ and $v \in U$ and $u, v = (u_1, v_1) + (u_2, v_2) + (u_3, v_3)$

$$\|u\|_U = \|u'\| \quad \|u\|^2 = (u,u)$$

$W^{k,p}(\bar{J}), H_0^1(\bar{J})$, is the usual sobolev space [26].

From Poincare inequality we know that the norm $\|\cdot\|_U$ is equivalent to norm $\|\cdot\|_1$. We define U^{-1} the dual space of U, and define the negative norm $\|\cdot\|_{-1}$.

$$\|f\|_{-1} = \sup_{u \in U} \frac{(f,u)}{\|u\|_U}$$

It is well known that the classical displacement finite element method is based on the following variational formulation:
Problem (P):Find $u \in U$ such that

$$a_\epsilon(u,v) = (E^{-1}L(\frac{d}{dx})u, L(\frac{d}{dx}v) = (f,v) \quad \forall v \in U \tag{24}$$

If we denote the associated norm $\|u\|_\epsilon = a_\epsilon(u,u)^{\frac{1}{2}}$, we can prove that $|||\cdot|||$ is an equivalent norm of $\|\cdot\|_U$. The skill used here is developped by P.G.Ciarlet [7], we omitte the proof. Onc can refer Zhimin Zhang [25].

Theorem 2.1. There exists a constant $\alpha > 0$ independent of ϵ such that

$$|||u|||_\epsilon \geq \alpha \|u\|_U \quad \forall u \in U \tag{25}$$

If we rewrite (25) in a more explicit form we have corallary 2.1.

Colloary 2.1. Denote $a_1(u,v) = (L(\frac{d}{dx})u, L(\frac{d}{dx})v)$, then there exists a constant $\gamma > 0$ such that

$$a_1(u,u) \geq \gamma \|u\|_U^2 \quad \forall u \in U. \tag{26}$$

By Theorem 2.1 and Lax-Milgram theorem, we have the following theorem.

Theorem 2.2. Let $f \in U^{-1}, 0 < \epsilon \ll 1$, there is a unique $u \in U$ such that

$$a_\epsilon(u,v) = f(v) \quad \forall v \in U \tag{27}$$

Moreover for $k = 0, 1, \cdots$, there exists a constants C_k depends only on k such that

$$\|u\|_{k+1} + \epsilon^{-2}\|u_1' + \lambda u_2\|_k + \epsilon^{-2}\|\lambda u_1 - u_2' + u_3\|_k \leq c_k \|f\|_{k-1} \tag{28}$$

We put off the proof of (28) after we introduce the mixed variational principle.

First we introduce a new mixed variational formulation which differs from it's counterpart in [11]. We will split the matrix E into two parts: F is the ϵ-dependent part and S is the ϵ-independent part.

$$F = \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the matrix L is splitted into C, D.

$$C = \begin{pmatrix} \frac{d}{dx} & \lambda & 0 \\ -\lambda & \frac{d}{dx} & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{d}{dx} \end{pmatrix}$$

Now we introduce the mixed problem.

Problem M: Find $(u,w) \in U \times W$ such that

$$a(u,v) + b(v,w) = (f,v) \quad \forall v \in U \tag{29}$$

$$b(u,\tau) - \epsilon^2((I-S)w, (I-S)\tau) = 0 \quad \forall \tau \in W \tag{30}$$

where the bilinear form $a : U \times U \to R$, and $b : U \times W \to R$ are:

$$a(u,v) = (u_3', v_3') = (Du, Dv) \tag{31}$$

$$b(u,w) = (Cv, (I-S)w) = (v_1' + \lambda v_3', w_1) + (-\lambda v_1 + v_2' - v_3, w_2) \tag{32}$$

For this mixed problem, we have the following theorem. Firstly we state the remarkable theorem of D.N.Arnold [2,Theorem 5.1] for the later's use.

Theorem 2.3. Let V, M be Hilbert spaces, let a:$V \times V \to R$ and $b : V \times M \to R$ be bounded linear form, and $\epsilon \in (0,1), \epsilon \ll 1$, if the following conditions are fulfilled:
1)a is a symmetric and positive semidefinite bilinerform.
2)there exists $c_1 > 0$ such that for all $z \in \{v \in V | b(v,w) = 0, \forall w \in M\}$
$C_1 a(z,z) \geq \|z\|_V^2$.

3)there exists $c_2 > 0$ such that for all $w \in M$ there exists $v \in V$ with $C_2 b(v,w) \geq \|v\|_V \|w\|_M$. Then for each pair $(f,g) \in V^{-1} \times M^{-1}$, there exists a unique pair $(u,w) \in V \times M$ such that

$$a(u,v) + b(v,w) = (f,v) \quad \forall v \in U \tag{33}$$

$$b(u,y) - \epsilon^2 (w,y) = (g,y) \quad \forall y \in M \tag{34}$$

$$\|u\|_V + \|w\|_M \leq C(\|f\|_{V^{-1}} + \|g\|_{M^{-1}}) \tag{35}$$

where C is independent of ϵ.

Lemma 2.1. There exists $C_1 > 0$ such that for all $u \in \{v \in U | b(v,w) = 0, \forall w \in M\}, C_1 a(u,u) \geq \|u\|_U^2$

This lemma is almost the same as the lemma 2.1 in [25], where $\lambda = 1$ is proved, this verifys the condition (2) of theorem 2.3. Condition (3) can also be verfied by the same trick used in [25], we will prove a more general case later. Applying theorem 2.3 we obtain the existence, uniqueness results of problem (M), more pricisely, there exists a unique solution of (M) such that

$$\|u\|_1 + \|w\| \leq C\|f\|_{-1} \tag{36}$$

from (30) we have $w_1 = \epsilon^{-2}(u_1' - \lambda u_2), w_2 = \epsilon^{-2}(\lambda u_1 - u_2' + u_3)$, substitute these two into (36), we have

$$\|u\|_1 + \epsilon^{-2}\|u_1' - \lambda u_2\| + \epsilon^{-2}\|\lambda u_1 - u_2' + u_3\| \leq C\|f\|_{-1} \tag{37}$$

Using (37) and induction method, we obtain (28).

3. Homotopy variational problem

From the bilinear form $a(u,v)$ and the proof of lemma 2.1[25], we will find it lacks K-ellipticity. To enhance it's K-ellipticity, Loula etc in [11] introduced a Prtrov-Galerkin method, their variational principle differs to ours. In fact our mixed variationed principle is superior to theirs' at least in light of symmetric.

In contrast with Loula's non variational principle approach, we will present a new variational principle under the name of Homotopy variational principle, we will find later our method enhancce it's K-ellipticity in a more nature way.

