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PARALLEL DYNAMIC ITERATION METHODS FOR SOLVING NONLINEAR TIME-PERIODIC DIFFERENTIAL-ALGEBRAIC EQUATIONS*1)

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Abstract

In this paper we presented a convergence condition of parallel dynamic iteration methods for a nonlinear system of differential-algebraic equations with a periodic constraint. The convergence criterion is decided by the spectral expression of a linear operator derived from system partitions. Numerical experiments given here confirm the theoretical work of the paper.

Key words: Nonlinear dynamic equations, Periodic solutions, Dynamic iterations, Engineering applications.

1. Introduction

In order to analyze physical characters of nonlinear dynamic equations issued from engineering applications we often need to compute their periodic solutions. Most models of mechanical systems and circuit simulation might be described by nonlinear differential-algebraic equations (DAEs) as follows

$$\begin{cases} \dfrac{d}{dt}x(t) = \tilde{f}(x(t), y(t), t), \quad x(0) = x(T), \\ y(t) = \tilde{g}(x(t), y(t), t), \quad t \in [0, T], \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ and $y(t) \in \mathbf{R}^m$ for $t \in [0,T]$, $\tilde{f} : \mathbf{R}^n \times \mathbf{R}^m \times [0,T] \mapsto \mathbf{R}^n$ and $\tilde{g} : \mathbf{R}^n \times \mathbf{R}^m \times [0,T] \mapsto \mathbf{R}^m$ satisfy $\tilde{f}(x,y,0) = \tilde{f}(x,y,T)$ and $\tilde{g}(x,y,0) = \tilde{g}(x,y,T)$ for any given $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$, and $y(0)$ satisfies $y(0) = \tilde{g}(x(0), y(0), 0)$.

It is a general knowledge that computation of periodic solutions for a dynamic system is very time-consuming owing to the unknown of initial values. The usual way to treat (1) is the shooting techniques and its variants [1, 2]. For example, if the time interval is broken into small pieces then the parallel shooting process is available. This parallel process is not direct since the standard shooting must be called for every small interval.

A direct parallel method for transient computation is dynamic iteration (see [3]) or waveform relaxation. The dynamic iteration method was originally presented to simulate VLSI in 1982 [4]. It decouples dynamic equations in system level, for example a system of ordinary differential equations (ODEs) or DAEs may be partitioned into some simplified systems of ODEs or DAEs [5, 6, 7]. We can also study dynamic iterations of linear integral-differential-algebraic equations (IDAEs) [8]. Numerical algorithms based on this approach could be conveniently implemented on multi-processor computer systems [9]. They are instinctive parallel algorithms.

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The dynamic iteration method has been adopted as a computational tool to study periodic solutions of linear systems [10, 11]. A simple form of nonlinear ODEs under a periodic excitation is studied in [12]. There is no theoretical work in this field to analyze the case of nonlinear DAEs as (1). In this paper for a class of nonlinear functions $\tilde{f}$ and $\tilde{g}$ we give a convergence condition for the dynamic iteration solutions of (1). The criterion comes from an analytically spectral expression of a linear operator. The linear operator is a periodically dynamic iteration operator resulted in system partitions. A nonlinear DAEs example is provided to illustrate the novel condition.

2. Main Results

The parallel dynamic iteration method of (1) is written as

$$\begin{cases} \frac{d}{dt}x^{(k+1)}(t) = f(x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), t), \quad x^{(k+1)}(0) = x^{(k+1)}(T), \\ y^{(k+1)}(t) = g(x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), t), \quad t \in [0,T], \quad k = 0, 1, \ldots, \end{cases} \tag{2}$$

where the function $x^{(0)}(t)$ is an initial guess with $x^{(0)}(0) = x^{(0)}(T)$ and the nonlinear splitting functions $f : (\mathbf{R}^n)^2 \times (\mathbf{R}^m)^2 \times [0,T] \mapsto \mathbf{R}^n$ and $g : (\mathbf{R}^n)^2 \times (\mathbf{R}^m)^2 \times [0,T] \mapsto \mathbf{R}^m$ satisfy

$$f(x, x, y, y, t) = \tilde{f}(x, y, t), \quad g(x, x, y, y, t) = \tilde{g}(x, y, t), \quad t \in [0,T], \tag{3}$$

in which $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$. Typical and important partitions in practical application are the Jacobi and Gauss-Seidel splitting functions.

