Journal of Computational Mathematics, Vol.21, No.3, 2002, 375–382.

STABILITIES OF (A,B,C) AND NPDIRK METHODS FOR SYSTEMS OF NEUTRAL DELAY-DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS *

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Abstract

Consider the following neutral delay-differential equations with multiple delays (NMDDE)

$$y'(t) = Ly(t) + \sum_{j=1}^{m} [M_j y(t-\tau_j) + N_j y'(t-\tau_j)], \quad t \geq 0, \tag{0.1}$$

where $\tau > 0, L, M_j$ and N_j are constant complex-value $d \times d$ matrices. A sufficient condition for the asymptotic stability of NMDDE system (0.1) is given. The stability of Butcher's (A,B,C)-method for systems of NMDDE are studied. In addition, we present a parallel diagonally-implicit iteration RK (PDIRK) methods(NPDIRK) for systems of NMDDE, which is easier to be implemented than fully implicit RK methods. We also investigate the stability of a special class of NPDIRK methods by analyzing their stability behaviors of the solutions of (0.1).

Key words: Neutral delay differential equations, (A,B,C)-method, RK method, Parallel diagonally-implicit iteration RK method.

1. Introduction

Consider the stability behavior in the numerical solution of neutral delay differential equations with multiple delays(NMDDE)

$$y'(t) = f(t, y(t), y(t-\tau_1), \cdots, y(t-\tau_m), y'(t-\tau_1), \cdots, y'(t-\tau_m)), \ t \geq 0, \tag{1.1}$$

$$y(t) = g(t), \qquad -\tau \leq t \leq 0, \tag{1.2}$$

where τ_j are some given positive constants for $j = 1, \cdots, m$, $\tau_m > \tau_{m-1} > \cdots > \tau_1 > 0$, f and g denote given vector-value functions and $y(t)$ is the unknown function to be solved for $t \geq 0$. The purpose of the present work is to investigate the stability properties of Butcher's (A,B,C) (see [2]) and NPDIRK methods by means of the linear test equations of the type (1.1), i.e.

$$\begin{cases} y'(t) = Ly(t) + \sum_{j=1}^{m} [M_j y(t-\tau_j) + N_j y'(t-\tau_j)], \quad t \geq 0, \\ y(t) = g(t), \ -\tau \leq t \leq 0, \end{cases} \tag{1.3}$$

where $\tau > 0, L, M_j$ and N_j are constant complex-value $d \times d$ matrices. The methods represented by matrices (A,B,C) are called as general linear method by Butcher(see([2]). It includes many numerical methods such as RK and LM methods and so on. As an example of (A,B,C) methods, we also obtain the stability result of RK methods. NPDIRK method is a new scheme for numerical solving NMDDE that is presented in this paper.

There have been a number of studies on the numerical stability analysis of system (1.3) with the cases of $m = 1$ and/or $d = 1$ and/or $N_j = 0$. The stability of RK and one-parameter

* Received January 15, 2001; final revised March 13, 2002.

methods for the case $m = 1$ have been investigated in [10,11]. [1] and [9] have studied stability properties of RK methods for $m = 1$ and $d = 1$. The stability analysis of RK methods and linear multistep methods for system (1.3) with $N_j = 0$ $(j = 1, \cdots, m)$ and/or $\tau_j = j\tau$, has been studied in [5,6]. In [15] Zhang & Zhou studied the numerical stability of multistep RK methods for system (1.3).

In this paper, we firstly give a sufficient condition for the asymptotic stability of the test NMDDE (1.3). Then, we extend and generalize the stability results of [1,5,6,9,10,11,15] to the general (A,B,C) methods for numerical solving NMDDE (1.1)(1.2). Finally, in section 5, we present NPDIRK scheme for NMDDE (1.1)(1.2), which is easier to be implemented than fully implicit RK methods on parallel computers for numerically solving NMDDE (1.1) and (1.2). We also study stability properties of a special class of PDIRK methods with respect to (1.3).