Problem (H): Find $(u,w) \in U \times W$ such that

$$((I-M)E^{-1}Lu, Lv) + (MDu, Dv) + (Cv, M(I-S)w) = (f,v) \quad \forall v \in U \tag{38}$$

$$(Cu, M(I-S)\tau) - \epsilon^2((I-S)w, M(I-S)\tau) = 0 \quad \forall \tau \in W \tag{39}$$

where I denotes the unit matrix.

$$M = \begin{pmatrix} 1-\alpha_1\epsilon^2 & 0 & 0 \\ 0 & 1-\alpha_2\epsilon^2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

Where $\alpha_i \in [0,1], i = 1,2,3$, are given as stabilized factors of new type. (38), (39) is equivalent to equations of (22), (23). In fact, we construct a homotopy of the primal variational principle and the mixed variational principle. To see this more clearly, we rewrite the mixed variational principle as follows: Find $(u,w) \in U \times W$ such that

$$B(u,w;v,\tau) = (f,v) \quad \forall (v,\tau) \in U \times W \tag{40}$$

where

$$B(u,w;v,\tau) = a(u,v) + b(v,w) - b(u,\tau) + \epsilon^2((I-S)w,(I-S)\tau)$$

To construct a homotopy of $a_\epsilon(u,v)$ and $B(u,w;v,\tau)$, we introduce a donation. If M is a matrix of 3×3, we denote $M^*(u,v) = (Mu,v)$, where (u,v) is defined as before, we can show our new variational principle is the homotopy variational principle of (24) and (40) in the sense of $*$ donation.

$$H(M;u,w;v,\tau) = (I-M) * a_\epsilon(u,v) + M * B(u,w;v,\tau) \tag{41}$$

So we have a variational-equivalent problem: Find $(u, w) \in U \times W$ such that Problfm (H):

$$H(M; u, w; v, \tau) = (f, v) \quad \forall (v, \tau) \in U \times W \tag{42}$$

It's easy to see this variational problem can rewrite as a mixed variational problem as follows: Problem $(\bar{H})$: Find $(u, w) \in U \times W$ such that

$$\bar{a}(u, v) + \bar{b}(v, w) = (f, v) \quad \forall v \in U \tag{43}$$

$$\bar{b}(u, \tau) - \epsilon^2 (M(I - S)\tau, (I - S)w) = 0 \quad \forall \tau \in W \tag{44}$$

where $\bar{a}(u, v) = ((I - M)E^{-1}Lu, Lv) + (MDu, Dv), \bar{b}(v, w) = (Cv, M(I - S)w)$

From another point of view for our variational form (43), (44), we will find the essence of the homotopy variational principle is to replace the equilibrium equation with a homotopy-family form of two different equilibrium equations.

Equilibrium equation 1:

$$L^*(\frac{d}{dx})w = f \tag{45}$$

Equilibrium equation 2:

$$L^*(\frac{d}{dx})E^{-1}L(\frac{d}{dx})u = f \tag{46}$$

Based upon these two equilibrium equations, we have another equilibrium equation of homotopy-family form. From (43), we have

$$((I - M)E^{-1}Lu, Lv) + (MDu, Dv) + (Cv, M(I - S)w) = (f, v)$$

from equation (18) we have

$$(MDu, Dv) = (Mw, Dv) = (Dv, MSw) = (Lv, MSw)$$

note $(Dv, M(I - S)v) = 0$ we have

$$((I - M)E^{-1}Lu, Lv) + (M(I - S)w, (D + C)v) + (Lv, MSw) = (f, v)$$

$$((I - M)E^{-1}Lu, Lv) + (Mw, Lv) = (f, v)$$

namely

$$L^*((I - M)E^{-1}Lu + Mw) = f \tag{47}$$

We can easily see that equilibrium equation (47) is the homotopy-family of the equilibrium equation (45) and (46). This is the mechanical essence of the Homotopy Finite Element Method(HFEM) for this problem.

From now on, we set $\alpha_1 = \alpha_2 = \alpha = (1 + \epsilon^2)^{-1}, \alpha_3 = \frac{1}{2}$. We can easily see $(I - M)E^{-1} = M$

Lemma 2.2. There exists $C_1 > 0$ such that for all $v \in V$

$$C_1 \bar{a}(v, v) \geq \|v\|_U^2$$

Proof. After a simple computation, we have

$$\bar{a}(v, v) = a_\alpha(v, v) \geq \frac{1}{2} a_1(v, v)$$

from corollary 2.1 we have $a_1(v, v) \geq \gamma \|v\|_U^2$. Set $C_1 = \frac{2}{\gamma}$, the proof is finished.

Remark. We can find the K-ellipticity in our new variational principle is enhanced really, since the K-ellipticity holds over the whole space U instead of only over the kernal space of $\bar{b}(v, w)$ which is presecved by all the previous papers. In fact, In all the previous works the proof of the ellipticity of the mixed problem is delicate for this kind of problem. By the way, it is easy to see their trick of proving the ellipticity essentially depends on the one-dimensional character. But our proof used here can extend to high-dimensional without any difficulty. We will find the enhanced K-ellipticity plays an essential role in our discussion.

Lemma 2.3. There exists a constants $C_2 > 0$ such that for all $w \in W$ there exists $v \in U$ with

$$C_2 \bar{b}(v, w) \geq \|v\|_U \|(I - S)w\|$$

Proof $\bar{b}(v,w) = (Cv, M(I-S)w)$, we set $\bar{w} = (I-S)w, \bar{w} \in L_2(0,1)^2$, Using the same trick of [25]. we can prove that for all $w \in W$ there exists $v \in U$ and a constant c such that $cb(v,\bar{w}) \geq \|v\|_V \|\bar{w}\|$, set $C_2 = c$, the proof is finished.

If we set $\bar{w} = (I-S)w, \bar{W} = (I-S)W$, As a results of theorem 2.3, lemma 2.2 and lemma 2.3, we can also obtain the existence, uniqueness and regularity results of problem $(\bar{H})$ as before.

3. Finite Element Approximation

We introduce some standard notations. Let $\Delta = \{\phi_0, \cdots, \phi_M\}$ be a partition of $(0,1), 0 = \phi_0 < \phi_1 < \cdots < \phi_M = 1, J_j = (\phi_{j-1}, \phi_j), L_j = \phi_j - \phi_{j-1}, h = max_{1\leq j \leq M}\{L_j\}$, and the partition is quasi-uniform. Denote $P^k_{-1}(\Delta)$ the space of piecewise polynomials of order k which are not continuous at nodal points.

Define $P^k(\Delta) = P^k_{-1}(\Delta) \cap C^0(J), P^k_0(\Delta) = P^k(\Delta) \cap H^1_0, U_h = P^k_0(\Delta)^3, W_h = P^{k-1}_{-1}(\Delta)^3$. Define L_2 projection $\Pi_{k-1} = \Pi_{k-1}(\Delta), \Pi_{k-1} : P^k_0(\Delta) \to P^{k-1}_{-1}(\Delta)$ we also have $\|v\|^2_{k,h} = \sum_{i=1}^{i=M} (v^k, v^k)_{J_i}, \|v\|^2_{k,h} = \sum_{j=0}^{j=k} |v|^2_{j,h}$. We consider the finite element approximation of problem $\bar{H}$.

$(\bar{H}_h)$: Find $(u_h, w_h) \in U_h \times W_h$ such that

$$\bar{a}(u_h, v) + \bar{b}(v, w_h) = (f, v) \quad \forall v \in U_h \tag{48}$$

$$\bar{b}(u_h, \tau) - \epsilon^2(M(I-S)\tau, (I-S)w_h) = 0 \quad \forall \tau \in W_h \tag{49}$$

In order to establish the existence, uniqueness and the error estimate of $(\bar{H}_h)$, we must verify all the assumptions of theorem 2.3. As U_h is a conforming approximation of U, we can establish the U_h-ellipticity as continous case does. For the condition (3), a special case is established In [25], if we carefully compare the forms of $b(v,w)$ in [25] and $\bar{b}(v,w)$, we can find $\bar{b}(v,w) = (Cv, M\bar{w}) = \alpha(Cv, \bar{w}) = \alpha b(v, \bar{w})$ in the sence of some parameter. So from [25] we find that for all $\bar{w} \in \bar{W}_h, \bar{W}_h = (I-S)W_h$, there exists $v \in U_h$ and $c(\lambda) > 0$ such that $c(\lambda)b(v,\bar{w}) \geq \|v\|_U \|\bar{w}\|$. So

$$\begin{aligned} c(\lambda)\alpha\bar{b}(v,\bar{w}) &\geq \alpha c \|v\|_U \|\bar{w}\| \\ &\geq \frac{1}{2}\|v\|_U \|w\| \end{aligned}$$

namely $2c(\lambda)\bar{b}(v,w) \geq \|v\|_U \|\bar{w}\|$. Now if we replace w_h, W_h by $\bar{w}_h, \bar{W}_h$ in (48), (49), set $C = 2c(\lambda)$, we have verified the conditon (3) of theorem 2.3. Now by virture of theorem 2.3 and the routine of error estimates of mixed methods [3], we get the following theorem.

