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JACOBI SPECTRAL METHODS FOR MULTIPLE-DIMENSIONAL SINGULAR DIFFERENTIAL EQUATIONS*[1)]

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Abstract

Jacobi polynomial approximations in multiple dimensions are investigated. They are applied to numerical solutions of singular differential equations. The convergence analysis and numerical results show their advantages.

Key words: Jacobi approximations, Multiple dimensions.

1. Introduction

The spectral method has high accuracy. However, it might be destroyed by the singularity of genuine solutions. Guo [1], and Guo and Wang [2] developed Jacobi approximations to singular differential equations. But so far, there is no work in multiple dimensions. This paper is devoted to Jacobi spectral method for multiple-dimensional singular differential equations. We first recall some basic results on Jacobi approximation, and then give the main results of this paper. They are used for numerical solutions of singular differential equations. The convergence analysis and numerical results show the efficiency of this approach.

2. Some Basic Results on Jacobi Approximations

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, $x \in \mathbb{R}^d$ and $\chi(x)$ be certain weight function. We define the weighted space $L^p_\chi(\Omega)$ and its norm $||v||_{L^p_\chi}$ in the usual way. Denote the inner product and the norm of the space $L^2_\chi(\Omega)$ by $(u,v)_\chi$ and $||v||_\chi$. We define the weighted Sobolev space $H^r_\chi(\Omega)$ as usual with the inner product $(u,v)_{r,\chi}$, the semi-norm $|v|_{r,\chi}$ and the norm $||v||_{r,\chi}$.

We recall some basic results on the Jacobi approximations. Let $d=1, \Omega \equiv \Lambda = (-1,1)$ and $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$. For $\alpha, \beta > -1$,

$$(J_l^{(\alpha,\beta)}, J_m^{(\alpha,\beta)})_{\chi^{(\alpha,\beta)}} = \gamma_l^{(\alpha,\beta)}\delta_{l,m}, \qquad \gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}. \tag{2.1}$$

Let $\mathbb{N}$ be the set of all non-negative integers. For any $N \in \mathbb{N}$, $\mathcal{P}_N$ stands for the set of all algebraic polynomials of degree at most N. Further let ${}_0\mathcal{P}_N = \{v | v \in \mathcal{P}_N, v(-1) = 0\}$ and $\mathcal{P}_N^0 = \{v | v \in \mathcal{P}_N, v(-1) = v(1) = 0\}$. Denote by c a generic positive constant independent of any function and N.

Lemma 2.1. (Lemma 3.7 of [2] and Lemma 2.4 of [1]). *If $-1 < \alpha, \beta < 1$, then for any $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$,*

$$||v||_{\chi^{(\alpha-2,\beta-2)}} \le c|v|_{1,\chi^{(\alpha,\beta)}}. \tag{2.2}$$

Moreover, if $\alpha > -1, \beta = 0$ or $\alpha = 0, \beta > -1$, then

$$||v||_{\chi^{(\alpha,\beta)}} \le c|v|_{1,\chi^{(\alpha,\beta)}}. \tag{2.3}$$

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Lemma 2.2. (Theorem 2.2 of [1]). *For any $\phi \in \mathcal{P}_N$ and $r \geq 0$,*

$$||\phi||_{r,\chi^{(\alpha,\beta)}} \leq cN^{2r}||\phi||_{\chi^{(\alpha,\beta)}}. \tag{2.4}$$

If, in addition, $\alpha, \beta > r-1$, then

$$||\phi||_{r,\chi^{(\alpha,\beta)}} \leq cN^{r}||\phi||_{\chi^{(\alpha-r,\beta-r)}}. \tag{2.5}$$

We now turn to some orthogonal projections. For any $r \in \mathbb{N}$, let (see [1])

$$H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) = \{v \mid v \text{ is measurable and } ||v||_{r,\chi^{(\alpha,\beta)},A} < \infty\}$$

where

$$||v||_{r,\chi^{(\alpha,\beta)},A} = \Big(\sum_{k=0}^{[\frac{r-1}{2}]} ||(1-x^2)^{\frac{r}{2}-k}\partial_x^{r-k}v||^2_{\chi^{(\alpha,\beta)}} + ||v||^2_{[\frac{r}{2}],\chi^{(\alpha,\beta)}}\Big)^{\frac{1}{2}}.$$

For any real $r > 0$, the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ is defined by space interpolation. Next, for any $\mu \in \mathbb{N}$,

$$H^r_{\chi^{(\alpha,\beta)},*,\mu}(\Lambda) = \{v \mid \partial_x^\mu v \in H^{r-\mu}_{\chi^{(\alpha,\beta)},A}(\Lambda)\}, \quad H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda) = \{v \mid v \in H^r_{\chi^{(\alpha,\beta)},*,k}(\Lambda), 0 \leq k \leq \mu\}$$

with the following norms

$$||v||_{r,\chi^{(\alpha,\beta)},*,\mu} = ||\partial_x^\mu v||_{r-\mu,\chi^{(\alpha,\beta)},A}, \qquad ||v||_{r,\chi^{(\alpha,\beta)},**,\mu} = \Big(\sum_{k=0}^{\mu} ||v||^2_{r,\chi^{(\alpha,\beta)},*,k}\Big)^{\frac{1}{2}}.$$

For any real $\mu > 0$, the spaces $H^r_{\chi^{(\alpha,\beta)},*,\mu}(\Lambda)$ and $H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda)$ are defined by space interpolation. In particular, $||v||_{r,\chi^{(\alpha,\beta)},*} = ||v||_{r,\chi^{(\alpha,\beta)},*,1}$.

Let $P_{N,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N$ be the $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$−orthogonal projection.

Lemma 2.3. (Theorem 2.3 of [1]). *For any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and $r \geq 0$,*

$$||P_{N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}} \leq cN^{-r}||v||_{r,\chi^{(\alpha,\beta)},A}. \tag{2.6}$$

Lemma 2.4. (Theorem 2.4 of [1]). *If $\alpha + r > 1$ or $\beta + r > 1$, then for any $v \in H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda), r \geq 1$ and $0 \leq \mu \leq r$,*

$$||P_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)}} \leq cN^{2\mu-r}||v||_{r,\chi^{(\alpha,\beta)},**,\mu}. \tag{2.7}$$

In particular, for any $\alpha = \beta > -1$,

$$||P_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)}} \leq cN^{\sigma(\mu,r)}||v||_{r,\chi^{(\alpha,\beta)},**,\mu} \tag{2.8}$$

where $\sigma(\mu, r) = 2\mu - r - \frac{1}{2}$ for $\mu > 1$, and $\sigma(\mu, r) = \frac{3}{2}\mu - r$ for $0 \leq \mu \leq 1$.

Now let $\alpha, \beta, \gamma, \delta > -1$. We define $H^0_{\alpha,\beta,\gamma,\delta}(\Lambda) = L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$, and

$$H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v \text{ is measurable and } ||v||_{1,\alpha,\beta,\gamma,\delta} < \infty\}$$

where

$$||v||_{1,\alpha,\beta,\gamma,\delta} = (|v|^2_{1,\chi^{(\alpha,\beta)}} + ||v||^2_{\chi^{(\gamma,\delta)}})^{\frac{1}{2}}.$$

For $0 < \mu < 1$, the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$ is defined by space interpolation.

Let

$$a_{\alpha,\beta,\gamma,\delta}(u, v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (u, v)_{\chi^{(\gamma,\delta)}}.$$

The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta} : H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}_N$ is a mapping such that

$$a_{\alpha,\beta,\gamma,\delta}(P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi) = 0, \qquad \forall \phi \in \mathcal{P}_N.$$

Lemma 2.5. (Theorem 2.5 of [1]). *If* $\alpha \le \gamma+2$ *and* $\beta \le \delta+2$, *then for any* $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *and* $r \ge 1$,

$$||P^1_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} \le cN^{1-r}||v||_{r,\chi^{(\alpha,\beta)},*}. \tag{2.9}$$

If, in addition,

$$\alpha \le \gamma+1, \quad \beta \le \delta+1, \tag{2.10}$$

then for all $0 \le \mu \le 1$,

$$||P^1_{N,\alpha,\beta,\gamma,\delta}v - v||_{\mu,\alpha,\beta,\gamma,\delta} \le cN^{\mu-r}||v||_{r,\chi^{(\alpha,\beta)},*}. \tag{2.11}$$

Next, let

$${}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } \ v(-1)=0\}.$$

The orthogonal projection ${}_0P^1_{N,\alpha,\beta,\gamma,\delta} : {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to {}_0\mathcal{P}_N$ is a mapping such that

$$a_{\alpha,\beta,\gamma,\delta}({}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi) = 0, \quad \forall \phi \in {}_0\mathcal{P}_N.$$

Lemma 2.6. (Theorem 2.6 of [1] and Lemma 3.11 of [2]). *If*

$$\alpha \le \gamma+1, \quad \beta \le \delta+2, \quad 0<\alpha<1, \quad \beta<1, \tag{2.12}$$

or

$$\alpha \le \gamma+2, \quad \beta \le 0, \quad \delta \ge 0, \tag{2.13}$$

then for any $v \in {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *and* $r \ge 1$,

$$||\,{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} \le cN^{1-r}||v||_{r,\chi^{(\alpha,\beta)},*}. \tag{2.14}$$

If, in addition, (2.10) *holds, then for all* $0 \le \mu \le 1$,

$$||\,{}_0P^1_{N,\alpha,\beta,\gamma,\delta}v - v||_{\mu,\alpha,\beta,\gamma,\delta} \le cN^{\mu-r}||v||_{r,\chi^{(\alpha,\beta)},*}. \tag{2.15}$$

Finally, let

$$H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) = \{v | v \in H_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } \ v(-1)=v(1)=0\}.$$

The orthogonal projection $P^{1,0}_{N,\alpha,\beta,\gamma,\delta} : H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}^0_N$ is a mapping such that

$$a_{\alpha,\beta,\gamma,\delta}(P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v, \phi) = 0, \qquad \phi \in \mathcal{P}^0_N.$$