We let $\mathbf{R}^l$ $(l = m, n)$ be the standard Euclid space equipped with an inner product $\langle \cdot, \cdot \rangle$. The 2-norm $\| \cdot \|$ in $\mathbf{R}^l$ is induced by the inner product. For the splitting functions f and g in (3) we assume that they obey the following condition.

Condition (L). *(1) For $f(\cdot, u_2, u_3, u_4, t)$, on $[0,T]$ there is a positive constant a_1 such that*

$$\langle f(u_1, u_2, u_3, u_4, t) - f(v_1, u_2, u_3, u_4, t), u_1 - v_1 \rangle \le -a_1 \|u_1 - v_1\|^2, \, u_1, v_1 \in \mathbf{R}^n; \tag{4}$$

(2) For $f(u, \cdot, \cdot, \cdot, t)$, on $[0,T]$ there are nonnegative constants a_j $(j = 2, 3, 4)$ such that

$$\|f(u, u_2, u_3, u_4, t) - f(u, v_2, v_3, v_4, t)\| \le \sum_{j=2}^{4} a_j \|u_j - v_j\|, \, u_2, v_2 \in \mathbf{R}^n, \, u_l, v_l \in \mathbf{R}^m (l = 3, 4); \tag{5}$$

(2) For $g(\cdot, \cdot, \cdot, \cdot, t)$, on $[0,T]$ there are nonnegative constants b_j $(j = 1, 2, 3, 4)$ such that

$$\begin{aligned} &\|g(u_1, u_2, u_3, u_4, t) - g(v_1, v_2, v_3, v_4, t)\| \\ &\le \sum_{j=1}^{4} b_j \|u_j - v_j\|, \, u_l, v_l \in \mathbf{R}^n (l = 1, 2), \, u_s, v_s \in \mathbf{R}^m (s = 3, 4). \end{aligned} \tag{6}$$

The inequality of (4) is a strongly dissipative condition. The inequalities (5) and (6) are classical Lipschitz conditions. In this paper we assume that (1) and each approximative system in (2) have periodic solutions.

We denote that $\widetilde{b}_1 = \frac{b_1}{1 - b_3}$, $\widetilde{b}_2 = \frac{b_2}{1 - b_3}$, and $\widetilde{b}_4 = \frac{b_4}{1 - b_3}$. Further, we also let $\widetilde{a}_1 = a_1 - a_3 \widetilde{b}_1$, $\widetilde{a}_2 = a_2 + a_3 \widetilde{b}_2$, and $\widetilde{a}_4 = a_4 + a_3 \widetilde{b}_4$. For $w \in C([0,T], \mathbf{C})$ or $L^2([0,T], \mathbf{C})$, we define a linear operator $\mathcal{R}$ as

$$(\mathcal{R}w)(t) = e^{-\widetilde{a}_1 t}(1 - e^{-\widetilde{a}_1 T})^{-1} \int_0^T e^{-\widetilde{a}_1 (T-s)} w(s) ds + \int_0^t e^{-\widetilde{a}_1 (t-s)} w(s) ds, \quad t \in [0,T]. \tag{7}$$

The operator $\mathcal{R}$ is compact (see [10, 11]), it is also positive due to $(\mathcal{R}w_+)(t) \geq 0$ if $w_+ \geq 0$ for $t \in [0,T]$. The error estimate of the dynamic iteration method (2) could be yielded by Condition (L), namely

Theorem 1. *Assume that $b_3 < 1$ and $a_1 b_3 + a_3 b_1 < a_1$ in (4) - (6), for (1) and (2) we have*

$$\|x^{(k+1)} - x\|(t) \leq \widetilde{a}_2(\mathcal{R}\|x^{(k)} - x\|)(t) + \widetilde{a}_4(\mathcal{R}\|y^{(k)} - y\|)(t), \quad k = 0,1,2,\dots \tag{8}$$

and

$$\begin{array}{rcl}\|y^{(k+1)} - y\|(t) & \leq & \widetilde{b}_2\|x^{(k)} - x\|(t) + \widetilde{b}_4\|y^{(k)} - y\|(t) + \widetilde{a}_2\widetilde{b}_1(\mathcal{R}\|x^{(k)} - x\|)(t) \\ & & \widetilde{a}_4\widetilde{b}_1(\mathcal{R}\|y^{(k)} - y\|)(t), \quad k = 0,1,\dots,\end{array} \tag{9}$$

where $t \in [0,T]$.