2. The Asymptotic Stability of Test Equation (1.3)

Definition 2.1. *The NMDDE (1.3) is called to be asymptotically stable, if for any continuous differentiable initial function $g(t)$ and for any delay $\tau_j > 0$, $(\tau_m > \tau_{m-1} > \cdots > \tau_1)$, the solution of (1.3) $y(t) \to 0$ as $t \to \infty$.*

Let

$$Q(v_1, v_2, \cdots, v_m) := (I - \sum_{j=1}^{m} N_j v_j)^{-1}(L + \sum_{j=1}^{m} M_j v_j), \tag{2.1}$$

$\lambda_i(F)$ and $Re\lambda_i(F)$ stand for the i-th eigenvalue and the real part of i-th eigenvalue, respectively, of any matrix F.

Lemma 2.1. *The system (1.3) is asymptotically stable if the following conditions*

$$\sup Re\lambda_i(Q(v_1, v_2, \cdots, v_m)) < 0, \quad for \ \ all \ \ i \ \ and \ \ v_j \in C \ and \ |v_j| \le 1 \tag{2.2}$$

and

$$\sum_{j=1}^{m} \| N_j \| < 1 \tag{2.3}$$

hold, where $||N_j|| = \sup_{||\xi||=1} ||N_j \xi||$.

The proof is analogous to that of Theorem in [15] we omit it here.

3. Stability of (A,B,C)-Methods for System (1.3)

For the initial-value problem of ODEs

$$y'(t) = f(t, y(t)), \quad t \ge 0, \quad y(0) = y_0, \tag{3.1}$$

An (A,B,C)-method for (3.1) is given in a standard form as (see [2])

$$Y_{n+1} = (A \otimes I)Y_n + h(B_0 \otimes I)F_n + h(B_1 \otimes I)F_{n+1}, \tag{3.2}$$

where I stands for the identity matrix; $A, B_0, B_1 \in R^{r\times r}, Y_{n+1} = (y_{n,1}^T, y_{n,2}^T, \cdots, y_{n,r}^T)^T$, $y_{n,i} \approx y(t_n + \alpha_i h)$, $\alpha_i > 0$, $\alpha_i \ne \alpha_j$ (if $i \ne j$), $F_{n+1} = (f^T(t_n, y_{n,1}), \cdots, f^T(t_n, y_{n,r}))^T$. Define

$$r(\overline{h}) := (I - \overline{h}B_1)^{-1}(A + \overline{h}B_0) \tag{3.3}$$

as the stability matrix of the (A,B,C) method. It is well known that (A,B,C)-method (3.2) is said to be A-stable for ODE (3.1) if

$$(I - \overline{h}B_1) \text{ is regular and } \rho(r(\overline{h})) < 1 \text{ for any } Re(\overline{h}) < 0. \tag{3.4}$$

Definition 3.1. *An (A,B,C)-method for NMDDE (1.3) is called to be asymptotically stable if, under conditions (2.2)(2.3), the numerical solution Y_n of (1.3) at the mesh points t_n satisfies* $\lim_{n\to\infty} Y_n = 0$ *for any stepsize $h > 0$.*

Application of (A,B,C)-method (3.2) to the test linear system (1.3), which can be written as

$$y'(t) = z(t),\ z(t) = Ly(t) + \sum_{j=1}^{m}[M_j y(t-\tau_j) + N_j z(t-\tau_j)], \tag{3.5}$$

yields

$$Y_{n+1} = (A \otimes I_d)Y_n + h(B_0 \otimes I_d)Z_n + h(B_1 \otimes I_d)Z_{n+1}, \tag{3.6}$$

$$Z_{n+i} = (I_r \otimes L)Y_{n+i} + \sum_{j=1}^{m}\left((I_r \otimes M_j)Y_{n-l_j+\delta j+i} + (I_r \otimes N_j)Z_{n-l_j+\delta_j+i}\right). \tag{3.7}$$