Theorem 3.1. Assume $kM \geq 3$, then problem $\bar{H}_h$ has a unique solution $(u_h, \bar{w}_h) \in U_h \times \bar{W}_h$ such that

$$\|u - u_h\|_U + \|\bar{w} - \bar{w}_h\| \leq \inf_{(v,\bar{\tau}) \in U_h \times \bar{W}_h} (\|u - v\|_U + \|\bar{w} - \bar{\tau}\|)$$

$$\|u - u_h\|_U + \|(I-S)(w - w_h)\| \leq c \inf_{(v,\tau) \in U_h \times W_h} (\|u - v\|_U + \|(I-S)(w - \tau)\|) \tag{50}$$

Theorem 3.2. Let u, u_h be solutions of $(\bar{H}), (\bar{H}_h)$ respectively, then there exists a constant c independent of $\epsilon \in (0,1]$ such that

$$\|u - u_h\| + h\|u - u_h\|_U \leq ch^{k+1}\|f\|_{k-1} \tag{51}$$

Proof. From the approximation results of U_h, W_h

$$\inf_{v \in U_h} \|u - v\|_U \leq ch^k \|u\|_{k+1} \leq ch^k \|f\|_{k-1}$$

$$\inf_{\tau \in W_h} \|(I-S)(w - \tau)\| \leq ch^k \|w\| k \leq ch^k \|f\|_{k-1}|$$

and by theorem 3.1, we have

$$h\|u - u_h\|_U \leq ch^{k+1}\|f\|_{k-1} \tag{51'}$$

we use a dual argument for L_2-estimate. (48) and (49) yield

$$(w_1, w_2) = \epsilon^{-2}(u_1' + \lambda u_2, -\lambda u_1 + u_2' - u_3) \tag{52}$$

$$(w_{h_1}, w_{h_2}) = \epsilon^{-2}(u_{h_1}' + \lambda\pi_{k-1}u_{h_2}, -\lambda\pi_{k-1}u_{h_1} + u_{h_2}' - \pi_{k-1}u_{h_3}) \tag{53}$$

we define

$$w_h^* = (\epsilon^{-2}(u_{h_1}' + \lambda u_{h_2}), \epsilon^{-2}(-\lambda u_{h_1} + u_{h_2}' - u_{h_3}), w_{h_3}) \tag{54}$$

Define an auxiliary problem: Find z in V such that

$$a_\epsilon(z, v) = (u - u_h, v) \quad \forall v \in U \tag{55}$$

$$\xi = \epsilon^{-2}Cz \tag{56}$$

where C is the matrix defined as before.
from theorem 2.1, we have

$$\|z\|_2 + \|\xi\|_1 \le c\|u - u_h\| \tag{57}$$

set $v = u - u_h$ in (55) then

$$\|u - u_h\|^2 = a_\epsilon(z, u - u_h) = a_\alpha(z, u - u_h) + \bar{b}(u - u_h, \xi) \tag{58}$$

$$\bar{b}(u - u_h, \xi) = \bar{b}(z, w - w_h^*) \tag{59}$$

substitute (59) into (58)

$$\begin{aligned} \|u - u_h\|^2 &= a_\alpha(z, u - u_h) + \bar{b}(z, w - w_h^*) \\ &= a_\alpha(z, u - u_h) + \bar{b}(z, w - w_h) + \bar{b}(z, w_h - w_h^*) \end{aligned} \tag{60}$$

for any $\eta \in W_h$

$$\begin{aligned} \bar{b}(z, w_h - w_h^*) &= \epsilon^2(\xi, M(I - S)(w_h - w_h^*)) \\ &= \epsilon^2(\xi - \eta, M(I - S)(w_h - w_h^*)) \\ &= \epsilon^2(\xi - \eta, M(I - S)(w_h - w)) + \epsilon^2(\xi - \eta, M(I - S)(w - w_h^*)) \end{aligned} \tag{61}$$

in (61) we have used the projection property of π_{k-1}

$$(\eta, (I - S)(w_h - w_h^*)) = 0, \quad \forall \eta \in W_h$$

and

$$(\eta, M(I - S)(w_h - w_h^*)) = 0, \quad \forall \eta \in W_h$$

substract (48) from (43) we have

$$a_\alpha(u - u_h, v) + \bar{b}(v, w - w_h) = 0 \tag{62}$$

Substitute (61),(62) into (60) we then have

$$\begin{aligned} \|u - u_h\|^2 &= a_\alpha(u - u_h, z - v) + \bar{b}(z - v, w - w_h) \\ &+\epsilon^2(\xi - \eta, M(I - S)(w_h - w)) + \epsilon^2(\xi - \eta, M(I - S)(w - w_h^*)) \quad \forall (v, \eta) \in U_h \times W_h \end{aligned} \tag{63}$$

$$\begin{aligned} \|u - u_h\|^2 &= \inf_{(v,\eta)\in U_h\times W_h} [a_\alpha(u - u_h, z - v) + \bar{b}(z - v, w - w_h) \\ &\quad +\epsilon^2(\xi - \eta, M(I - S)(w_h - w)) + \epsilon^2(\xi - \eta, M(I - S)(w - w_h^*))] \\ &\le a(u - u_h, z - v) + \|z - v\|_U\|w - w_h\| \\ &\quad +\epsilon^2\|\xi - \eta\|\|M(I - S)(w - w_h)\| + \epsilon^2\|\xi - \eta\|M(I - S)(w - w_h^*)\| \\ &\le c(\|u - u_h\|_U + \|M(I - S)(w - w_h)\|)(\|z - v\|_U + \|\xi - \eta\|) \\ &\le ch(\|u - u_h\|_U + \|M(I - S)(w - w_h)\|)(\|z\|_2 + \|\xi\|_1) \\ &\le ch \inf_{(v,\tau)\in U_h\times W_h} (\|u - v\|_U + \|(I - S)(w - \tau)\|)(\|z\|_2 + \|\xi\|_1) \\ &\le chh^k\|f\|_{k-1}\|u - u_h\| \\ &\le ch^{k+1}\|f\|_{k-1}\|u - u_h\| \end{aligned}$$

$$\|u-u_h\| \le ch^{k+1}\|f\|_{k-1} \tag{64}$$

combining (51') and (64) we have (51).

4. Superconvergence

In this section, we will point out our methods have point-superconvergence properties, in addition, we will establish a global superconvergence result. The point-superconvergence is of two kinds. One is the displacement at the nodal points, the other is the gradient of the displacement at the Gauss points.

We introduce some notations and lemmas. $(u,v)^* = \sum_{i=1}^{i=M}(u,v)^*_{ji}$

$((u,v)_{ji})^* = l_i\sum_{j=1}^{j=K} u(G_{ij})\cdot v(G_{ij})w_j$ Here $G_{ij} = \phi_{i-1}+l_iG_j, j=1,\cdots,K, i=1,\cdots,M, G_j$ is the j-th Gauss point in [-1,1] interval, and ω_j is the associated weighs $j=1,\cdots,K$.

$$\|u\|^* = \sqrt{(u,u)^*} \quad |u|_1^* = \sqrt{(u',u')^*}$$

$$a^*(u,v) = (u_1'+\lambda u_2, v_1'+\lambda v_2)^* + (-\lambda u_1+u_2'-u_3, -\lambda v_1+v_2'-v_3)^* + (u_3', v_3') \tag{65}$$

To prove the superconvergence results, we need the following lemma [8].