Proof. First we denote that $e_x^{(k+1)}(t) = x^{(k+1)}(t) - x(t)$ and $e_y^{(k+1)}(t) = y^{(k+1)}(t) - y(t)$, by use of (4) and (5) it holds

$$\begin{array}{rl} & \langle \frac{d}{dt}e_x^{(k+1)}(t), e_x^{(k+1)}(t)\rangle \\ = & \langle f(x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), t) - f(x(t), x(t), y(t), y(t), t), e_x^{(k+1)}(t)\rangle \\ = & \langle f(x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), t) \\ & -f(x(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), t), e_x^{(k+1)}(t)\rangle \\ & +\langle f(x(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), t) - f(x(t), x(t), y^{(k+1)}(t), y^{(k)}(t), t), e_x^{(k+1)}(t)\rangle \\ & +\langle f(x(t), x(t), y^{(k+1)}(t), y^{(k)}(t), t) - f(x(t), x(t), y(t), y^{(k)}(t), t), e_x^{(k+1)}(t)\rangle \\ & +\langle f(x(t), x(t), y(t), y^{(k)}(t), t) - f(x(t), x(t), y(t), y(t), t), e_x^{(k+1)}(t)\rangle \\ \leq & (-a_1\|e_x^{(k+1)}(t)\| + a_2\|e_x^{(k)}(t)\| + a_3\|e_y^{(k+1)}(t)\| + a_4\|e_y^{(k)}(t)\|)\|e_x^{(k+1)}(t)\|. \end{array} \tag{10}$$

Because $x^{(k+1)}(t)$ and $x(t)$ are continuously differential functions on $[0,T]$, $\frac{d}{dt}\|e_x^{(k+1)}(t)\|$ always exists for all t such that $\|e_x^{(k+1)}(t)\| \neq 0$ where $\|\cdot\|$ is the 2-norm in $\mathbf{R}^n$. Moreover, we have the relationship $\langle \frac{d}{dt}e_x^{(k+1)}(t), e_x^{(k+1)}(t)\rangle = \|e_x^{(k+1)}(t)\|\frac{d}{dt}\|e_x^{(k+1)}(t)\|$ for almost all t such that $\|e_x^{(k+1)}(t)\| \neq 0$. Thus, (10) can be simplified as

$$\frac{d}{dt}\|e_x^{(k+1)}\|(t) \leq -a_1\|e_x^{(k+1)}(t)\| + a_2\|e_x^{(k)}(t)\| + a_3\|e_y^{(k+1)}(t)\| + a_4\|e_y^{(k)}(t)\|. \tag{11}$$

On the other hand, by (6) we analogously acquire

$$\|e_y^{(k+1)}(t)\| \leq b_1\|e_x^{(k+1)}(t)\| + b_2\|e_x^{(k)}(t)\| + b_3\|e_y^{(k+1)}(t)\| + b_4\|e_y^{(k)}(t)\|, \tag{12}$$

that is

$$\|e_y^{(k+1)}\|(t) \leq \widetilde{b}_1\|e_x^{(k+1)}(t)\| + \widetilde{b}_2\|e_x^{(k)}(t)\| + \widetilde{b}_4\|e_y^{(k)}(t)\|. \tag{13}$$

Substituting (13) into (11), we obtain

$$\frac{d}{dt}\|e_x^{(k+1)}\|(t) \leq -\widetilde{a}_1\|e_x^{(k+1)}\|(t) + \widetilde{a}_2\|e_x^{(k)}\|(t) + \widetilde{a}_4\|e_y^{(k)}\|(t). \tag{14}$$

Now we pay attention to a known comparison theorem on differential inequalities (see also [13]), (14) becomes

$$\begin{array}{rcl}\|e_x^{(k+1)}\|(t) & \leq & e^{-\widetilde{a}_1 t}\|e_x^{(k+1)}\|(0) + \widetilde{a}_2\int_0^t e^{-\widetilde{a}_1(t-s)}\|e_x^{(k)}\|(s)ds \\ & & +\widetilde{a}_4\int_0^t e^{-\widetilde{a}_1(t-s)}\|e_y^{(k)}\|(s)ds.\end{array} \tag{15}$$