For $n = 1, 2, \cdots$ and $i = 0, 1, Y_{n-l_j+\delta_j+i} = g((n - l_j + \delta_j + i)h), Z_{n-l_j+\delta_j+i} = g'((n - l_j + \delta_j + i)h)$ with $(n - l_j + \delta_j + i) \le 0$, $Y_{n-l_j+\delta_j+i}$ and $Z_{n-l_j+\delta_j+i}$ with $(n - l_j + \delta_j + i) \ge 0$ are defined by the interpolation procedure which was proposed by in 't Hout[7]. That is

$$Y_{n-l_j+\delta_j+i} = \sum_{p=-v}^{s} L_p(\delta_j) Y_{n-l_j+p+i}, \tag{3.8}$$

$$Z_{n-l_j+\delta_j+i} = \sum_{p=-v}^{s} L_p(\delta_j) Z_{n-l_j+p+i} \tag{3.9}$$

for $0 \le \delta_j < 1,\ j = 1, \cdots, m$ and

$$L_p(\delta_j) = \prod_{k=-v,k\ne p}^{s} \frac{(\delta_j - k)}{(p-k)}, \tag{3.10}$$

where $v, s \ge 0$ are integer, $v \le s \le v + 2$, and $l_j = \lceil \tau_j h^{-1} \rceil, \delta_j = l_j - \tau_j h^{-1}, \delta_j \in [0, 1)$ for $j = 1, \cdots, m,\ l_m \ge \cdots \ge l_1 \ge s + 1$, $\lceil q \rceil$ denotes the smallest integer that is greater than or equal to $q \in R$.

Consider the following polynomial

$$\gamma(z, \delta) = \sum_{p=-v}^{s} L_p(\delta) z^{p+v},$$

$z \in C, \delta \in [0, 1)$ and $L_p(\delta)$ is given by (3.10). We quote the following lemma, which can be found in [5,6,8,11,15].

Lemma 3.1. *(i) The inequality $|\gamma(z, \delta)| \le 1$ holds if and only if $v \le s \le v + 2$, whenever $|z| = 1$ and $\delta \in [0, 1)$. (ii) If $v \le s \le v + 2,\ v + s > 0,\ |z| = 1, \delta \in (0, 1)$, then $|\gamma(z, \delta)| = 1$ if and only if $z = 1$.*

Lemma 3.2. (Lancaster, 1969) *Let φ be a polynomial in two variables*

$$\varphi(x, y) = \sum_{i,j=0}^{q} r_{ij} x^i y^j,$$

A, B are square matrices of dimension s and m, respectively, $\varphi(A, B)$ denotes the composite matrix

$$\varphi(A, B) = \sum_{i,j=0}^{q} r_{ij} A^i \otimes B^j.$$

If $\lambda_1, \lambda_2, \cdots, \lambda_s$ are the eigenvalues of A and $\mu_1, \mu_2, \cdots, \mu_s$ are the eigenvalues of B, then the eigenvalues of $\varphi(A, B)$ are the numbers $\varphi(\lambda_i, \mu_j), i = 1(1)s, j = 1(1)m$. Here $\otimes$ denotes the Kronecker product.

Theorem 3.1. *Assume that*

(i) the assumptions of Lemma 2.1 hold;

(ii) the underlying (A,B,C) method satisfies $Re\lambda_i(B_1) \geq 0$ for $i = 1, 2, \cdots, r$;

(iii) the interpolation procedures (3.8)(3.9) satisfy $v \leq s \leq v + 2$.

Then the underlying (A,B,C)-method for NMDDE (1.3) is asymptotically stable if and only if it is A-stable for ODEs.

Proof. The proof is similar to that of Theorem 4.1 in Hu et al.(1995). From (3.6) we get

$$\begin{aligned} Y_{n+1-l_j+p} = (A \otimes I_d)Y_{n-l_j+p} + h(B_0 \otimes I_d)Z_{n-l_j+p} \\ +h(B_1 \otimes I_d)Z_{n+1-l_j+p}. \end{aligned} \tag{3.11}$$

By multiplying $L_p(\delta_j)$ to (3.11) and summing about p we obtain

$$\begin{aligned} \textstyle\sum_{p=-v}^{s} L_p(\delta_j)Y_{n+1-l_j+p} = (A \otimes I_d) \sum_{p=-v}^{s} L_p(\delta_j)Y_{n-l_j+p} \\ +h \textstyle\sum_{i=0}^{1}(B_i \otimes I_d) \sum_{p=-v}^{s} L_p(\delta_j)Z_{n-l_j+p+i}. \end{aligned} \tag{3.12}$$