Lemma 2.7. (Theorem 2.7 of [1]). *If* $\gamma \le \alpha < \gamma+1, \delta \le \beta < \delta+1$ *and* $\gamma, \delta < 1$, *then for any* $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*,2}(\Lambda)$ *with* $r \ge 2$,

$$||P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} \le cN^{1-r}||v||_{r,\chi^{(\alpha,\beta)},*,2}.$$

If, in addition, $\alpha=\gamma>0$ *and* $\beta=\delta>0$, *then for all* $0 \le \mu \le 1$,

$$||P^{1,0}_{N,\alpha,\beta,\gamma,\delta}v - v||_{\mu,\alpha,\beta,\gamma,\delta} \le cN^{\mu-r}||v||_{r,\chi^{(\alpha,\beta)},*,2}.$$

The orthogonal projection $\widetilde{P}^{1,0}_{N,\alpha,\beta} : H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}^0_N$ is a mapping such that

$$(\partial_x(\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v), \partial_x\phi)_{\chi^{(\alpha,\beta)}} = 0, \qquad \forall \phi \in \mathcal{P}^0_N.$$

Lemma 2.8. (Lemma 3.16 of [2]). *If* $-1<\alpha,\beta<1$, *then for any* $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda) \cap H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ *and* $r \ge 1$,

$$||\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v||_{1,\chi^{(\alpha,\beta)}} \le cN^{1-r}||v||_{r,\chi^{(\alpha,\beta)},*}. \tag{2.16}$$

If, in addition, $\alpha,\beta \le 0$ *or* $0<\alpha,\beta<1$, *then for all* $0 \le \mu \le 1$,

$$||\widetilde{P}^{1,0}_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)}} \le cN^{\mu-r}||v||_{r,\chi^{(\alpha,\beta)},*}. \tag{2.17}$$

3. Jacobi Approximation in Two Dimensions

Let $d = 2$, $\Lambda_i = \{x_i \mid -1 < x_i < 1\}$, $\Omega = \Lambda_1 \times \Lambda_2$ and $x = (x_1, x_2)$. Further, let $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ and $l = (l_1, l_2)$, $\alpha_i, \beta_i > -1, l_i \in \mathbb{N}, i = 1, 2$. The two-dimensional Jacobi polynomial of degree l is

$$J_l^{(\alpha,\beta)}(x) = J_{l_1}^{(\alpha_1,\beta_1)}(x_1)J_{l_2}^{(\alpha_2,\beta_2)}(x_2).$$

Let $\chi^{(\alpha,\beta)}(x) = \chi^{(\alpha_1,\beta_1)}(x_1)\chi^{(\alpha_2,\beta_2)}(x_2)$. Then

$$(J_l^{(\alpha,\beta)}, J_m^{(\alpha,\beta)})_{\chi^{(\alpha,\beta)}} = \gamma_{l_1}^{(\alpha_1,\beta_1)}\gamma_{l_2}^{(\alpha_2,\beta_2)}\delta_{l_1,m_1}\delta_{l_2,m_2}. \tag{3.1}$$

Let $N = (N_1, N_2) \in \mathbb{N}^2$ and $\mathcal{P}_N$ be the set of all algebraic polynomials of degree at most N_i for x_i.

We first establish some inverse inequalities. Let $\mathcal{L}$ be a linear operator defined on $\mathcal{P}_N$. $\mathcal{L}$ is said to be of (p, q) type, if there exists a positive constant d depending only on p, q, N_1 and N_2 such that $\|\mathcal{L}\phi\|_{L^q_\chi} \le d\|\phi\|_{L^p_\chi}$ for any $\phi \in \mathcal{P}_N$. According to Riesz-Thorin Theorem (see Bergh and Löfström [3]), we know that if $\mathcal{L}$ is of both (p_1, q_1) type and (p_2, q_2) type for $1 \le p_1, p_2 \le \infty, 1 \le q_1, q_2 \le \infty$, then for

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \qquad 0 \le \theta \le 1, \tag{3.2}$$

the operator $\mathcal{L}$ is also of (p, q) type. If in addition $\|\mathcal{L}\phi\|_{L^{q_j}_\chi} \le d_j\|\phi\|_{L^{p_j}_\chi}, j = 1, 2$, then

$$\|\mathcal{L}\phi\|_{L^q_\chi} \le c(p_1, p_2)d_1^{1-\theta}d_2^\theta\|\phi\|_{L^p_\chi} \tag{3.3}$$

where $c(p_1, p_2)$ is a positive constant depending only on p_1 and p_2.

Lemma 3.1. *Let ϕ_l be certain algebraic polynomials of degree at most l_i with respect to variable $x_i, i = 1, 2$, and the set of ϕ_l be an orthogonal system in the space $L^2_\chi(\Omega)$. If for certain positive constant c_0 and real numbers η_1, η_2,*

$$||\phi_0||_\infty \le c_0, \quad ||\phi_l||_\infty \le c_0 l_1^{\eta_1} l_2^{\eta_2}||\phi_l||_\chi, \qquad l_1, l_2 \ge 1, \tag{3.4}$$

then for any $\phi \in \mathcal{P}_N$ and all $1 \le p \le q \le \infty$,

$$||\phi||_{L^q_\chi} \le c\sigma^{\frac{1}{p}-\frac{1}{q}}(N)||\phi||_{L^p_\chi}, \tag{3.5}$$

where

$$\sigma(N) = \begin{cases} N_1^{2\eta_1+1}N_2^{2\eta_2+1}, & \eta_1, \eta_2 > -\frac{1}{2}, \\ \ln N_1 \ln N_2, & \eta_1 = \eta_2 = -\frac{1}{2}, \\ N_i^{2\eta_i+1}\ln N_j, & \eta_i > -\frac{1}{2}, \eta_j = -\frac{1}{2}, i, j = 1, 2, i \ne j, \\ N_i^{2\eta_i+1}, & \eta_i > -\frac{1}{2}, \eta_j < -\frac{1}{2}, i, j = 1, 2, i \ne j, \\ \ln N_i, & \eta_i = -\frac{1}{2}, \eta_j < -\frac{1}{2}, i, j = 1, 2, i \ne j, \\ 1, & \eta_1, \eta_2 < -\frac{1}{2}. \end{cases}$$

Proof. Let us consider, for example, the case with $\eta_1, \eta_2 > -\frac{1}{2}$. For any $\phi \in \mathcal{P}_N$,

$$\phi = \sum_{l_1=0}^{N_1}\sum_{l_2=0}^{N_2}\widehat{\phi}_l\phi_l = \sum_{l_1=0}^{N_1}\sum_{l_2=0}^{N_2}\frac{\phi_l}{||\phi_l||^2_\chi}\int_\Omega \phi(x)\phi_l(x)\chi(x)dx.$$

Thus by (3.4),

$$||\phi||_\infty \le ||\phi||_{L^1_\chi}\sum_{l_1=0}^{N_1}\sum_{l_2=0}^{N_2}\frac{||\phi_l||^2_\infty}{||\phi_l||^2_\chi} \le c\sigma(N)||\phi||_{L^1_\chi}. \tag{3.6}$$

Hence the identity operator $\mathcal{L}$ is of $(1, \infty)$ type with $d_1 = c\sigma(N)$. Clearly, for any $r = \frac{pq-q+p}{p}, 1 \le p \le q \le \infty$, $\mathcal{L}$ is of (r, r) type with $d_2 = 1$. For $1 \le p < \infty$, we set $p_1 = 1, q_1 = \infty, p_2 = q_2 = r$ and $\theta = \frac{r}{q}$ in (3.2), then (3.3) implies the desired result. Moreover, $\mathcal{L}$ is also of $(1, 1)$ type and (∞, ∞) type. The proof is complete.

Theorem 3.1. *For any* $\phi \in \mathcal{P}_N$ *and* $1 \le p \le q \le \infty$,

$$||\phi||_{L^q_{\chi^{(\alpha,\beta)}}} \le c(N_1^{\rho(\alpha_1,\beta_1)} N_2^{\rho(\alpha_2,\beta_2)})^{\frac{1}{p}-\frac{1}{q}} ||\phi||_{L^p_{\chi^{(\alpha,\beta)}}}, \tag{3.7}$$

where $\rho(\alpha_i,\beta_i) = \max(2\alpha_i+1, 2\beta_i+1, 0)+1, i=1,2$.

Proof. We deduce from (2.1), (3.1) and the Stirling formula that $||J_l^{(\alpha,\beta)}||_{\chi^{(\alpha,\beta)}} = O(l_1^{-\frac{1}{2}} l_2^{-\frac{1}{2}})$. By Abramowitz and Stegun [4],

$$||J_l^{(\alpha,\beta)}||_\infty = ||J_{l_1}^{(\alpha_1,\beta_1)}||_\infty ||J_{l_2}^{(\alpha_2,\beta_2)}||_\infty \le c l_1^{\max(\alpha_1,\beta_1,-\frac{1}{2})} l_2^{\max(\alpha_2,\beta_2,-\frac{1}{2})}.$$

Therefore

$$||J_l^{(\alpha,\beta)}||_\infty \le c l_1^{\max(\alpha_1+\frac{1}{2},\beta_1+\frac{1}{2},0)} l_2^{\max(\alpha_2+\frac{1}{2},\beta_2+\frac{1}{2},0)} ||J_l^{(\alpha,\beta)}||_{\chi^{(\alpha,\beta)}}.$$

Taking $\phi_l(x) = J_l^{(\alpha,\beta)}(x), \chi(x) = \chi^{(\alpha,\beta)}(x), c_0 = c, \eta_1 = \max(\alpha_1+\frac{1}{2},\beta_1+\frac{1}{2},0)$ and $\eta_2 = \max(\alpha_2+\frac{1}{2},\beta_2+\frac{1}{2},0)$ in Lemma 3.1, we obtain the desired result.