The above inequality is also valid for $\|e_x^{(k+1)}(t)\| = 0$. We use the periodic constraint $\|e_x^{(k+1)}\|(T) = \|e_x^{(k+1)}\|(0)$, by (15) we yield

$$\|e_x^{(k+1)}\|(0) \le \widetilde{a}_2(1-e^{-\widetilde{a}_1 T})^{-1}\int_0^T e^{-\widetilde{a}_1(T-s)}\|e_x^{(k)}\|(s)ds$$
$$+\widetilde{a}_4(1-e^{-\widetilde{a}_1 T})^{-1}\int_0^T e^{-\widetilde{a}_1(T-s)}\|e_y^{(k)}\|(s)ds.$$

If we substitute this inequality into (15), then (8) is proven.

Finally the error estimate (9) automatically holds as long as we put (8) into (13). This completes the proof of Theorem 1. Q.E.D.

Let us define another linear operator $\mathcal{V}$ in $C([0,T],\mathbf{C}^2)$ or $L^2([0,T],\mathbf{C}^2)$ as

$$\mathcal{V} = \begin{bmatrix} 0 & 0 \\ \widetilde{b}_2 & \widetilde{b}_4 \end{bmatrix} + \begin{bmatrix} \widetilde{a}_2\mathcal{R} & \widetilde{a}_4\mathcal{R} \\ \widetilde{a}_2\widetilde{b}_1\mathcal{R} & \widetilde{a}_4\widetilde{b}_1\mathcal{R} \end{bmatrix}. \tag{16}$$

This operator is also positive. Thus, we may write (8) and (9) together as

$$\begin{bmatrix} \|x^{(k)}-x\| \\ \|y^{(k)}-y\| \end{bmatrix}(t) \le (\mathcal{V}^k \begin{bmatrix} \|x^{(0)}-x\| \\ \|y^{(0)}-y\| \end{bmatrix})(t), \quad t\in[0,T], \quad k=1,2,\ldots. \tag{17}$$

For the linear operator $\mathcal{V}$, as a special case of [11], by computing the Fourier series coefficients of the equalities of its eigenpairs we have an analytic expression about its spectrum.

Theorem 2. *Let $\omega = 2\pi/T$, the spectrum of $\mathcal{V}$ in (16) is*

$$\sigma(\mathcal{V}) = \overline{\bigcup\{\sigma(Q(ip\omega)) : p = 0, \pm 1, \ldots\}}, \tag{18}$$

where $i=\sqrt{-1}$ and

$$Q(ip\omega) = \begin{bmatrix} 0 & 0 \\ \widetilde{b}_2 & \widetilde{b}_4 \end{bmatrix} + \frac{1}{ip\omega+\widetilde{a}_1}\begin{bmatrix} \widetilde{a}_2 & \widetilde{a}_4 \\ \widetilde{a}_2\widetilde{b}_1 & \widetilde{a}_4\widetilde{b}_1 \end{bmatrix}. \tag{19}$$

Proof. First, let $\theta(t) = [\varphi(t),\psi(t)]^t$ and assume (λ,θ) be an eigenpair of $\mathcal{V}$ in $C([0,T],\mathbf{C}^2)$ or $L^2([0,T],\mathbf{C}^2)$. Based on the equality $\mathcal{V}\theta = \lambda\theta$, we compute the corresponding Fourier series coefficients. For $p = 0,\pm 1,\ldots$,

$$\begin{aligned}
\widetilde{(\mathcal{V}\theta)}_p &= \frac{1}{T}\int_0^T (\mathcal{V}\theta)(t)e^{-ip\omega t}dt \\
&= \begin{bmatrix} 0 & 0 \\ \widetilde{b}_2 & \widetilde{b}_4 \end{bmatrix}\begin{bmatrix} \tilde{\varphi}_p \\ \tilde{\psi}_p \end{bmatrix} + \begin{bmatrix} \widetilde{a}_2 & \widetilde{a}_4 \\ \widetilde{a}_2\widetilde{b}_1 & \widetilde{a}_4\widetilde{b}_1 \end{bmatrix}\begin{bmatrix} \widetilde{(\mathcal{R}\varphi)}_p \\ \widetilde{(\mathcal{R}\psi)}_p \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ \widetilde{b}_2 & \widetilde{b}_4 \end{bmatrix}\begin{bmatrix} \tilde{\varphi}_p \\ \tilde{\psi}_p \end{bmatrix} + \frac{1}{ip\omega+\tilde{a}_1}\begin{bmatrix} \widetilde{a}_2 & \widetilde{a}_4 \\ \widetilde{a}_2\widetilde{b}_1 & \widetilde{a}_4\widetilde{b}_1 \end{bmatrix}\begin{bmatrix} \tilde{\varphi}_p \\ \tilde{\psi}_p \end{bmatrix} \\
&= Q(ip\omega)\tilde{\theta}_p,
\end{aligned}$$