Lemma 4.1. For any $u,v\in V_{-1}^k(\Delta), (\pi_{k-1}u,v) = (u,v)^*$

We can write problem (P) in a more compact form

$$a(u,v) = \epsilon^2 a_\epsilon(u,v) = (\epsilon^2 f, v) \tag{65'}$$

(Don't mix the $a(u,v)$ here with that in the mixed problem.)

$$(Hu',v') + (Au,v') + (Bu,v) = (\epsilon^2 f,v)$$

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 2\lambda & 0 \\ -2\lambda & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \lambda^2 & 0 & \lambda \\ 0 & \lambda^2 & 0 \\ \lambda & 0 & 1 \end{pmatrix}$$

Clearly H is a symmetric positive definite matrix, A is a skew symmetric matrix and B is a symmetric semi-pisitive definite matrix.

Lemma 4.2. If we denote s the minimum eigenvalue of H, then $s=\epsilon^2$, If we denote $\|A\|_F, \|B\|_F$ the Frobenius norm [26] of matrix A and B, then we have $\|A\|_F \le \sqrt{8\lambda^2+2}, \|B\|_F \le \sqrt{2}(1+\lambda^2), \|H\|_F \le \sqrt{\epsilon^4+2}$.

From lemma 4.2, We have

$$\|H\|_F, \|A\|_F, \|B\|_F \le c(\lambda)$$

In section 3, we observed that $(\bar{H}_h)$ is equivalent to solve the problem:

Find $u_h \in U_h$ such that

$$((I-M)E^{-1}Lu_h, Lv) + (MDu_h, Dv) + \alpha\epsilon^{-2}(u'_{h_1}+\lambda\Pi_{k-1}u_{h_2}, v_1'+\lambda v_2)$$
$$+\alpha\epsilon^{-2}(-\lambda\Pi_{k-1}u_{h_1}+u'_{h_2}-\Pi_{k-1}u_{h_3}, -\lambda v_1+v_2'-v_3) = (f,v)$$

Use lemma 4.1, we can obtain the equivalent theorem.

Theorem 4.1. Problem $(\bar{H}_h)$ is equivalent to the problem $\bar{P}_h$

Problem $\bar{P}_h$:Find $u_h\in U_h$ such that

$$(1-\alpha)a(u_h,v) + \alpha a^*(u_h,v) = (\epsilon^2 f,v) \quad \forall v\in U_h \tag{66}$$

Now we have the following two superconvergence theorems.

Theorem 4.2. Let u and u_h be solutions of problem (P) and $\bar{P}_h$ respectively, then for any nodal points $\phi_j, j=1,\cdots,M$ there is a constant $c(k)$ independent of ϵ such that

$$|u(\phi_j)-u_h(\phi_j)| \le c(k)h^{k+s}\|f\|_{s-1} \quad 0\le s\le k \tag{67}$$

Proof. Define Green's function $G(\phi,\cdot)\in U$ as

$$(HG'(\phi,\cdot),z')+(AG(\phi,\cdot),z')+(BG(\phi,\cdot),z)=(\delta,z)\quad\forall z\in U \tag{68}$$

where $\delta\in H^{-1}(J)^3$ is the delta function in schwartz sense. Set $z=u-u_h$ in (68) then

$$u(\phi_j)-u_h(\phi_j)=(HG'(\phi_j,\cdot),u'-u_h')+(AG(\phi_j,\cdot),u'-u_h')+(BG(\phi_j,\cdot),u-u_h) \tag{69}$$

$$a(u,v)=(\epsilon^2 f,v)\Rightarrow(1-\alpha)a(u,v)+\alpha a(u,v)=(f,v) \tag{70}$$

(70)-(66)

$$(1-\alpha)a(u-u_h,v)+\alpha(a(u,v)-a^*(u_h,v))=0 \tag{71}$$

$$a(u-u_h,v)=(H(u'-u_h'),v')+(A(u-u_h),v')+(B(u-u_h),v)$$

for constant matrix H and A, we have

$$(Hu_h',v')^*=(Hu_h',v')\quad(Au_h,v')^*=(Au_h,v')$$

$$a(u,v)-a^*(u_h,v)=(H(u'-u_h'),v')+(A(u-u_h),v')+(B(u-u_h),v)+[(Bu_h,v)-(Bu_h,v)^*] \tag{72}$$

Substitute (71),(72) into (70) we have

$$(H(u'-u_h'),v')+(A(u-u_h),v')+(B(u-u_h),v)=\alpha[(Bu_h,v)^*-(Bu_h,v)] \tag{73}$$

Applying partial integration formula and note matrix H,A are onstant matrix.

$$((u'-u_h',Hv')+(u'-u_h',Av)+(u-u_h,Bv)=\alpha[(Bu_h,v)^*-(Bu_h,v)] \tag{74}$$

Combining (74) and (68) we have

$$\begin{aligned}u(\phi_j)-u_h(\phi_j)=&(H(G'(\phi_j,\cdot)-v'),u'-u_h')+(A(G(\phi_j,\cdot)-v),u'-u_h')+(B(G(\phi_j,\cdot)-v),u'-u_h')\\&+\alpha[(Bu_h,v)^*-(Bu_h,v)]\end{aligned} \tag{75}$$

From the regularity property of Green's function, we have

$$\|G(\phi_j,\cdot)\|_{H^{k+1}[0,\phi_j)}+\|G(\phi_j,\cdot)\|_{H^{k+1}(\phi_j,\Phi]}\le C \tag{76}$$

then

$$\inf_{v\in V_h}\|G(\phi_j,\cdot)-v\|_1\le ch^k$$

$$|(Bu_h,v)^*-(Bu_h,v)|\le c(k)h^{k+s}|u_h|_s|v|_{k,h}\le c(k)h^{k+s}\|u\|_s|v|_{k,h} \tag{77}$$

Considering estimates (51),(77) and lemma 4.2.

$$\begin{aligned}|u(\phi_j)-u_h(\phi_j)|&\le c(k)(\|u-u_h\|_U+\|u-u_h\|)\inf_{v\in U}\|G(\phi_j,\cdot)-v\|1\\&\quad+c\alpha|u|_{s,h}|v|_{k,h}h^{k+s}\\&\le ch^{k+s}\|u\|_{s+1}\\&\le c(k)h^{k+s}\|f\|_{s-1}\end{aligned}$$

We turn to the analysis of the superconvergence at the Gauss points. The following lemma is needed.

Lemma 4.3. For any $u\in U$, defined $\pi u\in U_h$, by

$$(\nabla u-\nabla(\pi u),\nabla v)=0\quad\forall v\in U_h \tag{78}$$

where ∇ is the gradient operator. Then we have [12,15]

$$|u'(G_{ij})-(\pi u)'(G_{ij})|\le c(k)h^{k+1-\frac{1}{p}}\|u\|_{k+2,p}\quad 1\le i\le M,1\le j\le K \tag{78'}$$

