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SYMMETRIC POINT STRUCTURE OF SUPERCONVERGENCE FOR CUBIC TRIANGULAR ELEMENTS -A CONSULTATION WITH ZHU [*1)]

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Abstract

Superconvergence structures for rectangular and triangular finite elements are summarized. Two debatable issues in Zhu's paper [18] are discussed. A superclose polynomial to cubic triangular finite element is constructed by area coordinate.

Key words: Cubic triangular element, Superconvergence, Symmetric points.

1. Summary on Superconvergence Structures

Suppose that domain Ω is a square with the boundary Γ and triangulation J^h in Ω is uniform. We shall discuss n-degree triangular family $P_n = \sum_{i+j\leq n} b_{ij}x^iy^j$ and n-degree rectangular family Q_n. Denote by $S_0^h = \{v \in H^1(\Omega), v|_\tau \in P_n(\text{or } Q_n),\ \tau \in J^h,\ v = 0 \text{ on } \Gamma\}$ the n-degree finite element subspace. The solution $u \in H_0^1(\Omega)$ of second order elliptic problem and its finite element approximation (Ritz-projection) $u_h \in S_0^h$ satisfy the orthogonal relation

$$A(u - u_h, v) = 0, \quad v \in S_0^h, \tag{1}$$

where the bilinear form $A(u,v) = \int_\Omega (a_{ij}D_iuD_jv + a_{00}uv)dx$ is assumed to be bounded and H_0^1-coercive. Denote by $W^{k,p}(\Omega)$ Sobolev space with norm $||u||_{k,p,\Omega}$. If $p = 2$, simply use $H^k(\Omega)$ and $||u||_{k,\Omega}$. It is well known that there are the error estimates

$$||u - u_h||_{l,\infty,\Omega} = O(h^{n+1-l}\ln h),\ l = 0, 1. \tag{2}$$

But, u_h or Du_h at some specific points possibly possess the higher rate of convergence (called superconvergence by Douglas).

In the conference on superconvergence in finite element method on March 15-30, 2000, at Berkeley, two chairmen Babuska and Wahlbin claimed that there are three present schools of superconvergence, i.e Ithaca (Locally symmetry theory [12,13,14]), Texas (Method based on the computer [1,2]) and China (Element othogonality analysis, see [6,7,11]). In another conference on three-dimensional finite elements on August 2000 at Jyvaskyla, Brandts and Krizek [3] also summarized three different methods of three schools.

From numerous researches on superconvergence up to now, we know that there are two basic structures of superconvergcne, i.e. Gauss-Lobatto points and symmetric points. Firstly, for regular rectangular element $u_h \in Q_\lambda(n) = \sum_{(i,j)\in I_{n,\lambda}} b_{ij}x^iy^j$, where $I_{n,\lambda} = \{(i,j)|0 \leq i,j \leq n, i+j \leq n+\lambda\}, 1 \leq \lambda \leq n$, we early known [5,6,10,19] that u_h and its gradient Du_h have superconvergence at $n+1$-order Lobatto points and n- order Gauss points, respectively. Besides, if n is odd, the average gradient $\bar{D}u_h$ has superconvergence at vertexes and n-order Gauss points on each side of the element. Secondly, if the number of parameter is decreased, it reduces to the rectangular defective (or serendipity) family $Q'(n) = P_n \oplus span\{x^ny, xy^n\}$ and

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n-degree triangular family $P_n = \sum_{i+j\le n} b_{ij}x^iy^j$. At this time, u_h (for even n) and the average gradient $\bar{D}u_h$ (for odd n) have superconvergence at symmetric points T_h, where T_h consists of four vertexes, four side midpoints and center for rectangular element (see [1,2,9,15]), and three vertexes and three side midpoints for triangular element (see [1,2, 6,7,8,12,13,14,16]).

Here, an interesting topic for us is that whether there exist other superconvergence points for triangular elements, besides symmetric points. We should point out that Wahlbin [12,13,14] first time proved superconvergence at locally symmetric points in quite extensive framework. Of course, their paper has not given the answer to the question mentioned above. However, Babuska [1,2] have calculated the derivative D_xu_h in a triangle for $1 \le n \le 7$ based on the computer and have pointed out that the midpoint of a side parallel to x-axis is only superconvergence point for D_xu_h (but the averaging have not been considered) for $n = 1, 3, 5, 7$, and have found no other points. And no superconvergence point of D_xu_h for $n = 4, 6$, but, $n = 2$ is an exceptional case, 2-Gauss points on this side are superconvergent. Recently, Babuska and Strouboulis have depicted a fig. 4.7*.8 for D_xu_h of the cubic triangular element in their new book [2] and especially emphasized that "Note that the mid-points of the sides which are parallel to the x-axis are the only superconvergence points in the case of the Poisson equation". We also proved [8] that $\bar{D}_xu_h$ for cubic triangular element u_h has no superconvergence points, besides symmetric points, and there are no superconvergence points for u_h itself at all. We exhibit the numerical examples in a square $\Omega = \{0, x, y < 1\}$ as follows.