it deduces that $Q(ip\omega)\tilde{\theta}_p = \lambda\tilde{\theta}_p$. That is, $\sigma(\mathcal{V}) \subseteq \overline{\bigcup\{\sigma(Q(ip\omega)) : p = 0,\pm 1,\ldots\}}$.

For any given $p \in \{0,\pm 1,\ldots\}$, let (λ,Θ_p) be an eigenpair of $Q(ip\omega)$. We define $\theta_p(t) =$

$e^{ip\omega t}\Theta_p$. It is easy to know

$$\begin{aligned}(\mathcal{V}\theta_p)(t) &= \begin{bmatrix} 0 & 0 \\ \widetilde{b}_2 & \widetilde{b}_4 \end{bmatrix}\theta_p(t) + (\mathcal{R}e^{ip\omega t})\begin{bmatrix} \widetilde{a}_2 & \widetilde{a}_4 \\ \widetilde{a}_2\widetilde{b}_1 & \widetilde{a}_4\widetilde{b}_1 \end{bmatrix}\Theta_p \\ &= e^{ip\omega t}\begin{bmatrix} 0 & 0 \\ \widetilde{b}_2 & \widetilde{b}_4 \end{bmatrix}\Theta_p + \frac{e^{ip\omega t}}{ip\omega + \tilde{a}_1}\begin{bmatrix} \widetilde{a}_2 & \widetilde{a}_4 \\ \widetilde{a}_2\widetilde{b}_1 & \widetilde{a}_4\widetilde{b}_1 \end{bmatrix}\Theta_p \\ &= e^{ip\omega t}Q(ip\omega)\Theta_p \\ &= \lambda_p\theta_p(t).\end{aligned}$$

Thus, $\overline{\bigcup\{\sigma(Q(ip\omega)) : p = 0, \pm 1, \ldots\}} \subseteq \sigma(\mathcal{V})$ since the spectral set $\sigma(\mathcal{V})$ is closed. This completes the proof of Theorem 2. Q.E.D.

A simple convergence condition of (2) can be retrieved from Theorem 2 because $\rho(\mathcal{V}) = \sup\{\rho(Q(ip\omega)) : p = 0, \pm 1, \ldots\}$.

Corollary. *For a given splitting or partition, the dynamic iteration method (2) converges to the exact periodic solution of (1) if*

$$\sup\{\rho(Q(ip\omega)) : p = 0, \pm 1, \ldots\} < 1. \tag{20}$$

3. Numerical Experiments

In this section we do numerical experiments for a five-dimensional test system of nonlinear DAEs as

$$\begin{cases} \dfrac{d}{dt}x_1(t) = -2x_1(t) + 0.25tanh(y_1(t) - x_2(t)), \\ \dfrac{d}{dt}x_2(t) = -2x_2(t) + 0.25tanh(y_1(t) - x_3(t)) + 0.25tanh(y_2(t) - x_3(t)), \\ \dfrac{d}{dt}x_3(t) = -2x_3(t) + 0.25tanh(y_2(t) - x_1(t)) + 1, \\ y_1(t) = 0.25tanh(y_2(t) - y_1(t)) + 0.25tanh(x_3(t) - y_1(t)) + 0.5cos(t), \\ y_2(t) = 0.25tanh(x_1(t) - y_2(t)) + 0.25tanh(x_2(t) - y_2(t)) - 1, \\ x_1(0) = x_1(2\pi), \quad x_2(0) = x_2(2\pi), \quad x_3(0) = x_3(2\pi), \quad t \in [0, 2\pi], \end{cases} \tag{21}$$