Theorem 4.3. Let u,u_h be the solutions of (P) and $\bar{P}_h$ respectively, then

$$|u-u_h|_1^*\le\epsilon^{-2}c(\lambda,k)h^{k+1}(\|f\|_{k,\infty}+\|f\|_{k-1})$$

Proof. Let $v=\pi u-u_h$,

$$(H(\pi u)-u_h)',v')=(H(u'-u_h'),v')=-(A(u-u_h),v')-(B(u-u_h),v)$$

$$+\alpha[(Bu_h,v)^*-(Bu_h,v)] \tag{79}$$

considering the positive definite of H and lemma 4.2 we have

$$\begin{aligned}\epsilon^2|\pi u-u_h|_1^2 &\le c(\lambda)(\|u-u_h\||v|_1+c(\lambda)\|u-u_h\|\|v\| \\ &\quad +c(k,\lambda)|u|_k|v|_1h^{k+1} \\ &\le c(\lambda)\|u-u_h\|+c(\lambda)h\|u-u_h\|+c(k,\lambda)\|u\|_kh^{k+1})|v|_1\end{aligned} \tag{80}$$

thus

$$|\pi u-u_h|_1\le\epsilon^{-2}h^{k+1}(c(\lambda,k)\|u\|_{k+1}\le c(k,\lambda)\epsilon^{-2}h^{k+1}\|f\|_{k-1} \tag{81}$$

using lemma 4.3

$$\begin{aligned}|u-u_h|_1^* &\le |u-\pi u|_1^*+|\pi u-u_h|_1 \\ &\le c(k)h^{k+1}\|f\|_{k,\infty}+c(k,\lambda)h^{k+1}\epsilon^{-2}\|f\|_{k-1} \\ &\le c(k,\lambda)\epsilon^{-2}h^{k+1}(\|f\|_{k,\infty}+\|f\|_{k-1})\end{aligned}$$

Remark 1. If $A=0$, then we have a better estimate

$$|u'(G_{ij})-u_h(G_{ij})|\le c(k,\lambda)\epsilon^{-2}h^{k+1}\|f\|_{k-1} \tag{82}$$

To see this we start from (79)

$$\epsilon^2|v|_1^2\le c(\lambda)\|u-u_h\|\|v\|+c(\lambda,k)h^{k+1}\|u\|_{k+1}\|v\| \tag{83}$$

Using imbedding theorem [28] and inverse estimate [7] we have

$$\|v'\|_\infty|v'|_{L'}\le ch^{-\frac{1}{2}}\|v'\|h^{\frac{1}{2}}\|v'\|\le c\|v'\|^2 \tag{84}$$

combining (83) and (84)

$$\begin{aligned}\|v'\|_\infty|v'|_{L^1} &\le c(\lambda)\epsilon^{-2}h^{k+1}\|u\|_{k+1}\|v\| \\ &\le c(\lambda)\epsilon^{-2}h^{k+1}\|f\|_{k-1}\|v\| \\ &\le c(k,\lambda)\epsilon^{-2}h^{k+1}\|f\|_{k-1}|v'|_{L^1}\end{aligned}$$

$$\|v'\|_\infty\le c(k,\lambda)\epsilon^{-2}h^{k+1}\|f\|_{k-1} \tag{85}$$

Use lemma 4.3 and (85), (82) can be obtained.

Remark 2. If u has lower regularity, theorem 4.3 can be rewrited in the following form:

$$|u-u_h|_1^*\le\epsilon^{-2}h^{k+\frac{1}{2}}\|f\|_k$$

In this case, (78') become

$$|u'(G_{ij})-(\pi u)'(G_{ij})|\le c(\lambda,k)h^{k+\frac{1}{2}}|f|_k$$

Use (78) and triangle inequlity, we have

$$\begin{aligned}|u-u_h|_1^* &\le |u-\pi u|_1^*+|\pi u-u_h|_1^* \\ &\le |u-\pi u|_1^*+|piu-u_h|_1 \\ &\le c(\lambda,k)\|f\|_kh^{k+\frac{1}{2}}+c(\lambda,k)\epsilon^{-2}\|f\|_{k-1}h^{k+1} \\ &\le c(\lambda,k)\epsilon^{-2}\|f\|_{k-1}h^{k+\frac{1}{2}}\end{aligned}$$

Remark 3. From (79), (85), we find the constant C is ϵ-dependent, but if we assume u has a higher regularity, through a post-process, we can get a ϵ-independent estimate.

In the computation, we can regulate the patition diameter h so that $h\le\epsilon$ holds. We have

$$\begin{aligned}|u-u_h|_1^* &\le |u-\pi u|_1^*+|\pi u-u_h| \\ &\le c(k)h^{k+1}\|u\|_{k+2,\infty}+c(\lambda,k)\epsilon^{-2}h^{k+3}\|u\|_{k+3} \\ &\le c(\lambda,k)h^{k+1}(\|f\|_{k,\infty}+\|f\|_{k+1})\end{aligned}$$

5. Posteriori and Pistprocess

The popularity of the finite element method has led to an increasing attention being paied to the problem of assessing the quatity of the computing cost. The priori error estimates are not appropriate in this case, then using the approximating result itself to estimate the error is needed. After the series paper of Babuska. I.E[4,5], the posterior estimates have been clear and becoming an interesting subject not only for engineers but also for mathematicians [13,21,6]. But we have found that many works have been done for the Poission problem, the Advection-Diffusion problem and Navier-Stokes equations. Legions of posteriori error eatimators have been introduced for these problems, unfortunate almost no one had been reasearched the circular-arch model, in this section we will investigate the posteriori estimate of this kind of problem.

First of all we use the variational problem presented in the introduction. Find $u \in U$ such that $a(u,v) = (f,v), \forall v \in U$, if U_h is a approximation space of U, we have the approximation problem, find $u_h \in U_h$ such that $a(u_h,v) = (f,v), \forall v \in U_h$. In common sense, a posteriori estemate is to use u_h, f(or other computable datas) to estimate $\|u-u_h\|$. But as we all know in most problem arising from physics and engineering fields, the norm $\|u-u_h\|_U$ is not a good metric of the error. An estimate of intrinsical physics meaning is more attractive, in some case this kind of norm is even superior to L_∞. But the most of this kind norm is huristical or intuitional (there have no strict mathmatical proof). So we consider an energy norm. We set $e = u - u_h, \|e\|_E = a(e,e)^{\frac{1}{2}}$ In the following part we will estimate $\|e\|_E$ using the computable datas, if we have been obtained a computable data $E(u_h, f)$ and we have the estimate $\|e\|_E \le CE(u_h, f)$ like that in [21], in most situation we can not precise the the constant C, so this kind of estimator is not satisfied. But the so called asymptotic exactness posteriori estimator introduced by Babuska and Rheinboldt [5] is interesting and attractive, for it had given a computable data in asymptotic exactness sense.

Definition 5.1. Asymptotic exactness posteriori error estimator. Let ϵ be an error estimator then if under reasonable assumptition on u Lu, f, and the patition Δ, we have that

$$\|e\|_E = \epsilon(1 + O(h^\gamma) \quad as \quad h \to 0 \tag{86}$$

where $\gamma \ge 0$ is independent of h and the constant in $O(h^\gamma)$ term dependent upon u, Lu and f only, then we say ϵ is an aymptotic exactness posteriori error estimator.

To derive the asymptotic exactness posteriori estimator, we have to derive some global superconvergence results and introduce some post-process first.

5.1. Global Superconvergence and Post-Process We introduce some foundmental superconvergence results.

Lemma 5.1. If the patition is regular [7,12] and $k \ge 1, I_h$ be a Lagrange interpolation then we have

$$|a(u - I_h u, v)| \le Ch^{k+1}\|u\|_{k+2}\|v\|_1 \quad \forall u \in H^{k+2} \cap H_0^1, v \in P(\Delta)_0^k, k \ge 1 \tag{87}$$

$$|a(u - I_h u, I_h v)| \le Ch^{k+2}\|u\|_{k+2}\|v\|_2 \quad \forall u \in H^{k+2} \cap H_0^1, v \in H^2 \cap P(\Delta)_0^k, k \ge 2 \tag{88}$$

Based on this lemma, we will have the global superconvergence results.