Theorem 3.2. *For any* $\phi \in \mathcal{P}_N$,

$$||\phi||_{1,\chi^{(\alpha,\beta)}} \le c(N_1^2+N_2^2)||\phi||_{\chi^{(\alpha,\beta)}}. \tag{3.8}$$

If, in addition, $\alpha_i, \beta_i > 0, i=1,2$, *then*

$$||\phi||_{1,\chi^{(\alpha,\beta)}} \le c(N_1+N_2)||\phi||_{\chi^{(\alpha-1,\beta-1)}}. \tag{3.9}$$

Proof. (3.8) comes from (2.4) immediately. If $\alpha_i,\beta_i > 0, i=1,2$, then by (2.5),

$$\begin{aligned}|\phi|_{1,\chi^{(\alpha,\beta)}} &\le c(N_1||\phi||_{L^2_{\chi^{(\alpha_1,\beta_1)}}(L^2_{\chi^{(\alpha_1-1,\beta_1-1)}})} + N_2||\phi||_{L^2_{\chi^{(\alpha_1-1,\beta_1-1)}}(L^2_{\chi^{(\alpha_1,\beta_1)}})})\\ &\le c(N_1+N_2)||\phi||_{\chi^{(\alpha-1,\beta-1)}}.\end{aligned}$$

For mixed Jacobi approximation, we need some non-isotropic spaces. Now, for $r,s \ge 0$, set

$$H^{r,s}_{\chi^{(\alpha,\beta)}}(\Omega) = L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2; H^r_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)) \cap H^s_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2; L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)) \tag{3.10}$$

with the norm

$$||v||_{H^{r,s}_{\chi^{(\alpha,\beta)}}} = (||v||^2_{L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2;H^r_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1))} + ||v||^2_{H^s_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2;L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1))})^{\frac{1}{2}}. \tag{3.11}$$

For simplicity, we drop the scripts Λ_1 and Λ_2. Note that $H^r_{\chi^{(\alpha,\beta)}}(\Omega) \subseteq H^{r,r}_{\chi^{(\alpha,\beta)}}(\Omega)$ and $H^1_{\chi^{(\alpha,\beta)}}(\Omega) \equiv H^{1,1}_{\chi^{(\alpha,\beta)}}(\Omega)$.

Remark 3.1. We can verify that for any $\phi \in \mathcal{P}_N$ and $r,s \ge 0$,

$$||\phi||_{H^{r,s}_{\chi^{(\alpha,\beta)}}} \le c(N_1^{2r}+N_2^{2s})||\phi||_{\chi^{(\alpha,\beta)}}.$$

Next, like (3.10) and (3.11), we define the non-isotropic spaces.

$$\begin{aligned}H^{r,s}_{\chi^{(\alpha,\beta)},A}(\Omega) &= L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2; H^r_{\chi^{(\alpha_1,\beta_1)},A}(\Lambda_1)) \cap H^s_{\chi^{(\alpha_2,\beta_2)},A}(\Lambda_2; L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)),\\ H^{r,s}_{\chi^{(\alpha,\beta)},*,\mu}(\Omega) &= L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2; H^r_{\chi^{(\alpha_1,\beta_1)},*,\mu}(\Lambda_1)) \cap H^s_{\chi^{(\alpha_2,\beta_2)},*,\mu}(\Lambda_2; L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)),\\ H^{r,s}_{\chi^{(\alpha,\beta)},**,\mu}(\Omega) &= L^2_{\chi^{(\alpha_2,\beta_2)}}(\Lambda_2; H^r_{\chi^{(\alpha_1,\beta_1)},**,\mu}(\Lambda_1)) \cap H^s_{\chi^{(\alpha_2,\beta_2)},**,\mu}(\Lambda_2; L^2_{\chi^{(\alpha_1,\beta_1)}}(\Lambda_1)),\end{aligned}$$

with the corresponding norms $||v||_{r,s;\chi^{(\alpha,\beta)},A}$, $||v||_{r,s;\chi^{(\alpha,\beta)},*,\mu}$ and $||v||_{r,s;\chi^{(\alpha,\beta)},**,\mu}$. In particular, $||v||_{r,s;\chi^{(\alpha,\beta)},*} = ||v||_{r,s;\chi^{(\alpha,\beta)},*,1}$ and $||v||_{r,s;\chi^{(\alpha,\beta)},**} = ||v||_{r,s;\chi^{(\alpha,\beta)},**,1}$.

Finally, for any $r,s \ge 1$, set

$$M^{r,s}_{\chi^{(\alpha,\beta)}}(\Omega) = H^{r,s}_{\chi^{(\alpha,\beta)},**}(\Omega) \cap H^1_{\chi^{(\alpha_2,\beta_2)}}(H^{r-1}_{\chi^{(\alpha_1,\beta_1)},A}) \cap H^{s-1}_{\chi^{(\alpha_2,\beta_2)},A}(H^1_{\chi^{(\alpha_1,\beta_1)}})$$

with the norm $||v||_{M^{r,s}_{\chi^{(\alpha,\beta)}}}$.

The $L^2_{\chi^{(\alpha,\beta)}}(\Omega)$–orthogonal projection $P_{N,\alpha,\beta}: L^2_{\chi^{(\alpha,\beta)}}(\Omega) \to \mathcal{P}_N$ is a mapping such that

$$(P_{N,\alpha,\beta}v - v, \phi)_{\chi^{(\alpha,\beta)}} = 0, \qquad \forall \phi \in \mathcal{P}_N.$$

Theorem 3.3. *For any $v \in H^{r,s}_{\chi^{(\alpha,\beta)},A}(\Omega)$ and $r,s \geq 0$,*

$$||P_{N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}} \leq c(N_1^{-r} + N_2^{-s})||v||_{r,s;\chi^{(\alpha,\beta)},A}. \tag{3.12}$$

Proof. Let P_{N_i,α_i,β_i} be the $L^2_{\chi^{(\alpha_i,\beta_i)}}(\Lambda_i)-$orthogonal projections, i=1,2. By Lemma 2.3,

$$\begin{aligned}||P_{N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}} &\leq ||P_{N_1,\alpha_1,\beta_1}v - v||_{\chi^{(\alpha,\beta)}} + ||P_{N_1,\alpha_1,\beta_1}(P_{N_2,\alpha_2,\beta_2}v - v)||_{\chi^{(\alpha,\beta)}}\\ &\leq c(N_1^{-r}||v||_{L^2_{\chi^{(\alpha_2,\beta_2)}}(H^r_{\chi^{(\alpha_1,\beta_1)},A})} + N_2^{-s}||v||_{H^s_{\chi^{(\alpha_2,\beta_2)},A}(L^2_{\chi^{(\alpha_1,\beta_1)}})})\\ &\leq c(N_1^{-r} + N_2^{-s})||v||_{r,s;\chi^{(\alpha,\beta)},A}.\end{aligned}$$

Theorem 3.4. *If for real $r,s \geq 1$, there hold the conditions*

$$\text{(i)}\quad \alpha_1 + r > 1 \quad or \quad \beta_1 + r > 1, \qquad \text{(ii)} \quad \alpha_2 + s > 1 \quad or \quad \beta_2 + s > 1,$$

then for any $v \in M^{r,s}_{\chi^{(\alpha,\beta)}}(\Omega)$ and all $0 \leq \mu \leq 1$,

$$||P_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)}} \leq c(N_1^{-r} + N_2^{-s})^{1-\mu}(N_1^{2-r} + N_1N_2^{1-s} + N_1^{1-r}N_2 + N_2^{2-s})^{\mu}||v||_{M^{r,s}_{\chi^{(\alpha,\beta)}}}. \tag{3.13}$$

If, in addition, $\alpha_i = \beta_i, i = 1,2$, then

$$||P_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)}} \leq c(N_1^{-r} + N_2^{-s})^{1-\mu}(N_1^{\frac{3}{2}-r} + N_1^{\frac{1}{2}}N_2^{1-s} + N_1^{1-r}N_2^{\frac{1}{2}} + N_2^{\frac{3}{2}-s})^{\mu}||v||_{M^{r,s}_{\chi^{(\alpha,\beta)}}}. \tag{3.14}$$

Proof. Theorem 3.3 gives the desired result (3.13) for $\mu = 0$. Next, for $\mu = 1$,

$$\begin{aligned}|P_{N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} \leq c(&||\partial_{x_1}(P_{N_1,\alpha_1,\beta_1} \circ P_{N_2,\alpha_2,\beta_2}v - v)||_{\chi^{(\alpha,\beta)}}\\ &+ ||\partial_{x_2}(P_{N_1,\alpha_1,\beta_1} \circ P_{N_2,\alpha_2,\beta_2}v - v)||_{\chi^{(\alpha,\beta)}}).\end{aligned}$$

Denote by G_1 and G_2 the first term and the second term at the right side of the above inequality, respectively. By taking $\mu = r = 1$ in (2.7) and using condition (i),

$$||\partial_{x_1}P_{N_1,\alpha_1,\beta_1}v||_{\chi^{(\alpha,\beta)}} \leq cN_1||v||_{L^2_{\chi^{(\alpha_2,\beta_2)}}(H^1_{\chi^{(\alpha_1,\beta_1)}})}. \tag{3.15}$$

By virtue of Lemma 2.4 and (3.15),

$$G_1 \leq c(N_1^{2-r}||v||_{L^2_{\chi^{(\alpha_2,\beta_2)}}(H^r_{\chi^{(\alpha_1,\beta_1)},**})} + N_1N_2^{1-s}||v||_{H^{s-1}_{\chi^{(\alpha_2,\beta_2)},A}(H^1_{\chi^{(\alpha_1,\beta_1)}})}).$$

A similar estimate is valid for G_2. The previous statements with space interpolation lead to (3.13). By (2.8), we can prove (3.14) similarly.