Consider an elliptic problem $-\triangle u = f$ in Ω, $u = 0$ on $\Gamma_0 = \{x = 0, 0 < y < 1\} \cup \{y = 0, 0 < x < 1\}$ and $D_nu = 0$ on $\Gamma_1 = \{x = 1, 0 < y < 1\} \cup \{y = 1, 0 < x < 1\}$. The exact solution $u = (13x - 8x^2 + x^3)(2y - y^2)$. Ω is subdivided into regular triangular uniform meshes J^h, $h = 1/N, N = 4, 8$. We have calculated the cubic finite element u_N and its error $e_N = u - u_N$ in the following table 1.

The table 1. The error e_4, e_8 at nodes and the ratio $e_4 : e_8$

	$x = 1/4$	1/2	3/4	1
$y = 1/4$	2.389E-4	1.913E-4	1.669E-4	1.436E-4
	1.207E-5	1.117E-5	9.489E-6	8.393E-6
	18.86	17.13	17.57	17.11
$y = 1/2$	2.399E-4	1.935E-4	1.699E-4	1.593E-4
	1.393E-5	1.243E-4	1.074E-5	9,521E-6
	17.22	15.57	15.82	16.73
$y = 3/4$	2.680E-4	2.142E-4	1.876E-4	1.783E-4
	1.503E-5	1.355E-5	1.187E-5	1.063E-5
	17.83	15.81	15.80	16.77
$y = 1$	2.510E-4	2.200E-4	1.910E-4	3.630E-4
	1.501E-5	1.421E-5	1.252E-5	2.274E-6
	15,78	15.48	15.26	16.00

We see that when triangulation is refined twice, the error ratio $e_4 : e_8 = 15.3 \sim 18.9$, thus the cubic triangular element has only the accuracy $O(h^4)$ at nodes, no superconvergence. A detailed data analysis shows that its accuracy at nodes is the worst. The facts mentioned above show that the cubic triangular elements do not possess Gauss-Lobatto point structure of superconvergence, which is of a great difference from the regular rectangular elements.

2. Discussion on Zhu's paper [18]

Early Chen [4] proved by the element analysis that the average gradient $\bar{D}u_h$ for triangular linear element has superconvergence at six symmetric points in each triangular element. Later, Zhu [16] proved by this method that the quadratic triangular element u_h itself has superconvergence at six symmetric points. Although the natural superconvergcne points within an element for the gradient Du_h has not found, but the tangent derivative $D_\tau u_h$ along each side has superconvergence at 2-Gauss points on this side. It is a pity that these results make Zhu produce an illusion that triangular elements on each side have same superconvergence structure as that of the rectangular elements.

In recent years Zhu proposed a proposition that any degree triangular elements have also Gauss-Lobatto structure of superconvergence along each side, like that of the regular rectangular elements. He claimed [17] that "we find third order Gauss points (for the tangent derivative) and third order Lobatto points" for cubic triangular elements. Recently, he repeated this conclusion in his paper [18] "Superconvergence analysis for cubic triangular element of the finite element" published in Journal of Computational Mathematics, Vol.18, No.5, 2000, pp.541-550. From the analysis in the last section, we see that Zhu's proposition is contradictory to present theoretical analysis and numerical results. He disregarded Babuska's results mentioned above.

Below, we shall point out another essential mistake of Zhu's proof in [18]. Assume that u is quartic polynomial and u_I is cubic superclose polynomial to u_h, then the remainder $R = u - u_I$ should consist of five independent terms, in which three terms can be defined by the values of u on three sides σ in a triangle $E = \{0 < s, t < 1, s + t < 1\}$, whereas other two terms E_1, E_2 will disappear on σ. Using the area coordinate $\lambda_1 = s, \lambda_2 = t, \lambda_3 = 1 - s - t$, one has four functions $\phi = \lambda_1\lambda_2\lambda_3, E_1 = \lambda_1\phi, E_2 = \lambda_2\phi, E_3 = \lambda_3\phi$. Zhu [18] wanted to construct two new bases l_1, l_2 by use of them and wrote in [18] p.544:

"Let $l_k = E_k + a_kE_3 + b_k\phi,\ k = 1, 2$, which satisfies $A(l_k, \phi)_E = 0, (l_k, 1)_E = 0$.

Suppose that the following condition holds*:

(A): $\{\hat{l}_1, \hat{l}_2\}$ is linear independent,

where symbol $\hat{l}$ means $\hat{l} = \sum_{i+j=4} c_{ij}s^it^j$, if $l = \sum_{i+j\le 4} c_{ij}s^it^j$.

* At least, condition (A) holds for equilateral triangular element.

Now we show that this assumption (A) is invalid.