Theorem 5.1. Let u and u_h be solution of (P) and $\bar{P}_h$, respectively, Assume $I_h u$ be a k-th lagrangue interpolation of u, if the partition is regular then we have the following basical superconvergence estimates:

$$\|I_h u - u_h\|_1 \le c(\lambda,k)\epsilon^{-2}h^{k+1}\|f\|_k \quad k \ge 1 \tag{89}$$

$$\|I_h u - u_h\| \le c(\lambda,k)\epsilon^{-2}h^{k+2}\|f\|_k \quad k \ge 2 \tag{90}$$

$$\|I_h u - u_h\|_{1,\infty} \le c(\lambda,k)\epsilon^{-2}h^{k+1}|lnh|\|f\|_k \quad k \ge 1 \tag{91}$$

$$\|I_h u - u_h\|_{0,\infty} \le c(\lambda,k)\epsilon^{-2}h^{k+2}|lnh|\|f\|_k \quad k \ge 2 \tag{92}$$

Furthermore, if $k \ge 2$ in (91), then the logrithem factor can be dedeleted [18].

Proof. From (65') and (66), we have

$$a(u_h, v) = (\epsilon^2 f, v) + \alpha[(Bu_h, v) - (Bu_h, v)^*] \tag{93}$$

Considering theorem 2.1 we have

$$Ca(I_h u - u_h, I_h u - u_h) \geq \epsilon^2 \|I_h u - u_h\|_1^2$$

But

$$\begin{aligned} Ca(I_h u - u_h, I_h u - u_h) &= Ca(I_h u - u, I_h u - u_h) + Ca(u - u_h, I_h u - u_h) \\ &= Ca(I_h u - u, I_h u - u_h) + C\alpha[(Bu_h, I_h u - u_h)^* \\ &\quad -(Bu_h, I_h u - u_h)] \\ &= I_1 + I_2 \end{aligned}$$

From the k-th truncation error estimate of Gauss quadrature, we can estimate I_2.

$$I_2 \leq c(\lambda, k) h^{k+1} \|u\|_k \|I_h u - u_h\|_1$$

Now we turn to the trem I_1.

$$I_1 = Ca(I_h u - u, I_h u - u_h) \leq c(\lambda, k) h^{k+1} \|u\|_{k+2} \|I_h u - u_h\|_1$$

In the last step of the estimate of I_1, we have used (88). So

$$\|I_h u - u_h\|_1 \leq c(\lambda, k) \epsilon^{-2} h^{k+1} \|u\|_{k+2} \quad k \geq 1$$

We use the standard Aubin-Nitsche trick to obtain (87). Define an auxiliary problem. Find $w \in H^2(\bar{J}) \cap H_0^1(\bar{J})$ such that

$$a(w, v) = (I_h u - u_h, v) \quad \forall v \in H_0^1(\bar{J})$$

with the elliptic regularity theory, we have

$$\|w\|_2 \leq C \|I_h u - u_h\|_0$$

Set $v = I_h u - u_h$ in the auxiliary problem,

$$\begin{aligned} \|I_h u - u_h\|^2 &= a(w, I_h u - u_h) \\ &= a(w - I_h w, I_h u - u_h) + a(I_h w, I_h u - u_h) \\ &= I_1 + I_2 \end{aligned}$$

Using (88),

$$\begin{aligned} I_1 &\leq c(\lambda, k) \|w - I_h w\|_1 \|I_h u - u_h\|_1 \\ &\leq c(\lambda, k) \epsilon^{-2} h^{k+2} \|u\|_{k+2} \|w\|_2 \end{aligned}$$

Like the privious estimate of I_2, we have

$$I_2 \leq c(\lambda, k) h^{k+2} \|u\|_{k+2} \|w\|_2 \tag{94}$$

So

$$\|I_h u - u_h\|_0^2 \leq c(\lambda, k) \epsilon^{-2} h^{k+2} \|u\|_{k+2} \|w\|_2 \leq c(\lambda, k) \epsilon^{-2} h^{k+2} \|u\|_{k+2} \|I_h u - u_h\|_0^2$$

$$\|I_h u - u_h\|_0 \leq c(\lambda, k) \epsilon^{-2} h^{k+2} \|u\|_{k+2} \quad k \geq 2 \tag{95}$$

To obtain (89), (90), we use a different method compare with the way in section 4. We define the discrete Green's function [12][8].
Find $G_z^h \in U_h$ such that

$$a(G_z^h, v) = v(z) \quad \forall v \in U_h \tag{96}$$

where $z \in \Delta$, so we have

$$\begin{aligned} |\partial I_h(u - u_h)(z)/\partial z| &= |a(I_h u - u_h, \partial G_z^h/\partial z)| \\ &\leq |a(I_h u - u, \partial G_z^h/\partial z| + |a(u - u_h, \partial G_z^h/\partial z)| \\ &\leq C\epsilon^{-2} h^{k+1} \|u\|_{k+2,\infty} \|G_z^h\|_{1,1} + I_2 \\ &\leq C\epsilon^{-2} h^{k+1} |lnh| \|u\|_{k+2,\infty} + I_2 \end{aligned}$$

$$\begin{aligned} I_2 &= \alpha|(B(u-u_h),\partial G_z^h/\partial z)^* - (B(u-u_h),\partial G_z^h/\partial z)| \\ &\leq c(\lambda,k)h^{k+1}\|u\|_{k,\infty}\|G_z^h\|_{1,1} \\ &\leq c(\lambda,k)h^{k+1}|lnh|\|u\|_{k,\infty} \end{aligned}$$

So

$$|I_h u - u_h|_{1,\infty} \leq c(\lambda,k)\epsilon^{-2}h^{k+1}|lnh|\|u\|_{k+2,\infty} \tag{97}$$

In light of (95) and lemma 5.1, we can get (92)

$$\begin{aligned} |I_h u - u_h(z)| &= a(I_h u - u_h, G_z^h)| \\ &\leq |a(I_h u - u, G_z^h - I_h G_z^h)| + |a(I_h u - u_h, I_h G_z^h)| \\ &\quad +\alpha|(B(u-u_h),G_z^h)^* - (B(u-u_h),G_z^h)| \\ &\leq c(\lambda,k)\epsilon^{-2}h^{k+1}\|u\|_{k+2,\infty}[\|G_z^h - I_h G_z^h\|_{1,1} + h\|G_z^h\|_{2,1}] \\ &\leq c(\lambda,k)\epsilon^{-2}h^{k+2}|lnh|\|u\|_{k+2,\infty} \quad k\geq 2 \end{aligned} \tag{97'}$$

Now we are in a situation to do some postprocess to u_h by using theorem 5.1 to get higher accuracy. More important is we want to use this processed u_h to define an asymptotic exactness posteriori error estimator. First of all, we introduce the two-level interpolation functions as in [12]. Δ^H is a partition of diameter $H, H = 2h.\Delta^h$ is a patition sub-divided by mid-point patition. $S^h(\bar{J}) = \{v \in C(\bar{J}) : v|_e \in P_k, e \in \Delta^h\}, V^h(\bar{J}) = \{v \in C(\bar{J}) : v|_E \in P_{2k}, e \in \Delta^H\}$. Now we have the Lagrange interpolation I_h^k, I_h^{2k} of $S^h(\bar{J}), V^h(\bar{J})$ respectively. where the upper index and the lower index denote the order of interpolation polynomial and the patition diameter.