In many cases, the coefficients of derivatives of different orders degenerate in different ways. So we need the approximation in non-isotropic Hilbert spaces. Let $\gamma = (\gamma_1, \gamma_2), \delta = (\delta_1, \delta_2)$ and $\gamma_i, \delta_i > -1, i = 1,2$. Further, $H^0_{\alpha,\beta,\gamma,\delta}(\Omega) = L^2_{\chi^{(\gamma,\delta)}}(\Omega)$, and

$$H^1_{\alpha,\beta,\gamma,\delta}(\Omega) = \{v|v \text{ is measurable and } \ ||v||_{1,\alpha,\beta,\gamma,\delta} < \infty\}$$

where

$$||v||_{1,\alpha,\beta,\gamma,\delta} = (||\nabla v||^2_{\chi^{(\alpha,\beta)}} + ||v||^2_{\chi^{(\gamma,\delta)}})^{\frac{1}{2}}.$$

For $0 < \mu < 1$, the space $H^{\mu}_{\alpha,\beta,\gamma,\delta}(\Omega)$ and its norm $||v||_{\mu,\alpha,\beta,\gamma,\delta}$ are defined by space interpolation. Next, let

$$a_{\alpha,\beta,\gamma,\delta}(u,v) = (\nabla u, \nabla v)_{\chi^{(\alpha,\beta)}} + (u,v)_{\chi^{(\gamma,\delta)}}, \qquad \forall u,v \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega).$$

In particular, $a_{\alpha,\beta}(u,v) = a_{\alpha,\beta,\alpha,\beta}(u,v)$. The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta} : H^1_{\alpha,\beta,\gamma,\delta}(\Omega) \to \mathcal{P}_N$ is a mapping such that

$$a_{\alpha,\beta,\gamma,\delta}(P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi) = 0, \qquad \forall \phi \in \mathcal{P}_N.$$

For simplicity, we introduce the space $Y^{r,s}_{\alpha,\beta,\gamma,\delta}(\Omega)$ with $r,s \geq 1$. For $r,s = 1$, $Y^{1,1}_{\alpha,\beta,\gamma,\delta}(\Omega) = H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$. For $r,s \geq 2$, we define

$$Y^{r,s}_{\alpha,\beta,\gamma,\delta}(\Omega) = H^{r,s}_{\chi^{(\alpha,\beta)},*}(\Omega) \cap H^1_{\alpha_2,\beta_2,\gamma_2,\delta_2}(H^{r-1}_{\chi^{(\alpha_1,\beta_1)},*}) \cap H^{s-1}_{\chi^{(\alpha_2,\beta_2)},*}(H^1_{\alpha_1,\beta_1,\gamma_1,\delta_1}),$$

with the corresponding norm $||v||_{Y^{r,s}_{\alpha,\beta,\gamma,\delta}}$. For any $r, s \geq 1$, the space $Y^{r,s}_{\alpha,\beta,\gamma,\delta}(\Omega)$ is defined by space interpolation. In particular, $||v||_{Y^{r,s}_{\chi^{(\alpha,\beta)}}} = ||v||_{Y^{r,s}_{\alpha,\beta,\alpha,\beta}}$.

Theorem 3.5. *If*

$$\gamma_i \leq \alpha_i \leq \gamma_i + 1, \quad \delta_i \leq \beta_i \leq \delta_i + 1, \quad i = 1, 2, \tag{3.16}$$

then for any $v \in Y^{r,s}_{\alpha,\beta,\gamma,\delta}(\Omega)$ *and* $r, s \geq 1$,

$$||P^1_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} \leq c(N_1^{1-r} + N_2^{1-s})||v||_{Y^{r,s}_{\alpha,\beta,\gamma,\delta}}. \tag{3.17}$$

Proof. We first consider the cases $r, s \geq 2$. By the projection theorem,

$$||P^1_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} \leq c(W_1 + W_2 + W_3) \tag{3.18}$$

where

$$\begin{aligned} W_1 &= ||\partial_{x_1}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1} \circ P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2}v - v)||_{\chi^{(\alpha,\beta)}}, \\ W_2 &= ||\partial_{x_2}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1} \circ P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2}v - v)||_{\chi^{(\alpha,\beta)}}, \\ W_3 &= ||P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1} \circ P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2}v - v||_{\chi^{(\gamma,\delta)}}. \end{aligned}$$

We get from (3.16) and Lemma 2.5 that

$$||\partial_{x_1}P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1}v||_{\chi^{(\alpha,\beta)}} \leq c||\partial_{x_1}v||_{\chi^{(\alpha,\beta)}}, \tag{3.19}$$

$$||P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1}v||_{\chi^{(\gamma,\delta)}} \leq c(||v||_{\chi^{(\gamma,\delta)}} + N_1^{-1}||\partial_{x_1}v||_{L^2_{\chi^{(\gamma_2,\delta_2)}}(L^2_{\chi^{(\alpha_1,\beta_1)}})}). \tag{3.20}$$

Therefore by (3.16), (3.19) and Lemma 2.5,

$$\begin{aligned} W_1 &\leq ||\partial_{x_1}(P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1}v - v)||_{\chi^{(\alpha,\beta)}} + ||\partial_{x_1}(P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2}v - v)||_{\chi^{(\alpha,\beta)}} \\ &\leq c(N_1^{1-r}||v||_{L^2_{\chi^{(\alpha_2,\beta_2)}}(H^r_{\chi^{(\alpha_1,\beta_1)},*})} + ||P^1_{N_2,\alpha_2,\beta_2,\gamma_2,\delta_2}v - v||_{L^2_{\chi^{(\gamma_2,\delta_2)}}(H^1_{\alpha_1,\beta_1,\gamma_1,\delta_1})}) \\ &\leq c(N_1^{1-r}||v||_{L^2_{\chi^{(\alpha_2,\beta_2)}}(H^r_{\chi^{(\alpha_1,\beta_1)},*})} + N_2^{1-s}||v||_{H^{s-1}_{\chi^{(\alpha_2,\beta_2)},*}(H^1_{\alpha_1,\beta_1,\gamma_1,\delta_1})}). \end{aligned}$$

Similarly, we have from (3.16), (3.19), (3.20) and Lemma 2.5 that

$$W_2 \leq c(N_1^{1-r}||v||_{H^1_{\alpha_2,\beta_2,\gamma_2,\delta_2}(H^{r-1}_{\chi^{(\alpha_1,\beta_1)},*})} + N_2^{1-s}||v||_{H^s_{\chi^{(\alpha_2,\beta_2)},*}(L^2_{\chi^{(\alpha_1,\beta_1)}})}),$$

$$W_3 \leq c(N_1^{-r} + N_2^{-s} + N_1^{-1}N_2^{1-s})||v||_{Y^{r,s}_{\alpha,\beta,\gamma,\delta}}.$$

The above estimates lead to the desired result for $r, s \geq 2$. Obviously, for $r = s = 1$,

$$||P^1_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} \leq c||v||_{1,\alpha,\beta,\gamma,\delta} = c||v||_{Y^{1,1}_{\alpha,\beta,\gamma,\delta}}.$$

On the other hand,

$$||P^1_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta} \leq c(N_1^{-1} + N_2^{-1})||v||_{Y^{2,2}_{\alpha,\beta,\gamma,\delta}}.$$

So we get (3.17) for $1 < r, s < 2$ by space interpolation. The rest of the proof is clear.

In some cases, the unknown functions may vanish in some parts of the boundary. Denote by V_1, V_2, V_3 and V_4 the corners $(-1,-1), (1,-1), (1,1)$ and $(-1,1)$ of the square Ω, respectively. $\Gamma_j (j = 1, 2, 3, 4)$ stand for the edges with the end points V_{j-1} and V_j $(V_0 = V_4)$. Let

$$H^{1,\Gamma_j}_{0,\alpha,\beta,\gamma,\delta}(\Omega) = \{v | v \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega), v|_{\Gamma_j} = 0\}, \qquad \mathcal{P}^{\Gamma_j,0}_N = H^{1,\Gamma_j}_{0,\alpha,\beta,\gamma,\delta}(\Omega) \cap \mathcal{P}_N.$$

The orthogonal projection $P^{1,\Gamma_j}_{N,\alpha,\beta,\gamma,\delta} : H^{1,\Gamma_j}_{0,\alpha,\beta,\gamma,\delta}(\Omega) \to \mathcal{P}^{\Gamma_j,0}_N$ is a mapping such that

$$a_{\alpha,\beta,\gamma,\delta}(P^{1,\Gamma_j}_{N,\alpha,\beta,\gamma,\delta}v - v, \phi) = 0, \quad \forall \phi \in \mathcal{P}^{\Gamma_j,0}_N.$$

We can estimate $||P^{1,\Gamma_j}_{N,\alpha,\beta,\gamma,\delta}v - v||_{1,\alpha,\beta,\gamma,\delta}, 1 \leq j \leq 4$. For instance, we have the following result.

Theorem 3.6. *Let* $\gamma_2 \le \alpha_2 \le \gamma_2 + 1$ *and* $\delta_2 \le \beta_2 \le \delta_2 + 1$. *If one of the following conditions holds,*

(i) $\gamma_1 \le \alpha_1 \le \gamma_1 + 1,\ \delta_1 \le \beta_1 \le \delta_1 + 1,\ 0 < \alpha_1 < 1,\ \beta_1 < 1,$

(ii) $\gamma_1 \le \alpha_1 \le \gamma_1 + 1,\ \beta_1 = \delta_1 = 0,$

then for any $v \in H^{1,\Gamma_1}_{0,\alpha,\beta,\gamma,\delta}(\Omega) \cap Y^{r,s}_{\alpha,\beta,\gamma,\delta}(\Omega)$ *and* $r, s \ge 1$,

$$||P^{1,\Gamma_1}_{N,\alpha,\beta,\gamma,\delta} v - v||_{1,\alpha,\beta,\gamma,\delta} \le c(N_1^{1-r} + N_2^{1-s})||v||_{Y^{r,s}_{\alpha,\beta,\gamma,\delta}}. \tag{3.21}$$

The proof of this theorem is similar to the proof of Theorem 3.5. But $P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1}$ in (3.18)-(3.20) is now replaced by ${}_0P^1_{N_1,\alpha_1,\beta_1,\gamma_1,\delta_1}$. So by Lemmas 2.5 and 2.6, we reach the desired conclusion.