Multiplying $(I_r \otimes N_j)$ on both sides of the equation (3.12) and summing about j for $j = 1, \cdots, m$, then the above equation becomes

$$\begin{aligned} \textstyle\sum_{j=1}^{m}(I \otimes N_j) \sum_{p=-v}^{s} L_p(\delta_j)Y_{n+1-l_j+p} = \sum_{j=1}^{m}(A \otimes N_j) \sum_{p=-v}^{s} L_p(\delta_j)Y_{n-l_j+p} \\ +h \textstyle\sum_{i=0}^{1} \sum_{j=1}^{m}(B_i \otimes N_j) \sum_{p=-v}^{s} L_p(\delta_j)Z_{n-l_j+p+i}. \end{aligned} \tag{3.13}$$

From (3.6)-(3.9) we obtain

$$\begin{aligned} Y_{n+1} = (A \otimes I)Y_n + h \textstyle\sum_{i=0}^{1} \sum_{j=1}^{m}(B_i \otimes M_j) \sum_{p=-v}^{s} L_p(\delta_j)Y_{n+i-l_j+p} \\ +h \textstyle\sum_{i=0}^{1}(B_i \otimes L)Y_{n+i} + h \sum_{i=0}^{1} \sum_{j=1}^{m}(B_i \otimes N_j) \sum_{p=-v}^{s} L_p(\delta_j)Z_{n-l_j+p+i}, \end{aligned}$$

which, together with (3.13), implies

$$\begin{aligned} Y_{n+1} = (A \otimes I)Y_n + h \textstyle\sum_{i=0}^{1} \sum_{j=1}^{m}(B_i \otimes M_j) \sum_{p=-v}^{s} L_p(\delta_j)Y_{n+i-l_j+p} \\ +h \textstyle\sum_{i=0}^{1}(B_i \otimes L)Y_{n+i} + \sum_{j=1}^{m}(I \otimes N_j) \sum_{p=-v}^{s} L_p(\delta_j)Y_{n-l_j+p+1} \\ - \textstyle\sum_{j=1}^{m}(A \otimes N_j) \sum_{p=-v}^{s} L_p(\delta_j)Y_{n-l_j+p}. \end{aligned}$$

The characteristic equation of the above difference equation is

$$\det P(z) = 0, \tag{3.14}$$

where

$$\begin{aligned} P(z) = z^{l_m+1}I_{rd} - (A \otimes I)z^{l_m} - h \textstyle\sum_{i=0}^{1} \sum_{j=1}^{m}(B_i \otimes M_j) \sum_{p=-v}^{s} L_p(\delta_j)z^{l_m-l_j+p+i} \\ -h \textstyle\sum_{i=0}^{1}(B_i \otimes L)z^{l_m+i} - \sum_{j=1}^{m}(I \otimes N_j) \sum_{p=-v}^{s} L_p(\delta_j)z^{l_m-l_j+p+1} \\ + \textstyle\sum_{j=1}^{m}(A \otimes N_j) \sum_{p=-v}^{s} L_p(\delta_j)z^{l_m-l_j+p}. \end{aligned}$$

Assume that z ($|z| \geq 1$) is a root of the characteristic equation (3.14). Let

$$\widehat{v}_j = \sum_{p=-v}^{s} L_p(\delta_j)z^{p-l_j} = \gamma(z, \delta_j)z^{-l_j-v},$$

then for $l_j \geq s+1$, $(j = 1, \cdots, m)$, $\widehat{v}_j = \sum_{p=-v}^{s} L_p(\delta_j)z^{p-l_j}$ is an analytic function for $|z| \geq 1$. When $|z| = 1$, we get $|\widehat{v}_j| \leq 1$ for $\delta_j \in [0, 1)$ by Lemma 3.1; when $z = \infty$, $|\widehat{v}_j| = 0$ since $l_j \geq s+1$. Thus by employing the maximum modulus principle for analytic functions we obtain

$$|\widehat{v}_j| \leq 1 \text{ for } |z| \geq 1,\ \delta_j \in [0, 1). \tag{3.15}$$