Lemma 5.2. [12] there exists constant $c > 0$ such that

$$\|I_{2h}^{2k}u\|_{m,p} \leq c\|u\|_{m,p} \quad 1 \leq p \leq \infty, m = 0,1 \forall u \in S^h(\bar{J}) \tag{98}$$

$$(I_{2h}^{2k})^2 = I_{2h}^{2k} \quad I_{2h}^{2k}I_h^1 = I_{2h}^{2k} \quad I_h^1 I_{2h}^{2k} = I_h^1 \tag{99}$$

$$\|u - I_{2h}^{2k}u\|_{m,p,E} \leq ch^{2k+1-m}\|u\|_{2k+1,p,E} \tag{100}$$

$$\forall u \in W^{2k+1,p}(E), E \in \Delta^H, m = 0,1, 1 \leq p \leq \infty$$

Now we can start our post-process to u_h If we denote

$$\bar{u}_h = I_{2h}u_h \tag{101}$$

where $I_{2h} = I_{2h}^2(I_h = I_h^1)$. For simplicity, we only investigate the linear element (but the results are the same to the high-oder element). From the following theorem we will find $\bar{u}_h$ has one-order higher accuracy than u_h.

Theorem 5.2. If u and u_h are solutions of $(P), (\bar{P}_h)$ respectively, $\bar{u}_h$ is defined as (101), then there exists a constant C such that

$$\|u - \bar{u}_h\|_1 \leq C\epsilon^{-2}h^{-2}\|f\|_1 \tag{102}$$

$$\|u - \bar{u}_h\|_{1,\infty} \leq C\epsilon^{-2}h^2|lnh|\|f\|_{1,\infty} \tag{103}$$

Proof. From theorem 5.1 and lemma 5.2 we have

$$\begin{aligned} \|I_{2h}u - I_{2h}u_h\|_1 &= \|I_{2h}(I_h u - u_h)\|_1 \\ &\leq C\|I_h u - u_h\|_1 \\ &\leq C\epsilon^{-2}h^2\|u\|_3 \end{aligned}$$

by means of triangle inequality

$$\begin{aligned} \|u - \bar{u}_h\|_1 &= \|u - I_{2h}u_h\|_1 \\ &\leq \|u - I_{2h}u\|_1 + \|I_{2h}u - I_{2h}u_h\|_1 \\ &\leq C\epsilon^{-2}h^2\|u\|_3 \\ &\leq C\epsilon^{-2}h^2\|f\|_1 \end{aligned}$$

Using the same technical we can prove (103).

Remark. It is easy to see we can derive the estimation of $\|u-\bar{u}_h\|_0, \|u-\bar{u}_h\|_{0,\infty}$. Furthermore we can get the estimates under the four norms when $k>1$,in this case, we have the estimation

$$\|u-\bar{u}_h\|_1 \le c\epsilon^{-2}h^{k+1}(h^{k-1}\|u\|_{2k+1}+\|u\|_{k+2}) \tag{102'}$$

Now we turn to estimate $|\nabla u-\nabla\bar{u}_h|_1$. This ascribe to the approximating theorem of Scott. R and Dupont. T.F[29] and our construction of the $\bar{u}_h$.

Lemma 5.3. Let u and $\bar{u}_h$ efined as before, then we have

$$|\nabla u-\nabla\bar{u}_h|_1 \le C\epsilon^{-2}h|u|_3 \tag{104}$$

Proof.

$$\begin{aligned}
|\nabla u-\nabla\bar{u}_h|_1 &= |\nabla(u-\phi)-\nabla(\bar{u}_h-\phi)|_1\\
&= |\nabla(u-\phi)-\nabla(I_{2h}u_h-I_{2h}\phi)|_1 \quad \forall\phi\in P_2\\
&= |\nabla(u-\phi)-\nabla(I_{2h}(I_{2h}u_h-\phi))|_1\\
&\le |u-\phi|_2+C|I_{2h}u_h-\phi|_2\\
&\le |u-\phi|_2+Ch^{-1}|I_{2h}u_h-\phi|_1\\
&\le |u-\phi|_2+Ch^{-1}|I_{2h}u_h-u|_1+Ch^{-1}|u-\phi|_1\\
&\le C\epsilon^{-2}h\|u\|_3
\end{aligned}$$

where we haver used the regular assumption of the partition.

5.2. Asymptotic Exactness Posteriori Error Estimate

we have introduced all the prelimiliary results for deriving the posteriori error estimate. In order to obtain the main result (theorem 5.3) of this section, we also ned to establish the following lemmas. For simplicity we only deal with the linear finite element in this section.

Lemma 5.4. Let u, u_h be solutions of $(P), (\bar{P}_h)$, so we have the two estimations

$$\|e\|_0 \le ch\|e\|_E \tag{105}$$

$$\|e\|_E \ge c(u)h^p \tag{106}$$

Remark. the proof of (105) is a standard Aubin-Nitsche trick, as to (106) we refer [20].

Now we shall use a device suggested by Babuska and Rheinboldt [5]. First we define a new bilinear form $\hat{a}(\cdot,\cdot): U\times U\to R$.

$$\hat{a}(y,w)=((H+Ax)y',w')+\beta(y,w) \tag{107}$$

where β is a parameter to be determined later. A new variational problem is prsented.
Find $y\in U$ such that

$$\hat{a}(y,w)=(\epsilon^2 f,w)-a(u_h,w) \quad \forall w\in U \tag{108}$$

Lemma 5.5. Let $\aleph(p)$, be a quadratic functional on $H(\text{div}, \bar{J})$, given by

$$\begin{aligned}
\aleph(p) \;=\; & ((H+Ax)^{-1}(p-(H+Ax)\nabla u_h), p-(H+Ax)\nabla u_h)\\
&+\beta^{-1}(\epsilon^2 f+\nabla\cdot p-Bu_h, \epsilon^2 f+\nabla\cdot p-Bu_h)
\end{aligned}$$

then the following bound holds

$$\aleph(p)\ge\aleph((H+Ax)\nabla(u_h+y))=\hat{a}(y,y) \quad \forall y\in H(div,\bar{J}) \tag{109}$$

Proof. The strong form of the variational problem (107) is given by

$$\nabla\cdot(H+Ax)\nabla(u_h+y)+\epsilon^2 f-Bu_h=\beta y$$

where we have used the following equation:$(Axu_h', y')=-((Axu_h)', y)=-(Au_h', y)-(Axu_h'', y)=-((Au_h)', y)=(Au_h, y')$, set

$$p=(H+Ax)\nabla(y+u_h) \quad \nabla\cdot p=Bu_h-\epsilon^2 f+\beta y\in L^2(\bar{J})$$

so $\aleph(p) = \hat{a}(y,y)$ Now let $\epsilon \in (0,1)$, and $q, r \in H(div, \bar{J})$, it is easy to show that

$$\aleph((1-\epsilon)r + \epsilon q) \le (1-\epsilon)\aleph(r) + \epsilon\aleph(q)$$

so $\aleph(p)$ is a convex functional. Moreover if $p = (H + Ax)\nabla(y + u_h)$

$$\frac{\partial\aleph((1-\epsilon)p + \epsilon q)}{\partial\epsilon}|_{\epsilon=0} = 0$$

Thus $\aleph$ is stationary at p and results hold.