Now, let

$$H^1_{0,\alpha,\beta,\gamma,\delta}(\Omega) = \{v \mid v \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega) \text{ and } v|_{\partial\Omega} = 0\}.$$

The orthogonal projection $P^{1,0}_{N,\alpha,\beta,\gamma,\delta} : H^1_{0,\alpha,\beta,\gamma,\delta}(\Omega) \to \mathcal{P}^0_N$ is a mapping such that

$$a_{\alpha,\beta,\gamma,\delta}(P^{1,0}_{N,\alpha,\beta,\gamma,\delta} v - v, \phi) = 0, \quad \forall \phi \in \mathcal{P}^0_N.$$

Next, let $Z^{1,1}_{\alpha,\beta,\gamma,\delta}(\Omega) = H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$ and for $r, s \ge 3$,

$$Z^{r,s}_{\alpha,\beta,\gamma,\delta}(\Omega) = H^{r,s}_{\chi^{(\alpha,\beta)},*,2}(\Omega) \cap H^1_{\alpha_2,\beta_2,\gamma_2,\delta_2}(H^{r-1}_{\chi^{(\alpha_1,\beta_1)},*,2}) \cap H^{s-1}_{\chi^{(\alpha_2,\beta_2)},*,2}(H^1_{\alpha_1,\beta_1,\gamma_1,\delta_1})$$

with the corresponding norm $||v||_{Z^{r,s}_{\alpha,\beta,\gamma,\delta}}$. For any $r, s \ge 1$, we define the space $Z^{r,s}_{\alpha,\beta,\gamma,\delta}(\Omega)$ and its norm by space interpolation.

By a similar argument as in the proof of Theorem 3.5 and using Lemma 2.7, we have

Theorem 3.7. *If* $0 < \alpha_i = \gamma_i < 1$ *and* $0 < \beta_i = \delta_i < 1$, $i = 1, 2$, *then for any* $v \in H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \cap Z^{r,s}_{\alpha,\beta,\gamma,\delta}(\Omega)$ *with* $r, s \ge 1$,

$$||P^{1,0}_{N,\alpha,\beta,\gamma,\delta} v - v||_{1,\alpha,\beta,\gamma,\delta} \le c(N_1^{1-r} + N_2^{1-s})||v||_{Z^{r,s}_{\alpha,\beta,\gamma,\delta}}. \tag{3.22}$$

We now turn to some orthogonal projections in the space $H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$. For any $u, v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$, set

$$\widetilde{a}_{\alpha,\beta}(u, v) = (\nabla u, \nabla v)_{\chi^{(\alpha,\beta)}}, \quad \widehat{a}_{\alpha,\beta}(u, v) = (\nabla u, \nabla(\chi^{(\alpha,\beta)} v)).$$

Let $\mathcal{P}^0_N = H^1_{0,\chi^{(\alpha,\beta)}}(\Omega) \cap \mathcal{P}_N$. The projection $\widetilde{P}^{1,0}_{N,\alpha,\beta} : H^1_{0,\chi^{(\alpha,\beta)}}(\Omega) \to \mathcal{P}^0_N$ is a mapping such that

$$\widetilde{a}_{\alpha,\beta}(\widetilde{P}^{1,0}_{N,\alpha,\beta} v - v, \phi) = 0, \quad \forall \phi \in \mathcal{P}^0_N.$$

While the projection $\widehat{P}^{1,0}_{N,\alpha,\beta} : H^1_{0,\chi^{(\alpha,\beta)}}(\Omega) \to \mathcal{P}^0_N$ is a mapping such that

$$\widehat{a}_{\alpha,\beta}(\widehat{P}^{1,0}_{N,\alpha,\beta} v - v, \phi) = 0, \quad \forall \phi \in \mathcal{P}^0_N.$$

We focus on some specific cases in which the parameters satisfy one of the following conditions.

(i) $\alpha_i > -1,\ \beta_i = 0$ *or* $\alpha_i = 0,\ \beta_i > -1,\ i = 1, 2,$ (3.23)

(ii) $-1 < \alpha_i, \beta_i < 1,\ i = 1, 2.$ (3.24)

We can use Lemma 2.1 to prove the following result.

Lemma 3.2. *If* (3.23) *or* (3.24) *holds, then the semi-norm* $|\cdot|_{1,\chi^{(\alpha,\beta)}}$ *is a norm of the space* $H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$, *which is equivalent to the usual norm* $||\cdot||_{1,\chi^{(\alpha,\beta)}}$.

Lemma 3.3. *If* (3.23) *or* (3.24) *holds, then the bilinear form* $\widetilde{a}_{\alpha,\beta}(\cdot,\cdot)$ *is continuous and elliptic on* $H^1_{0,\chi^{(\alpha,\beta)}}(\Omega) \times H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$.

Lemma 3.4. *If* (3.24) *holds, then the mapping* $T^{\alpha,\beta}: v(x) \to v(x)\chi^{(\alpha,\beta)}(x)$ *is an isomorphism from* $H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$ *onto* $H^1_{0,\chi^{(-\alpha,-\beta)}}(\Omega)$.

Proof. For any $v \in \mathcal{D}(\Omega)$, we get from Lemma 2.1 that

$$||\partial_{x_1}(v\chi^{(\alpha,\beta)})||^2_{\chi^{(-\alpha,-\beta)}} \le c\int_\Omega (\partial_{x_1}v(x))^2\chi^{(\alpha,\beta)}(x)dx+$$
$$4(\alpha_1^2+\beta_1^2)\int_\Omega v^2(x)\chi^{(\alpha_1-2,\beta_1-2)}(x_1)\chi^{(\alpha_2,\beta_2)}(x_2)dx \le c||\partial_{x_1}v||^2_{\chi^{(\alpha,\beta)}}.$$

Similarly,

$$||\partial_{x_2}(v\chi^{(\alpha,\beta)})||_{\chi^{(-\alpha,-\beta)}} \le c||\partial_{x_2}v||_{\chi^{(\alpha,\beta)}}.$$

Consequently,

$$||v\chi^{(\alpha,\beta)}||_{1,\chi^{(-\alpha,-\beta)}} \le c||v||_{1,\chi^{(\alpha,\beta)}}.$$

So $T^{\alpha,\beta}$ is a linear continuous mapping from $H^1_{0,\chi^{(-\alpha,-\beta)}}(\Omega)$ into $H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$. Conversely, we also can show that the inverse mapping $T^{-\alpha,-\beta}: v \to v\chi^{(-\alpha,-\beta)}$ is also a linear continuous mapping from $H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$ into $H^1_{0,\chi^{(-\alpha,-\beta)}}(\Omega)$. Finally, a density argument leads to the desired result.

Lemma 3.5. *If* $-1 < \alpha_i, \beta_i \le 0$ *or* $0 < \alpha_i, \beta_i < 1, i = 1, 2$, *then the bilinear form* $\widehat{a}_{\alpha,\beta}(\cdot,\cdot)$ *is continuous and elliptic on* $H^1_{0,\chi^{(\alpha,\beta)}}(\Omega) \times H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$.

Proof. For any $u, v \in \mathcal{D}(\Omega)$, we use Lemma 2.1 to get that

$$\begin{aligned}
&\widehat{a}_{\alpha,\beta}(u,v) \le ||\nabla u||_{\chi^{(\alpha,\beta)}}||\nabla u||_{\chi^{(\alpha,\beta)}}\\
&\quad + 4(\alpha_1^2+\beta_1^2)||\partial_{x_1}u||_{\chi^{(\alpha,\beta)}}(\int_\Omega v^2\chi^{(\alpha_1-2,\beta_1-2)}(x_1)\chi^{(\alpha_2,\beta_2)}(x_2)dx)^{\frac{1}{2}}\\
&\quad + 4(\alpha_2^2+\beta_2^2)||\partial_{x_2}u||_{\chi^{(\alpha,\beta)}}(\int_\Omega v^2\chi^{(\alpha_1,\beta_1)}(x_1)\chi^{(\alpha_2-2,\beta_2-2)}(x_2)dx)^{\frac{1}{2}}\\
&\le c(||\nabla u||_{\chi^{(\alpha,\beta)}}||\nabla u||_{\chi^{(\alpha,\beta)}} + ||\partial_{x_1}u||_{\chi^{(\alpha,\beta)}}||\partial_{x_1}v||_{\chi^{(\alpha,\beta)}} + ||\partial_{x_2}u||_{\chi^{(\alpha,\beta)}}||\partial_{x_2}v||_{\chi^{(\alpha,\beta)}})\\
&\le c||u||_{1,\chi^{(\alpha,\beta)}}||v||_{1,\chi^{(\alpha,\beta)}}.
\end{aligned} \tag{3.25}$$

On the other hand, integrating by parts yields

$$\widehat{a}_{\alpha,\beta}(u,u) = ||\nabla u||^2_{\chi^{(\alpha,\beta)}} + \frac{1}{2}\int_\Omega u^2W_1(x_1)\chi^{(\alpha,\beta)}(x)dx + \frac{1}{2}\int_\Omega u^2W_2(x_2)\chi^{(\alpha,\beta)}(x)dx, \tag{3.26}$$

where

$$W_1(x_1) = -\partial^2_{x_1}(\chi^{(\alpha_1,\beta_1)}(x_1))\chi^{(-\alpha_1,-\beta_1)}(x_1), \qquad W_2(x_2) = -\partial^2_{x_2}(\chi^{(\alpha_2,\beta_2)}(x_2))\chi^{(-\alpha_2,-\beta_2)}(x_2).$$

We now determine the ranges of α_i and β_i such that $W_i(x_i) \ge 0$ for all $x_i \in \Lambda_i, i = 1, 2$. Let $f(x_1) = (1-x_1^2)^2W_1(x_1)$. A calculation shows that

$$f(x_1) = -(\alpha_1+\beta_1)(\alpha_1+\beta_1-1)x_1^2 + 2(\beta_1-\alpha_1)(\alpha_1+\beta_1-1)x_1 + \alpha_1+\beta_1-(\beta_1-\alpha_1)^2.$$

By the properties of the quadratic function, we find that $f(x_1) \ge 0$ for all $x_1 \in \Lambda_1$, if

$$(\alpha_1+\beta_1)(\alpha_1+\beta_1-1) \ge 0, \quad f(-1) = -4\beta_1^2 + 4\beta_1 \ge 0, \quad f(1) = -4\alpha_1^2 + 4\alpha_1 \ge 0, \tag{3.27}$$

or

$$\begin{cases} (\alpha_1+\beta_1)(\alpha_1+\beta_1-1) \le 0, \\ 4(\beta_1-\alpha_1)^2(\alpha_1+\beta_1-1)^2 + 4(\alpha_1+\beta_1)(\alpha_1+\beta_1-1)(\alpha_1+\beta_1-(\beta_1-\alpha_1)^2) \le 0. \end{cases} \tag{3.28}$$

Solving (3.27) and (3.28) yields that $0 \le \alpha_1, \beta_1 \le 1$. Similarly, we have $W_2(x_2) \ge 0$ in the case of $0 \le \alpha_2, \beta_2 \le 1$. Therefore, if $0 \le \alpha_i, \beta_i \le 1, i = 1, 2$, then we obtain from (3.26) that

$$\widehat{a}_{\alpha,\beta}(u,u) \ge c||u||^2_{1,\chi^{(\alpha,\beta)}}. \tag{3.29}$$

Furthermore, if $-1 < \alpha_i, \beta_i \le 0, i = 1, 2$, then we set $w = u\chi^{(\alpha,\beta)}$. So we know from Lemma 3.4 that $w \in H^1_{0,\chi^{(-\alpha,-\beta)}}(\Omega)$. Hence, by (3.29),

$$\widehat{a}_{\alpha,\beta}(u,u) = \widehat{a}_{-\alpha,-\beta}(w,w) \ge c||w||^2_{1,\chi^{(-\alpha,-\beta)}} \ge c||u||^2_{1,\chi^{(\alpha,\beta)}} \tag{3.30}$$

Finally, (3.25), (3.29) and (3.30) lead to the conclusion.