Noticing the condition (2.2) and the inequality (3.15), we have $Re\lambda_i(Q(\widehat{v}_1,\cdots,\widehat{v}_m))<0$ for all $i=1,\cdots,d,\ 0\le\delta_i<1,\ |\widehat{v}_i|\le 1$ whenever $|z|\ge 1$. From condition (ii) we can obtain that $\det(I-hB_1\otimes Q(\widehat{v}_1,\cdots,\widehat{v}_m))\ne 0$. Using the notations about $\widehat{v}_j$ and $Q(v_1,v_2,\cdots,v_m)$, $P(z)$ can be written as

$$\begin{aligned}
P(z) &= z^{l_m+1}(I_{rd}-\textstyle\sum_{j=1}^m (I\otimes N_j)\widehat{v}_j) - z^{l_m}(A\otimes(I-\sum_{j=1}^m N_j\widehat{v}_j))\\
&\quad -h\textstyle\sum_{i=0}^1 B_i\otimes(L+\sum_{j=1}^m M_j\widehat{v}_j)z^{l_m+i}\\
&= z^{l_m}\left(I\otimes(I-\textstyle\sum_{j=1}^m N_j\widehat{v}_j)\right)((zI-A\otimes I-h(B_0+B_1z)\otimes Q(\widehat{v}_1,\widehat{v}_2,\cdots,\widehat{v}_m))\\
&= z^{l_m+1}\left(I\otimes(I-\textstyle\sum_{j=1}^m N_j\widehat{v}_j)\right)(I-hB_1\otimes Q(\widehat{v}_1,\cdots,\widehat{v}_m))\\
&\quad -z^{l_m}\left(I\otimes(I-\textstyle\sum_{j=1}^m N_j\widehat{v}_j)\right)(A\otimes I+hB_0\otimes Q(\widehat{v}_1,\cdots,\widehat{v}_m))\\
&= z^{l_m}\left(I\otimes(I-\textstyle\sum_{j=1}^m N_j\widehat{v}_j)\right)(I-hB_1\otimes Q(\widehat{v}_1,\cdots,\widehat{v}_m))\\
&\quad (zI-r(hQ(\widehat{v}_1,\cdots,\widehat{v}_m))),
\end{aligned}$$

where $r(hQ(\widehat{v}_1,\cdots,\widehat{v}_m)):=(I-hB_1\otimes Q)^{-1}\ (A\otimes I+hB_0\otimes Q)$. So,

$$\begin{aligned}
\det P(z) = z^{rdl_m}\left(\det[I-\textstyle\sum_{j=1}^m N_j\widehat{v}_j]\right)^r\cdot\det[I-hB_1\otimes Q(\widehat{v}_1,\cdots,\widehat{v}_m)]\\
\cdot\det[zI-r(hQ(\widehat{v}_1,\cdots,\widehat{v}_m))].
\end{aligned}$$

Since $\det[I-\sum_{j=1}^m N_j\widehat{v}_j]\ne 0$ and $\det[I-hB_1\otimes Q(\widehat{v}_1,\cdots,\widehat{v}_m)]\ne 0$ and z ($|z|\ge 1$) is a root of the characteristic equation $\det P(z)=0$, this means that the root z of the characteristic equation should satisfy

$$\det[zI_d-r\,(hQ(\widehat{v}_1,\cdots,\widehat{v}_m))]=0.$$

From the spectral mapping theorem we have the identity

$$\lambda_i(r(hQ(\widehat{v}_1,\cdots,\widehat{v}_m)))=r(\lambda_i(hQ(\widehat{v}_1,\cdots,\widehat{v}_m))).$$

Henceforth from condition (ii) of the Theorem, we can conclude

$$|r(\lambda_i(hQ(\widehat{v}_1,\cdots,\widehat{v}_m)))|<1\Longrightarrow|\lambda_i(r(hQ(\widehat{v}_1,\cdots,\widehat{v}_m)))|<1,$$

which means

$$\det[zI_d-r\,(hQ(\widehat{v}_1,\cdots,\widehat{v}_m))]=0\Longrightarrow|z|<1.$$

This contradicts with the assumption $|z|\ge 1$.