Lemma 5.6. Suppose $\beta = Mh^{-\alpha}$, where $\alpha \in (0,2)$ and $M \ge 0$ are constant. Then the following bounds holds for any $p \in H(div, \bar{J})$

$$\|e\|_E^2 \le (1 + O(h^\gamma))\aleph(p) \quad as\ h \to 0 \tag{110}$$

where $\gamma = 1 - \frac{\alpha}{2}$

Proof.

$$a(e,w) = \aleph(y,w) \tag{111}$$

set $w = e$ in (110) and $\beta = Mh^{-\alpha}$,

$$\begin{aligned} a(e,e) &= \aleph(y,e) \\ &= a(y,e) + (\beta - 1)(y,e) \\ &= \hat{a}(y,y) + (\beta - 1)(y,e) \\ &\le \hat{a}(y,y) + |\beta - 1|\|y\|_0\|e\|_0 \\ &\le \hat{a}(y,y) + ch|\beta - 1||\beta|^{-\frac{1}{2}}\|y\|_{\hat{E}}\|e\|_E \end{aligned}$$

$$\le \hat{a}(y,y) + ch|\beta - 1||\beta|^{\frac{-1}{2}}\|y\|_{\hat{E}}\|e\|_E \tag{111'}$$

when h is small enough

$$|\beta - 1|(y,e) \le 2ch|\beta|^{\frac{1}{2}}\|y\|_{\hat{E}}\|e\|_E$$

Replacing β with $Mh^{-\alpha}$ and using mean inequality

$$|(\beta - 1)(y,e)| \le cM^{\frac{1}{2}}h^{1-\frac{\alpha}{2}}(\|e\|_E^2 + \|y\|_{\hat{E}}^2 \tag{112}$$

Substitute (112) into (111') we have

$$\|e\|_E^2 \le \|y\|_{\hat{E}}^2(1 + O(h^\gamma))$$

where $\gamma = min(1 - \frac{\alpha}{2}, k + \frac{\alpha}{2} - p)$ using lemma 5.5 we have

$$\|e\|_E \le \aleph(p)(1 + O(h^\gamma))$$

Now we define an asymptotic exactness posteriori error estimator.

$$\epsilon^2 = ((H + Ax)\nabla(\bar{u}_h - u_h), \nabla(\bar{u}_h - u_h)) \tag{113}$$

and $\bar{u}_h$ is defined in (101). In the following theorem we will prove ϵ is an asymptotical exactness posteriori error estimator.

Theorem 5.3. Let ϵ be a posteriori error estimator defined as above, then if $k + 1 > p$, we have

$$\|e\|_E = \epsilon(1 + O(h^\gamma)) \quad as \quad h \to 0 \tag{114}$$

where γ is the same as in lemma 5.6, and c in $O(h^\gamma)$ is a constant independent of h.

proof. Let $\aleph(p)$ be the definition as above, and take $p = (H + Ax)\nabla\bar{u}_h$ in lemma 5.2. This is valid because $(H + Ax)\nabla\bar{u}_h \in C(\bar{J})$, so we have

$$\aleph(p) = \epsilon^2 + \frac{h^\alpha \bigwedge^2}{M} \tag{115}$$

where

$$\epsilon^2 = ((H + Ax)\nabla(\bar{u}_h - u_h), \nabla(\bar{u}_h - u_h)) \tag{116}$$

and

$$\bigwedge^2 = (\epsilon^2 f + \nabla \cdot (H+Ax)\nabla \bar{u}_h - Bu_h, \epsilon^2 f + \nabla \cdot (H+Ax)\nabla \bar{u}_h - Bu_h) \tag{117}$$

First, we consider the term $\bigwedge$

$$\begin{aligned} \bigwedge &= \|\epsilon^2 f + \nabla \cdot (H+Ax)\nabla \bar{u}_h - Bu_h\|_0 \\ &= \| - \nabla \cdot (H+Ax)\nabla(u - \bar{u}_h) - Be\|_0 \\ &\leq \|A\|_{0,\infty}\|\nabla(u-\bar{u}_h)\|_1 + \|B\|_F\|e\|_0 \\ &\leq (1+\lambda^2)[Ch^k|u|_{k+2} + h\|e\|_E] \end{aligned}$$

where we use lemma 5.7 and lemma 4.2. In light of lemma 5.4, we have

$$\begin{aligned} \frac{h^\alpha \bigwedge^2}{M} &\leq c(u,\lambda)h^{2k+\alpha-2p}\|e\|_E^2 + c(\lambda)h^{2+\alpha}\|e\|_E^2 \\ &\leq c(u,\lambda)h^\alpha\|e\|_E^2 \end{aligned}$$

Applying lemma 5.6, we get

$$\begin{aligned} \|e\|_E^2 &\leq (1+O(h^\gamma))(\|\epsilon\|^2 + Ch^\alpha\|e\|_E^2) \\ &\leq (1+O(h^\gamma))(\|\epsilon\|^2 + Ch^\alpha\|e\|_E^2) \\ &\leq (1+O(h^\gamma))\|\epsilon\|^2 \end{aligned} \tag{118}$$

Considering now the term ϵ

$$\begin{aligned} \epsilon &= ((1+Ax)(\nabla\bar{u}_h - \nabla u_h), \nabla\bar{u}_h - \nabla u_h)^{\frac{1}{2}} \\ &\leq ((1+Ax)\nabla e, \nabla e)^{\frac{1}{2}} + ((1+Ax)(\nabla\bar{u}_h - \nabla u_h), \nabla u_h - \nabla u)^{\frac{1}{2}} \\ &\leq \|e\|_E + C\|\nabla\bar{u}_h - \nabla u\| \end{aligned}$$

Applying theorem 5.1, we obtain

$$\begin{aligned} \epsilon &\leq \|e\|_E + C\epsilon^{-2}h^{k+1}(\|u\|_{2k+1} + \|u\|_{k+2}) \\ &\leq \|e\|_E + C\epsilon^{-2}h^{k+1-p}C(u) \\ &\leq \|e\|_E + C(u,\epsilon)h^{k+1-p}\|e\|_E \\ &\leq (1+O(h^{k+1-p}))\|e\|_E \\ &\leq (1+O(h^\gamma)\|e\|_E \end{aligned} \tag{119}$$

together with (118) and (119) we have $\|e\|_E = (1+O(h^\gamma))\epsilon$ as $h \to 0$, where $\gamma = min(1 - \frac{1}{2}\alpha, k + \frac{\alpha}{2} - p)$.

Remark. We can easily see that the C in theorem 5.3 in ϵ-depent. but if we want to get the ϵ-indepent asymptotic exactness posteriori error estimate, we have to lose some sharpness as we have done in section 4. To see this clearly, we find the ϵ-dependent constant results from theorem 5.1 and 5.2. But we can obtain (102) in a more simple way by lose one-order accuracy. By means of theorem 5.1, we have

$$\begin{aligned} \|I_h u - u_h\|_1 &\leq \|u - u_h\|_1 + \|u - I_h u\|_1 \\ &\leq Ch\|u\|_2 \end{aligned}$$

$$\begin{aligned} \|u - \bar{u}_h\|_1 &= \|u - I_{2h}u_h\|_1 \\ &\leq \|u - I_{2h}u\|_1 + \|I_{2h}u - I_{2h}u_h\|_1 \\ &\leq C(h^2\|u\|_3 + h\|u\|_2) \\ &\leq Ch\|u\|_3 \end{aligned}$$

for $k > 1$, we have $\|u - u_h\|_1 \leq Ch^k\|u\|_{2k+1}$, substitute this instead of (102') into the proof of theorem 5.3, we can also obtain $\|e\|_E = \epsilon(1+O(h^\gamma))$, where $\gamma = min(1 - \frac{1}{2}\alpha, k-p)$, but this is not essential in our discussion.

6. Conclusion

In this paper, Homotopy Finite Element Method(HFEM) is presented for the arch beam model lacking of K-ellipticity which is a kind of locking phenomenon. It can be considered as a generalized mixed method, and it enhance the K-ellipticity by adding some terms having mechanical meaning. From this point of view, we can easily see the HFEM can include the remarkable Least-Square-Petrov-Galerkin methods(LSPG)[23,24]. As a result of simulating the arch more accurately, this method can successfully overcome locking phenomenon preserving the superconvergence results. This is the same as our previous paper [16], but furthermore in this paper we find out our method have the global superconvergence phenomenon which have never been discovered before, based upon it we propose some postprocessing in section 5, the most important is we have found an asymptotical exactness posteriori estimator and it is a trivial thing to extend it to more models especially for arch beam model (for example the model in [16]).

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