Theorem 3.8. *If* $-1 < \alpha_i, \beta_i \le 0$ *or* $0 < \alpha_i, \beta_i < 1, i = 1, 2$, *then for any* $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Omega) \cap Y^{r,s}_{\chi^{(\alpha,\beta)}}(\Omega)$ *and* $r, s \ge 1$,

$$||\widetilde{P}^{1,0}_{N,\alpha,\beta} v - v||_{1,\chi^{(\alpha,\beta)}} \le c(N_1^{1-r} + N_2^{1-s})||v||_{Y^{r,s}_{\chi^{(\alpha,\beta)}}}, \tag{3.31}$$

and

$$||\widehat{P}^{1,0}_{N,\alpha,\beta} v - v||_{1,\chi^{(\alpha,\beta)}} \le c(N_1^{1-r} + N_2^{1-s})||v||_{Y^{r,s}_{\chi^{(\alpha,\beta)}}}. \tag{3.32}$$

Proof. By Lemma 2.8 and a similar argument as in the proof of Theorem 3.5, we obtain (3.31). We now prove (3.32). By Lemma 3.5, for any $\phi \in \mathcal{P}^0_N$,

$$\begin{aligned} ||\widehat{P}^{1,0}_{N,\alpha,\beta} v - v||^2_{1,\chi^{(\alpha,\beta)}} &\le c\widehat{a}_{\alpha,\beta}(\widehat{P}^{1,0}_{N,\alpha,\beta} v - v, \widehat{P}^{1,0}_{N,\alpha,\beta} v - v) \\ &= c\widehat{a}_{\alpha,\beta}(\widehat{P}^{1,0}_{N,\alpha,\beta} v - v, \phi - v) \le c||\widehat{P}^{1,0}_{N,\alpha,\beta} v - v||_{1,\chi^{(\alpha,\beta)}} ||\phi - v||_{1,\chi^{(\alpha,\beta)}}. \end{aligned}$$

Therefore, by the above fact and Lemma 3.2,

$$||\widehat{P}^{1,0}_{N,\alpha,\beta} v - v||_{1,\chi^{(\alpha,\beta)}} \le c \inf_{\phi \in \mathcal{P}^0_N} ||\phi - v||_{1,\chi^{(\alpha,\beta)}} \le c|\widetilde{P}^{1,0}_{N,\alpha,\beta} v - v|_{1,\chi^{(\alpha,\beta)}},$$

and so the conclusion follows from (3.31).

Theorem 3.9. *If* $-\frac{1}{2} \le \alpha_i, \beta_i \le 0$ *or* $0 < \alpha_i, \beta_i \le \frac{1}{2}$, $i = 1, 2$, *then for any* $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Omega) \cap Y^{r,s}_{\chi^{(\alpha,\beta)}}(\Lambda), r, s \ge 1$ *and* $0 \le \mu \le 1$,

$$||\widehat{P}^{1,0}_{N,\alpha,\beta} v - v||_{\mu,\chi^{(\alpha,\beta)}} \le c(N_1^{1-r} + N_2^{1-s})^{\mu} (N_1^{-1} + N_2^{-1})^{1-\mu} ||v||_{Y^{r,s}_{\chi^{(\alpha,\beta)}}}. \tag{3.33}$$

Proof. Theorem 3.8 implies (3.33) for $\mu = 1$. We now prove (3.33) with $\mu = 0$. Let $g \in L^2_{\chi^{(\alpha,\beta)}}(\Omega)$ and consider the auxiliary problem

$$\widehat{a}_{\alpha,\beta}(w, z) = (g, z)_{\chi^{(\alpha,\beta)}}, \qquad \forall z \in H^1_{0,\chi^{(\alpha,\beta)}}(\Omega). \tag{3.34}$$

Taking $z = w$ in (3.34), we get from Lemma 3.2 that $||w||_{1,\chi^{(\alpha,\beta)}} \le c||g||_{\chi^{(\alpha,\beta)}}$. Next, let $w(x)$ vary in $\mathcal{D}(\Omega)$, and so in the sense of distributions, $\Delta w(x) = g(x)$. Now, we prove that

$$|w|_{2,\chi^{(\alpha,\beta)}} \le c||\Delta w||_{\chi^{(\alpha,\beta)}}. \tag{3.35}$$

Indeed,

$$\begin{aligned} \int_\Omega (\Delta w(x))^2 \chi^{(\alpha,\beta)}(x) dx \quad &= \int_\Omega (\frac{\partial^2 w}{\partial x_1^2}(x))^2 \chi^{(\alpha,\beta)}(x) dx + \int_\Omega (\frac{\partial^2 w}{\partial x_2^2}(x))^2 \chi^{(\alpha,\beta)}(x) dx \\ &+2\int_\Omega \frac{\partial^2 w}{\partial x_1^2}(x) \frac{\partial^2 w}{\partial x_2}(x) \chi^{(\alpha,\beta)}(x) dx. \end{aligned} \tag{3.36}$$

For any $i \ne j$, the derivatives $\dfrac{\partial^q w}{\partial x_i^q}$ vanish on the edges with $x_j = \pm 1$. Thus by integration by parts,

$$\begin{aligned} &\int_\Omega \frac{\partial^2 w}{\partial x_1^2}(x) \frac{\partial^2 w}{\partial x_2^2}(x) \chi^{(\alpha,\beta)}(x) dx \\ &= \int_\Omega \frac{\partial^2 (w\chi^{(\alpha_1,\beta_1)}(x_1))}{\partial x_1 \partial x_2} \frac{\partial^2 (w\chi^{(\alpha_2,\beta_2)}(x_2))}{\partial x_1 \partial x_2} dx \\ &= \int_\Omega (\frac{\partial^2 w}{\partial x_1 \partial x_2}(x))^2 \chi^{(\alpha,\beta)}(x) dx - \frac{1}{2}\int_\Omega (\frac{\partial w}{\partial x_2}(x))^2 (\chi^{(\alpha_1,\beta_1)}(x_1))'' \chi^{(\alpha_2,\beta_2)}(x_2) dx \\ &\quad - \frac{1}{2}\int_\Omega (\frac{\partial w}{\partial x_1}(x))^2 \chi^{(\alpha_1,\beta_1)}(x_1) (\chi^{(\alpha_2,\beta_2)}(x_2))'' dx \\ &\quad + \int_\Omega \frac{\partial w}{\partial x_1}(x) \frac{\partial w}{\partial x_2}(x) (\chi^{(\alpha_1,\beta_1)}(x_1))' (\chi^{(\alpha_2,\beta_2)}(x_2))' dx. \end{aligned} \tag{3.37}$$

Moreover, by the Cauchy-Schwartz inequality,

$$
\begin{aligned}
|\int_\Omega \frac{\partial w}{\partial x_1}(x)\frac{\partial w}{\partial x_2}(x)\ \ (\chi^{(\alpha_1,\beta_1)}(x_1))'(\chi^{(\alpha_2,\beta_2)}(x_2))'dx| \\
\le \frac{1}{2}\int_\Omega(\frac{\partial w}{\partial x_1}(x))^2((\chi^{(\alpha_2,\beta_2)}(x_2))')^2\chi^{(\alpha_1,\beta_1)}(x_1)\chi^{(-\alpha_2,-\beta_2)}(x_2)dx \\
+\frac{1}{2}\int_\Omega(\frac{\partial w}{\partial x_2}(x))^2((\chi^{(\alpha_1,\beta_1)}(x_1))')^2\chi^{(-\alpha_1,-\beta_1)}(x_1)\chi^{(\alpha_2,\beta_2)}(x_2)dx.
\end{aligned}
$$

Inserting the above estimate into (3.37), we get that

$$
\begin{aligned}
\int_\Omega \frac{\partial^2(w\chi^{(\alpha_1,\beta_1)}(x_1))}{\partial x_1\partial x_2}\frac{\partial^2(w\chi^{(\alpha_2,\beta_2)}(x_2))}{\partial x_1\partial x_2}dx \ge \int_\Omega(\frac{\partial^2 w}{\partial x_1\partial x_2}(x))^2\chi^{(\alpha,\beta)}(x)dx \\
+\frac{1}{2}\int_\Omega(\frac{\partial w}{\partial x_1}(x))^2W_2(x_2)\chi^{(\alpha,\beta)}(x)dx+\frac{1}{2}\int_\Omega(\frac{\partial w}{\partial x_2}(x))^2W_1(x_1)\chi^{(\alpha,\beta)}(x)dx,
\end{aligned}
\tag{3.38}
$$

where

$$
\begin{aligned}
W_1(x_1)&=-(\chi^{(\alpha_1,\beta_1)}(x_1))''\chi^{(-\alpha_1,-\beta_1)}(x_1)-((\chi^{(\alpha_1,\beta_1)}(x_1))')^2\chi^{(-2\alpha_1,-2\beta_1)}(x_1),\\
W_2(x_2)&=-(\chi^{(\alpha_2,\beta_2)}(x_2))''\chi^{(-\alpha_2,-\beta_2)}(x_2)-((\chi^{(\alpha_2,\beta_2)}(x_2))')^2\chi^{(-2\alpha_2,-2\beta_2)}(x_2).
\end{aligned}
$$