Suppose that an (A,B,C)-method for NMDDE (1.3) is asymptotically stable. Let $m=1$, $M_j=N_j=0$. We can obtain that it also is A-stable for ODEs and the proof is completed.

4. Stability of RK Methods for System (1.3)

A general s-stage implicit RK method for ODEs can be expressed as

$$\begin{cases} K_n=e\otimes y_n+hA_1\otimes F_1(K_n),\\ y_{n+1}=y_n+hb^T\otimes F_1(K_n),\end{cases}\tag{4.1}$$

where $t_n=nh$ ($n=0,1,2,\cdots$) are the grid points of the discretization, h is the stepsize, $y_n\approx y(t_n)$, $K_n=(K_{n,1},K_{n,2},\cdots,K_{n,s})^T\approx(y(t_n+c_1h),y(t_n+c_2h),\cdots,y(t_n+c_sh))^T$, $e=(1,\cdots,1)^T$, $F_1(K_n)=(f(K_{n,1}),\cdots,f(K_{n,s}))^T$, $b=(b_i)$ and $c=(c_i)$ are s-dimensional vectors, $A_1=(a_{ij})$ is an $s\times s$ matrix. The notations $A_1\otimes F$ and $b^T\otimes F$ stand for $(A_1\otimes I)F$ and $(b^T\otimes I)F$, respectively.

Let

$$Y_{n+1} := \begin{pmatrix} K_n \\ y_{n+1} \end{pmatrix}, \; F_{n+1} := \begin{pmatrix} F_1(K_n) \\ f(y_{n+1}) \end{pmatrix}, \; A = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \otimes I.$$

Then s-stage RK method (4.1) can be seen as an (A,B,C) method

$$Y_{n+1} = AY_n + hB_1F_{n+1}, \tag{4.2}$$

where

$$B_1 = \begin{pmatrix} A_1 \otimes I & 0 \\ b^T \otimes I & 0 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ b^T & 0 \end{pmatrix} \otimes I.$$

Lemma 4.1. *The eigenvalues and spectral abscissa of the composite matrix $A_1 \otimes I$ are*

$$\lambda_i(A_1 \otimes I) = \lambda_i(A_1),$$
$$\rho(A_1 \otimes I) = \rho(A_1).$$

By simple calculation, Runge-Kutta method as a special case of (A,B,C) method (3.2), we can obtain

$$r(hQ(\widehat{v}_1, \cdots, \widehat{v}_m)) = \begin{bmatrix} 0 & (I - hA_1 \otimes Q)^{-1}(e \otimes I) \\ 0 & I + (hb^T \otimes Q)(I - hA_1 \otimes Q)^{-1}(e \otimes I) \end{bmatrix}.$$

Its eigenvalues are the same as those of the stability matrix of Runge-Kutta method (4.1). Therefore, from Theorem 3.1, Lemma 4.1 and (4.2) we can obtain the following conclusion.

Theorem 4.1. *Assume that*

(i) the assumptions of Lemma 2.1 hold;

(ii) the underlying RK method satisfies $Re\lambda_i(A_1) \geq 0$ for $i = 1, 2, \cdots, s$;

(iii) the interpolation procedures (3.8)(3.9) satisfy $\upsilon \leq s \leq \upsilon + 2$.

Then the RK method applied to NMDDE (1.3) is asymptotically stable if and only if it is A-stable for ODEs.

Remark. There are a lot of RK methods satisfying the conditions of Theorem 4.1.

5. NPDIRK Scheme for NMDDE (1.1) and Stability of a Special Class of NPDIRK Methods

The so-called PDIRK methods are parallel diagonally-implicitly iterated RK methods for the parallel numerical integration of ODEs. These methods have many important computational advantages over fully implicit RK methods: they preserve their stability properties and stage order, while the computational cost involved in their implementation is similar to diagonally implicit RK(DIRK) methods (cf. [3,4,12,14]). Another attractive feature of PDIRK method is the availability of embedded formulae of lower orders which make them an ideal starting point for developing variable-order/variable-step code.