where $tanh(z) = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$. We take the following dynamic partition to numerically compute the periodic transient response of (21). For $k = 0, 1, \ldots$, the dynamic iteration method of (21) is

$$\begin{cases} \dfrac{d}{dt}x_1^{(k+1)}(t) = -2x_1^{(k+1)}(t) + 0.25tanh(y_1^{(k)}(t) - x_2^{(k)}(t)), \\ \dfrac{d}{dt}x_2^{(k+1)}(t) = -2x_2^{(k+1)}(t) + 0.25tanh(y_1^{(k)}(t) - x_3^{(k)}(t)) + 0.25tanh(y_2^{(k)}(t) - x_3^{(k)}(t)), \\ \dfrac{d}{dt}x_3^{(k+1)}(t) = -2x_3^{(k+1)}(t) + 0.25tanh(y_2^{(k)}(t) - x_1^{(k)}(t)) + 1, \\ y_1^{(k+1)}(t) = 0.25tanh(y_2^{(k)}(t) - y_1^{(k+1)}(t)) + 0.25tanh(x_3^{(k+1)}(t) - y_1^{(k+1)}(t)) + 0.5cos(t), \\ y_2^{(k+1)}(t) = 0.25tanh(x_1^{(k+1)}(t) - y_2^{(k+1)}(t)) + 0.25tanh(x_2^{(k+1)}(t) - y_2^{(k+1)}(t)) - 1, \\ x_1^{(k+1)}(0) = x_1^{(k+1)}(2\pi), \quad x_2^{(k+1)}(0) = x_2^{(k+1)}(2\pi), \quad x_3^{(k+1)}(0) = x_3^{(k+1)}(2\pi), \quad t \in [0, 2\pi]. \end{cases} \tag{22}$$

This partition satisfies the conditions (4), (5), and (6) where the parameters may be taken as $a_1 = 2$, $a_2 = 0.5$, $a_3 = 0$, $a_4 = 0.25\sqrt{3}$, $b_1 = 0.25\sqrt{2}$, $b_2 = 0$, $b_3 = 0.5$, and $b_4 = 0.25$. The convergence criterion (20) is safeguarded since

$$Q(ip) = \frac{1}{ip+2}\begin{bmatrix} 0.5 & 0.25\sqrt{3} \\ 0.175\sqrt{2} & 0.25ip + 0.5 + 0.0625\sqrt{6} \end{bmatrix}, \quad p = 0, \pm 1, \ldots$$

and $\sup\{\rho(Q(ip)) : p = 0, \pm 1, \ldots\} = 0.4564$. For any fixed k, the differential part of (22) is linear and easily solved at the present form. At each iteration the algebraic part of (22) is nonlinear. It could be computed by the well-known Newton iteration method in function spaces (see also [7]).

Table 1. Numerical results of the dynamic iterations.

Number of the dynamic iterations	Iterative error
1 – 3	$2.2465,\ 5.2250\times10^{-1},\ 1.5833\times10^{-2}$
4 – 6	$1.1596\times10^{-3},\ 1.3952\times10^{-4},\ 1.4676\times10^{-5}$
7 – 9	$1.2354\times10^{-6},\ 1.4945\times10^{-7},\ 1.6320\times10^{-8}$