We now determine the ranges of α_i and β_i such that $W_i(x_i)\ge 0$ for all $x_i\in\Lambda_i, i=1,2$. To do this, let $f(x_1)=(1-x_1)^2W_1(x_1)$. A calculation shows that

$$
f(x_1)=-(\alpha_1+\beta_1)(2\alpha_1+2\beta_1-1)x_1^2+2(\beta_1-\alpha_1)(2\alpha_1+2\beta_1-1)x_1+\alpha_1+\beta_1-2(\beta_1-\alpha_1)^2.
$$

Following the same lines as in the proof of Lemma 3.4, we can prove that $f(x_1)\ge 0$ for all $x_1\in\Lambda_1$ in the case of $0\le\alpha_1,\beta_1\le\frac{1}{2}$. Therefore, we know from (3.36)-(3.38) that (3.35) is valid for $0\le\alpha_i,\beta_i\le\frac{1}{2}, i=1,2$. If α_i and β_i are negative, then we take $u=w\chi^{(\alpha,\beta)}$ and find that

$$
\int_\Omega \frac{\partial^2(w\chi^{(\alpha_1,\beta_1)}(x_1))}{\partial x_1\partial x_2}\frac{\partial^2(w\chi^{(\alpha_2,\beta_2)}(x_2))}{\partial x_1\partial x_2}dx=\int_\Omega \frac{\partial^2(u\chi^{(-\alpha_1,-\beta_1)}(x_1))}{\partial x_1\partial x_2}\frac{\partial^2(u\chi^{(-\alpha_1,-\beta_1)}(x_1))}{\partial x_1\partial x_2}dx.
$$

Hence, we can show that (3.35) is also valid for $-\frac{1}{2}\le\alpha_i,\beta_i<0, i=1,2$. By the definition of the space $Y^{2,2}_{\chi^{(\alpha,\beta)}}(\Omega)$, we have that $H^2_{\chi^{(\alpha,\beta)}}(\Omega)\subseteq Y^{2,2}_{\chi^{(\alpha,\beta)}}(\Omega)$. Thus by (3.35),

$$
||w||^2_{Y^{2,2}_{\chi^{(\alpha,\beta)}}}\le c||w||^2_{2,\chi^{(\alpha,\beta)}}\le c(||w||^2_{1,\chi^{(\alpha,\beta)}}+|w|^2_{2,\chi^{(\alpha,\beta)}})\le c||g||^2_{\chi^{(\alpha,\beta)}}.
\tag{3.39}
$$

Taking $z=\widehat{P}^{1,0}_{N,\alpha,\beta}v-v$ in (3.34), we obtain that

$$
\begin{aligned}
|(\widehat{P}^{1,0}_{N,\alpha,\beta}v-v,g)_{\chi^{(\alpha,\beta)}}| &=|\widehat{a}_{\alpha,\beta}(\widehat{P}^{1,0}_{N,\alpha,\beta}v-v,\widehat{P}^{1,0}_{N,\alpha,\beta}w-w)|\\
&\le c(N_1^{1-r}+N_2^{1-s})(N_1^{-1}+N_2^{-1})||g||_{\chi^{(\alpha,\beta)}}||v||_{Y^{r,s}_{\chi^{(\alpha,\beta)}}}.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&||\widehat{P}^{1,0}_{N,\alpha,\beta}v-v||_{\chi^{(\alpha,\beta)}}\\
=&\sup_{\substack{g\in L^2_{\chi^{(\alpha,\beta)}}\\ g\ne 0}}\frac{|(\widehat{P}^{1,0}_{N,\alpha,\beta}v-v,g)_{\chi^{(\alpha,\beta)}}|}{||g||_{\chi^{(\alpha,\beta)}}}\le c(N_1^{1-r}+N_2^{1-s})(N_1^{-1}+N_2^{-1})||v||_{Y^{r,s}_{\chi^{(\alpha,\beta)}}}.
\end{aligned}
$$

Finally, we complete the proof by space interpolation.

4. Some Applications

We first consider the problem

$$
-\nabla(a(x)\nabla U(x))+c(x)U(x)=f(x),\qquad x\in\Omega,
\tag{4.1}
$$

where $\nabla=(\partial_{x_1},\partial_{x_2})$, $a(x)\ge 0$, $c(x)\ge 0$ and $f(x)$ are given functions. Assume that $a(x)=a_1(x)\chi^{(\alpha,\beta)}(x)$, $c(x)=c_1(x)\chi^{(\gamma,\delta)}(x)$, $a_1(x)\in L^\infty(\Omega)$, $c_1(x)\in L^\infty(\Omega), a_1(x)\ge a_{\min}>0$ and

$c_1(x) \geq c_{\min} > 0$, for $x \in \bar{\Lambda}$. We look for solution of (4.1) such that $a(x)\nabla U(x) \to 0$ as x tends to $\partial\Omega$. Let

$$b_{\alpha,\beta,\gamma,\delta}(u,v) = (a_1 \nabla u, \nabla v)_{\chi^{(\alpha,\beta)}} + (c_1 u, v)_{\chi^{(\gamma,\delta)}}, \quad \forall u, v \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega).$$

A weak formulation of (4.1) is to find $U \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega)$ such that

$$b_{\alpha,\beta,\gamma,\delta}(U,v) = (f,v), \qquad \forall v \in H^1_{\alpha,\beta,\gamma,\delta}(\Omega). \tag{4.2}$$

If $f \in L^2_{\chi^{(-\gamma,-\delta)}}(\Omega)$, then (4.2) has a unique solution such that $||U||_{1,\alpha,\beta,\gamma,\delta} \leq c||f||_{\chi^{(-\gamma,-\delta)}}$.

Next, let $u_N \in \mathcal{P}_N$ be the approximation to U, satisfying

$$b_{\alpha,\beta,\gamma,\delta}(u_N,\phi) = (f,\phi), \quad \forall \phi \in \mathcal{P}_N. \tag{4.3}$$

The scheme (4.3) is unisolvent, and $||u_N||_{1,\alpha,\beta,\gamma,\delta} \leq c||f||_{\chi^{(-\gamma,-\delta)}}$.

Theorem 4.1. *If* (3.16) *holds, and* $U \in Y^{r,s}_{\alpha,\beta,\gamma,\delta}(\Omega)$ *with* $r, s \geq 1$, *then*

$$||U - u_N||_{1,\alpha,\beta,\gamma,\delta} \leq d(N_1^{1-r} + N_2^{1-s})||U||_{Y^{r,s}_{\alpha,\beta,\gamma,\delta}},$$

where the constant d *depends only on the norms* $||a_1||_{L^\infty}$ *and* $||c_1||_{L^\infty}$.

Proof. Let $U_N = P^1_{N,\alpha,\beta,\gamma,\delta}U$. By (4.2) and Theorem 3.5,

$$\begin{aligned} c||u_N - U_N||^2_{1,\alpha,\beta,\gamma,\delta} &\leq b_{\alpha,\beta,\gamma,\delta}(u_N - U_N, u_N - U_N) = b_{\alpha,\beta,\gamma,\delta}(U - U_N, u_N - U_N) \\ &\leq d(N_1^{1-r} + N_2^{1-s})||U||_{Y^{r,s}_{\alpha,\beta,\gamma,\delta}}||u_N - U_N||_{1,\alpha,\beta,\gamma,\delta}. \end{aligned}$$

Thus

$$||U - u_N||_{1,\alpha,\beta,\gamma,\delta} \leq ||U - U_N||_{1,\alpha,\beta,\gamma,\delta} + ||u_N - U_N||_{1,\alpha,\beta,\gamma,\delta} \leq d(N_1^{1-r} + N_2^{1-s})||U||_{Y^{r,s}_{\alpha,\beta,\gamma,\delta}}.$$

We next deal with the Poisson equation

$$\begin{cases} -\Delta U(x) = f(x), & x \in \Omega, \\ U(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.4}$$

where $f(x) = f_1(x)\chi^{(-\alpha,-\beta)}(x)$, $f_1 \in L^\infty(\Omega)$, and α_i, $\beta_i < 1, i = 1,2$. A weak formulation of (4.4) is to find $U \in H^1_{0,\chi^{(\alpha,\beta)}}(\Omega)$ such that

$$\widehat{a}_{\alpha,\beta}(U,v) = (f_1, v), \quad \forall v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Omega). \tag{4.5}$$

We know from Lemma 3.5 that

$$||U||^2_{1,\chi^{(\alpha,\beta)}} \leq c|\widehat{a}_{\alpha,\beta}(U,U)| = c|(f_1,U)| \leq c||f_1||_{L^\infty}||U||_{L^1} \leq c||f_1||_{L^\infty}||U||_{\chi^{(\alpha,\beta)}}. \tag{4.6}$$

Thus if $f_1 \in L^\infty(\Omega)$, then (4.5) has a unique solution such that $||U||_{1,\chi^{(\alpha,\beta)}} \leq c||f_1||_{L^\infty}$.

The Jacobi spectral approximation to (4.5) is to find $u_N \in \mathcal{P}^0_N$ such that

$$\widehat{a}_{\alpha,\beta}(u_N,\phi) = (f_1,\phi), \quad \forall \phi \in \mathcal{P}^0_N. \tag{4.7}$$

It is unisolvent and $||u_N||_{1,\chi^{(\alpha,\beta)}} \leq c||f_1||_{L^\infty}$. For error estimate, let $U_N = \widehat{P}^{1,0}_{N,\alpha,\beta}U$. By (4.5) and (4.7),

$$\widehat{a}_{\alpha,\beta}(u_N - U_N, \phi) = \widehat{a}_{\alpha,\beta}(U - U_N, \phi), \quad \forall \phi \in \mathcal{P}^0_N.$$

Taking $\phi = u_N - U_N$ and using Lemma 3.5, Theorem 3.8 and Theorem 3.9, we have the following result.