PDIRK methods have above many advantages over fully implicit RK methods, however, the question that they whether or not can be applied to NDDEs and preserve the asymptotic stability of the analytical solution of system (1.3) has not been investigated. This paper gives a partial answer for the question.

In the following we shall studied the stability of a special class of PDIRK methods (see, for example, [3, 4, 12, 14]), which satisfies

$$D = dI \quad and \quad \rho(I - D^{-1}A) = 0, \tag{5.1}$$

where D is a diagonal matrix , d is real constants and $\rho(\cdot)$ stands for the spectral radius.

Define NPDIRK scheme for system (1.1) by:

$$\begin{array}{c} K_n^{(k+1)} - F(et_n + ch,\ y_n + hD\otimes K_n^{(k+1)},\ y_{n-l_1+\delta_1} + hD\otimes K_{n-l_1+\delta_1}, \\ \cdots,\ y_{n-l_m+\delta_m} + hD\otimes K_{n-l_m+\delta_m}, K_{n-l_1+\delta_1},\ \cdots,\ K_{n-l_m+\delta_m}) \\ = F(et_n + ch,\ y_n + hA\otimes K_n^{(k)},\ y_{n-l_1+\delta_1} + hA\otimes K_{n-l_1+\delta_1}, \\ \cdots,\ y_{n-l_m+\delta_m} + hA\otimes K_{n-l_m+\delta_m},\ K_{n-l_1+\delta_1},\ \cdots,\ K_{n-l_m+\delta_m}) \\ -F(et_n + ch,\ y_n + hD\otimes K_n^{(k)},\ y_{n-l_1+\delta_1} + hD\otimes K_{n-l_1+\delta_1}, \cdots, \\ y_{n-l_m+\delta_m} + hD\otimes K_{n-l_m+\delta_m},\ K_{n-l_1+\delta_1},\ \cdots,\ K_{n-l_m+\delta_m}), \\ (k = 0, 1, \cdots, k_0), \end{array} \tag{5.2}$$

$$y_{n+1}^{(k_0)} = y_n + hb^T\otimes K_n^{(k_0)}, \tag{5.3}$$

where $D\otimes K_n = (D\otimes I)K_n,\ A\otimes K_n = (A\otimes I)K_n,\ b^T\otimes K = (b^T\otimes I)K_n,\ K_n = (K_{n,1}, \cdots K_{n,s})^T$, $F(et_n+ch,\ y_n+hA\otimes K_n, y_{n-l_1+\delta_1}+hA\otimes K_{n-l_1+\delta_1}, \cdots, y_{n-l_m+\delta_m}+hA\otimes K_{n-l_m+\delta_m}, K_{n-l_1+\delta_1}, \cdots,\ K_{n-l_m+\delta_m}) = (f(t_n + c_1h, y_n + h\sum_{j=1}^{s} a_{1,j}K_{n,j},\ y_{n-l_1+\delta_1} + h\sum_{j=1}^{s} a_{1,j}K_{n-l_1+\delta_1,j}, \cdots, y_{n-l_m+\delta_m} + h\sum_{j=1}^{s} a_{1,j}\ K_{n-l_m+\delta_m,j},\ K_{n-l_1+\delta_1,1},\ \cdots,\ K_{n-l_m+\delta_m,1}),\ \cdots,\ f(t_n + c_sh,\ y_n + h\sum_{j=1}^{s} a_{sj}K_{n,j},\ y_{n-l_1+\delta_1} + h\sum_{j=1}^{s} a_{s,j}K_{n-l_1+\delta_1,j},\ \cdots,\ y_{n-l_m+\delta_m} + h\sum_{j=1}^{s} a_{s,j}K_{n-l_m+\delta_m,j}, K_{n-l_1+\delta_1,m}, \cdots, K_{n-l_m+\delta_m,m})^T$. For the linear system (1.3), we have

$$\begin{array}{c} K_n^{(k+1)} - (I\otimes L)(e\otimes y_n + hD\otimes K_n^{(k+1)}) = (I\otimes L)(e\otimes y_n + hA\otimes K_n^{(k)}) \\ +\sum_{j=1}^{m}(I\otimes M_j)(y_{n-l_j+\delta_j} + hA\otimes K_{n-l_j+\delta_j}) + \sum_{j=1}^{m}(I\otimes N_j)K_{n-l_j+\delta_j} \\ -(I\otimes L)(e\otimes y_n + hD\otimes K_n^{(k)}),\ (k = 0, 1, \cdots, k_0). \end{array} \tag{5.4}$$