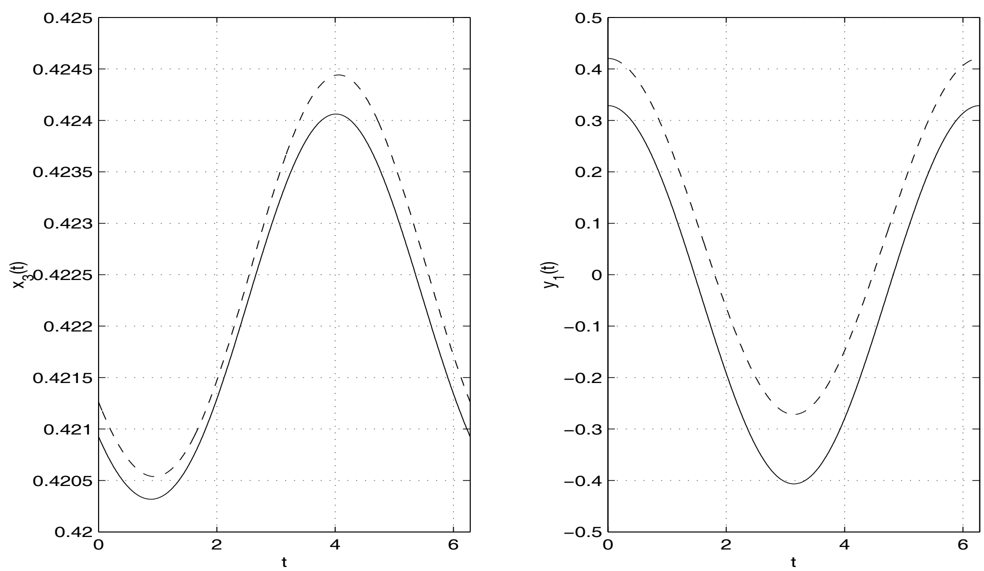

Figure 1. Iterative waveforms $x_3^{(3)}(t)$ and $y_1^{(1)}(t)$ (dashed lines) of the periodic solution functions $x_3(t)$ and $y_1(t)$ (solid lines) in Example.

For the example we let the time-step is $0.005\pi(=\Delta t)$ and the initial guess is the zero function. The number of the Newton iteration is taken as 3 in our code for solving (22). The iterative error is defined as the sum of the squared difference of successive waveforms taken over all time-points, namely for computed waveforms $w^{(l)}(t)(l = k, k+1)$ the iterative error is $E_{ie}^{(k+1)} = \sqrt{\sum_{j=0}^{399} \|w^{(k+1)}(t_j) - w^{(k)}(t_j)\|^2}$ where $t_j = j\Delta t$ $(j = 0, 1, \ldots, 399)$. The computed results of the first nine iterations are given in Table 1. The third iteration waveform $x_3^{(3)}(t)$ and the first iteration waveform $y_1^{(1)}(t)$ for the true responses $x_3(t)$ and $y_1(t)$ of (21) is shown in Figure 1. The approximative phase pictures (dashed lines) of x_1 versus x_3 ($k = 3$) and y_1 versus y_2 ($k = 1$) for their exact phase relationships (solid lines) are drawn in Figure 2.

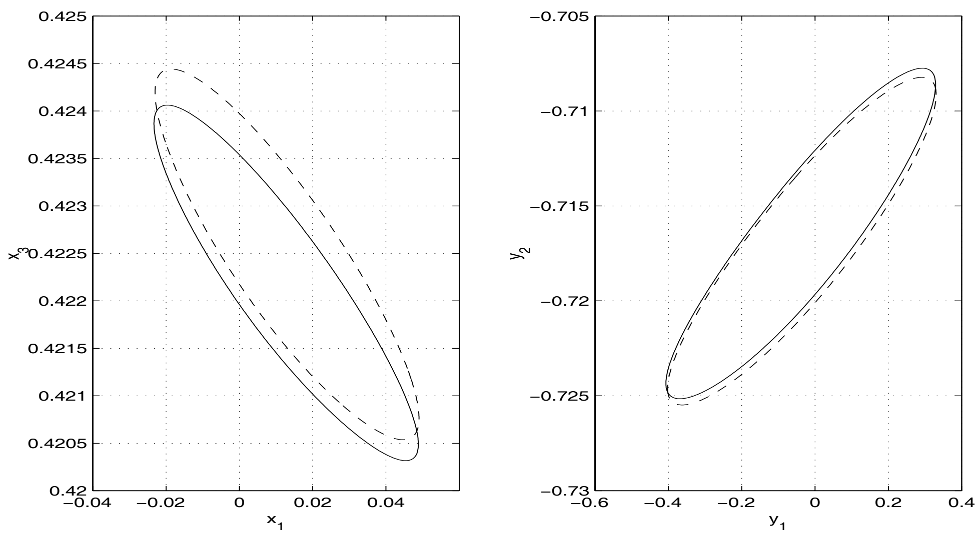

Figure 2. Approximative (dashed lines) and exact (solid lines) phase drawings for $k = 3$ (left) and $k = 1$ (right).

4. Conclusion

By invoking a linear operator theory we successfully gained a simple convergence condition of parallel dynamic iteration method for a class of nonlinear DAEs. The condition reported in this paper is easy to be checked in practical application. This approach suits for computer simulation.

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