Theorem 4.2. *If* α *and* β *fulfill the conditions in Theorem* 3.8, *and* $U \in H^1_{0,\chi^{(\alpha,\beta)}}(\Omega) \cap Y^{r,s}_{\chi^{(\alpha,\beta)}}(\Omega)$ *with* $r, s \geq 1$, *then*

$$||U - u_N||_{1,\chi^{(\alpha,\beta)}} \leq c(N_1^{1-r} + N_2^{1-s})||U||_{Y^{r,s}_{\chi^{(\alpha,\beta)}}}.$$

If, in addition, $-\frac{1}{2} \leq \alpha_i, \beta_i \leq 0$ *or* $0 < \alpha_i, \beta_i \leq \frac{1}{2}, i = 1,2$, *then for all* $0 \leq \mu \leq 1$,

$$||U - u_N||_{\mu,\chi^{(\alpha,\beta)}} \leq c(N_1^{1-r} + N_2^{1-s})^\mu (N_1^{-1} + N_2^{-1})^{1-\mu}||U||_{Y^{r,s}_{\chi^{(\alpha,\beta)}}}.$$

We now consider the singularity of the boundary value. Let Γ_3 be the same as in Section 3, and consider the following boundary value problem

$$\begin{cases} -\Delta U(x) = f(x), & \text{in } \Omega, \\ U(x) = g(x), & \text{on } \Gamma_3, \\ U(x) = 0, & \text{on } \Gamma = \partial\Omega/\Gamma_3, \end{cases} \tag{4.8}$$

where $f(x)$ is regular, $g \in L^2_\omega(\Gamma_3)$ with $\omega(x) = (1-x_1)^{\alpha_1} (\alpha_1 < 1)$, and $g(x) = 0$ on $\bar{\Gamma}_3 \cap \bar{\Gamma}$. By a similar argument as in the proof of Theorem 18.7 of Bernardi and Maday [5], we have that for any $v \in H^1_\omega(\Omega)$, its trace on Γ_3 belongs to $H^{(1-\alpha_1)/2}_\omega(\Gamma_3)$. Further, denote by $U_B \in H^1_\omega(\Omega)$ the trace lifting of g. Then

$$||U_B||_{1,\omega} \le c||g||_{H^{(1-\alpha_1)/2}_\omega(\Gamma_3)}. \tag{4.9}$$

Next, let

$$H^{1,\Gamma}_{0,\omega}(\Omega) = \{v | v \in H^1_\omega(\Omega), v = 0 \text{ on } \Gamma\}, \qquad a_\omega(u,v) = \int_\Omega (\nabla u \cdot \nabla(\omega v))dx.$$

A weak formulation of (4.8) is to find $U \in H^{1,\Gamma}_{0,\omega}(\Omega)$ and $U^* = U - U_B \in H^1_{0,\omega}(\Omega)$ such that

$$a_\omega(U,v) = (f,v)_\omega, \qquad \forall v \in H^1_{0,\omega}(\Omega), \tag{4.10}$$

or equivalently

$$a_\omega(U^*,v) = (f,v)_\omega - a_\omega(U_B,v), \qquad \forall v \in H^1_{0,\omega}(\Omega). \tag{4.11}$$

We can show that $a_\omega(u,v)$ is continuous on $H^1_\omega(\Omega) \times H^1_{0,\omega}(\Omega)$ and elliptic on $H^1_{0,\omega}(\Omega)$. Let $H^{-1}_\omega(\Omega) = (H^1_{0,\omega}(\Omega))'$ with the norm $||v||_{-1,\omega}$. By the above fact, (4.11) and the Lax-Milgram Lemma, we know that if $f \in H^{-1}_\omega(\Omega)$, then (4.10) has a unique solution such that

$$||U||_{1,\omega} \le c(||U_B||_{1,\omega} + ||f||_{-1,\omega}) \le c(||g||_{H^{(1-\alpha_1)/2}_\omega(\Gamma_3)} + ||f||_{-1,\omega}). \tag{4.12}$$

Let $\mathcal{P}^{\Gamma,0}_N = H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega) \cap \mathcal{P}_N$ and $P^{1,\Gamma}_{N,\alpha,\beta,\gamma,\delta} : H^{1,\Gamma}_{0,\alpha,\beta,\gamma,\delta}(\Omega) \to \mathcal{P}^{\Gamma,0}_N$ be the mapping such that

$$a_{\alpha,\beta,\gamma,\delta}(P^{1,\Gamma}_{N,\alpha,\beta,\gamma,\delta} v - v, \phi) = 0, \qquad \forall \phi \in \mathcal{P}^{\Gamma,0}_N.$$

For simplicity, let $P^1_{N,\omega} = P^1_{N,\alpha,\beta,\alpha,\beta}$ and $P^{1,\Gamma}_{N,\omega} = P^{1,\Gamma}_{N,\alpha,\beta,\alpha,\beta}$ with $\alpha = (\alpha_1, 0)$ and $\beta = (0,0)$. The spectral scheme for (4.10) is to find $u_N \in \mathcal{P}^{\Gamma,0}_N$ and $u^*_N = u_N - P^1_{N,\omega} U_B \in \mathcal{P}^0_N$ such that

$$a_\omega(u_N, \phi) = (f,\phi)_\omega, \qquad \forall \phi \in \mathcal{P}^0_N, \tag{4.13}$$

or equivalently

$$a_\omega(u^*_N, \phi) = (f,\phi)_\omega - a_\omega(P^1_{N,\omega} U_B, \phi), \qquad \forall \phi \in \mathcal{P}^0_N. \tag{4.14}$$

By Theorem 3.6 and a similar argument as in the derivation of (4.12), (4.13) is unisolvent, and

$$||u_N||_{1,\omega} \le c(||P^1_{N,\omega} U_B||_{1,\omega} + ||f||_{-1,\omega}) \le c(||U_B||_{1,\omega} + ||f||_{-1,\omega}) \le c(||g||_{H^{(1-\alpha_1)/2}_\omega(\Gamma_3)} + ||f||_{-1,\omega}). \tag{4.15}$$

Next, let $U_N = P^{1,\Gamma}_{N,\omega} U$ and $U^*_N = U_N - P^1_{N,\omega} U_B$. By the ellipticity and the continuity of $a_\omega(\cdot,\cdot)$, we have from (4.11) and (4.14) that for certain $c_0 > 0$,

$$\begin{aligned} c_0 ||u^*_N - U^*_N||^2_{1,\omega} &\le a_\omega(u^*_N - U^*_N, u^*_N - U^*_N) \\ &= a_\omega(U^* - U^*_N, u^*_N - U^*_N) + a_\omega(U_B - P^1_{N,\omega} U_B, u^*_N - U^*_N) \\ &\le c(||U^* - U^*_N||_{1,\omega} + ||U_B - P^1_{N,\omega} U_B||_{1,\omega}) ||u^*_N - U^*_N||_{1,\omega} \\ &\le c(||U - U_N||_{1,\omega} + 2||U_B - P^1_{N,\omega} U_B||_{1,\omega}) ||u^*_N - U^*_N||_{1,\omega}. \end{aligned}$$

By Theorem 3.5 and a result like Theorem 3.6,

$$||u^*_N - U^*_N||_{1,\omega} \le c((N_1^{1-r} + N_2^{1-s})||U||_{Y^{r,s}_\omega} + (N_1^{1-r'} + N_2^{1-s'})||U||_{Y^{r',s'}_\omega}),$$

where $Y^{r,s}_\omega(\Omega) = Y^{r,s}_{\alpha,\beta,\alpha,\beta}(\Omega)$ and $Y^{r',s'}_\omega(\Omega) = Y^{r',s'}_{\alpha,\beta,\alpha,\beta}(\Omega)$ with $\alpha = (\alpha_1, 0)$ and $\beta = (0,0)$. Since $u_N - U_N = u^*_N - U^*_N$, we get the following result.

Theorem 4.3. *If* $\alpha_1 < 1$, $U \in H^{1,\Gamma}_{0,\omega}(\Omega) \cap Y^{r,s}_\omega(\Omega)$ *and* $U_B \in Y^{r',s'}_\omega(\Omega)$ *with* $r, s, r', s' \geq 1$, *then*

$$||u_N - U_N||_{1,\omega} \leq c((N_1^{1-r} + N_2^{1-s})||U||_{Y^{r,s}_\omega} + (N_1^{1-r'} + N_2^{1-s'})||U_B||_{Y^{r',s'}_\omega}.$$

Finally, we present some numerical results. We first consider problem (4.1) with $a(x) = (1-x_1^2)(1-x_2^2)$ and $c(x) = 1$. Take the test function

$$U(x) = \arcsin(x_1 x_2) e^{x_1 x_2}.$$

Clearly, $|\partial_{x_i} U| \to \infty, i = 1, 2$, as x tends to $\partial\Omega$. We use (4.3) to solve (4.1) numerically. $E(U - u_N)$ is the discrete L^2-norm of the error $U - u_N$ based on the Gauss-Legendre nodes and weights. The errors $E(U - u_N)$ with $N_1 = N_2$ are listed in Table 1, which show the high accuracy and convergence of the scheme (4.3).

We next consider problem (4.4), and take the test function

$$U(x) = \frac{1}{4}(1+x_1)(1+x_2)\ln\frac{1+x_1}{2}\ln\frac{1+x_2}{2}.$$

$E(U - u_N)$ is the discrete L^2-norm of the error $U - u_N$ based on the Gauss-Legendre-Lobatto nodes and weights. The errors $E(U - u_N)$ with $N_1 = N_2$ are listed in Table 2, showing the effectiveness of this approach.

Table 1. The errors $E(U - u_N)$.

$N_1 = N_2$	$E(U - u_N)$
8	3.06E-3
16	4.97E-4
32	7.15E-5
48	2.23E-5

Table 2. The errors $E(U - u_N)$.

$N_1 = N_2$	$E(U - u_N)$
8	5.55E-4
16	5.42E-5
32	4.77E-6
64	3.99E-7

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