That is

$$\begin{array}{l} K_n^{(k+1)} = XK_n^{(k)} + (I - hD\otimes L)^{-1}[(e\otimes L)y_n + \sum_{j=1}^{m}(e\otimes M_j)y_{n-l_j+\delta_j} + \\ \quad +h\sum_{j=1}^{m}(A\otimes M_j)K_{n-l_j+\delta_j} + \sum_{j=1}^{m}(I\otimes N_j)K_{n-l_j+\delta_j}, \end{array}$$

where $X = h(I - hD\otimes L)^{-1}((A-D)\otimes L)$ is the iterative matrix of NPDIRK method (5.3)(5.4). It is easy to see that the condition (5.1) implies $X^{k_0} = 0,\ (k_0 \geq s)$. Under the condition (5.1), a calculation yields

$$\begin{array}{l} (I - hA\otimes L)^{-1}K_n^{(k_0)} = (e\otimes L)y_n + \sum_{j=1}^{m}(e\otimes M_j)y_{n-l_j+\delta_j} + \\ \quad +h\sum_{j=1}^{m}(A\otimes M_j)K_{n-l_j+\delta_j} + \sum_{j=1}^{m}(I\otimes N_j)K_{n-l_j+\delta_j}. \end{array} \tag{5.5}$$

Since (5.3)(5.5) can be rewritten as

$$\begin{array}{l} \begin{bmatrix} I - h(A\otimes L) & 0 \\ -hb^T\otimes I & I \end{bmatrix} \begin{bmatrix} K_n^{(k_0)} \\ y_{n+1}^{(k_0)} \end{bmatrix} = \begin{bmatrix} 0 & e\otimes L \\ 0 & I \end{bmatrix} \begin{bmatrix} K_{n-1} \\ y_n \end{bmatrix} \\ + \sum_{i=1}^{m} \begin{bmatrix} h(A\otimes M_i) + I\otimes N_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sum_{p=-r}^{s} L_p(\delta_i)K_{n-l_i+p} \\ \sum_{p=-r}^{s} L_p(\delta_i)y_{n-l_i+1+p} \end{bmatrix} \\ \quad + \sum_{i=1}^{m} \begin{bmatrix} 0 & e\otimes M_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sum_{p=-r}^{s} L_p(\delta_i)K_{n-l_i-1+p} \\ \sum_{p=-r}^{s} L_p(\delta_i)y_{n-l_i+p} \end{bmatrix}, \end{array}$$

its characteristic equation is the same as RK method (4.1), hence we obtain

Theorem 5.1. *Assume that*

(i) the assumptions of Lemma 2.1 hold;

(ii) the corrector method of PDIRK methods, that is the underlying RK, is A-stable for ODEs and $Re\lambda_i(A) \geq 0$ *for* $i = 1, 2, \cdots, s$;

(iii) the interpolation procedure (3.8)(3.9) satisfies $r \leq s \leq r + 2$;

(iv) $D = dI$ and $\rho(I - D^{-1}A) = 0$ in NPDIRK method (5.2)(5.3).

Then the resulting NPDIRK difference systems of (5.3)(5.4) corresponding to (1.3) is asymptotically stable.

6. Conclusions

In this paper, we firstly give a sufficient condition for the asymptotic stability of the system (1.3). Then we extend and complement the results of Zhang & Zhou[15], Kuang & Xiang & Tian [10,11,13], Hu et.al. [1,5,6,9] about stability properties of numerical methods for DDEs or NDDEs to the systems of NMDDE (1.3). We observe that under some conditions, they possess the same stability properties. Finally, we give a NPDIRK scheme for NMDDE, and we also study the stability properties of a class of NPDIRK methods with respect to (1.3). It can be found that the characteristic equation for this class of method is the same as that for RK method and, as a consequence, it has the same stability with respect to (1.3) as the underlying RK method.

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