First, as $\lambda_1 + \lambda_2 + \lambda_3 = 1$, then only three functions in four ϕ, E_1, E_2, E_3 are linear independent, and their linear combinations l_1, l_2 are demanded to satisfy two constrains $A(l_k, \phi)_E = 0, (l_k, 1)_E = 0$. Therefore, l_1 and l_2 will be surely linear dependent. The assumption (A) is theoretically impossible.

Second, we can numerically calculate these functions l_1, l_2 by an integral formula

$$\int_E \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma dsdt = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!}.$$

Obviously, $(E_1, 1) = (E_2, 1) = 2/6!$. Using Green formula and noting $E_k = 0$ on σ, from $A(E_k, \phi)_E = \int_\sigma E_k D_n\phi dl - \int_E E_k \triangle\phi dsdt = 0$, we know that the condition $A(E_k, \phi)_E = 0$ is equivalent to $(E_k, \triangle\phi)_E = 0$. As $\triangle\phi = -2(x + y)$, we can calculate

$$A(\phi, \phi)_E = 2(\phi, x + y)_E = 8/6!,\ A(E_k, \phi)_E = 2(\lambda_k\phi, x + y)_E = 20/7!.$$

Now consider new bases (as ϕ, E_1, E_2, E_3 are linear dependent, E_3 is omitted)

$$l_1 = E_1 + a_1E_2 + b_1\phi = (x + a_1y + b_1)\phi,\ l_2 = E_2 + a_2E_1 + b_2\phi = (y + a_2x + b_2)\phi,$$

so we directly have a linear system of two equations

$$A(l_k, \phi)_E = \frac{20}{7!}(1 + a_k) + \frac{8}{6!}b_k = 0, (l_k, 1)_E = \frac{2}{6!}(1 + a_k) + \frac{1}{5!}b_k = 0,\ k = 1, 2.$$

Its solution is $b_k = 0, a_k = -1, k = 1, 2$, i.e.$l_1 = E_1 - E_2, l_2 = E_2 - E_1$ are linear dependent.

Should point out that finding linear independent basis $\{l_1, l_2\}$ is possible , if add other lower terms. However, at that time, $l_1, l_2 \neq 0$ on the side σ, i.e. the superclose polynomial u_I does not possess Gauss-Lobatto structure. This is just EOA to be done by author in [8]. A superclose u_I for cubic triangular element is constructed in the next section.

3. A Superclose Polynomial for Cubic Element

By integrating Legendre's polynomials $l_j(t) = \partial_t(t^2-t)^j/j!$ in $(0,1)$, we can define M-type polynomials: $M_0 = 1, M_1 = 2t-1, M_2 = t(t-1), M_3 = t(t-1)(2t-1), M_4 = t(t-1)(5t^2-5t+1)$,..., which have $M_{j+1} = \partial_t^{j-1}(t^2-t)^j/j!, j \geq 1$. Obviously $M_j(1-t) = (-1)^j M_j(t), j \geq 0$, $M_j \perp P_{j-3}, j \geq 3$ and $M_j \perp M_{j-1}$. Now we construct basis functions in $E = \{0 < s,t < 1, s+t < 1\}$: $p_1 = \{1,t,s\}, p_j = \{\phi_{j0}, \phi_{j1}, \phi_{j2}\}$, where $\phi_{j0} = M_j(t) - (-1)^j\phi_{j2}$, $\phi_{j1} = M_j(s) - \phi_{j2}$, $j \geq 2, \phi_{22} = -st, \phi_{32} = -st(s-t), \phi_{42} = st(5st-1), \phi_{33} = \phi = st(1-s-t), \phi_{43} = s\phi, \phi_{44} = t\phi$. Note that $\phi = \phi_{43} = \phi_{44} = 0$ on σ. When $t = 0, s = 0, t = 1-s$, we have $p_j = \{0, M_j(s), 0\}, p_j = \{M_j(t), 0, 0\}, p_j = \{0, 0, M_j(s)\}, j \geq 2$, respectively.

To avoid the trouble on the boundary Γ, assume that $u(x,y)$ is any smooth support function in Ω. Denote still by $u(s,t)$ the $u(x,y)$ in a standard element $\tau = \{0 < x, y < h, x+y < h\}, x = hs, y = ht$. Obviously, $\partial_s^i\partial_t^j u = h^{i+j}D_x^iD_y^j u = O(h^{i+j})$. Any quartic polynomial P_4 has $3\times 4+3 = 15$ parameters, and the function u can be expanded in τ as

$$u = L(t,s,u) + \sum_{j=2}^{4}\sum_{i=0}^{2} b_{ji}\phi_{ji} + \sum_{3\leq i\leq j\leq 4} b_{ji}\phi_{ji} + O(h^5), \tag{3}$$

where $L(t,s,u) = u_3 + (u_2-u_3)t + (u_1-u_3)s$ is a linear interpolation of u. We have on three sides $s = 0, t = 0, t = 1-s$,

$$u = \frac{u_3+u_2}{2} + (u_2-u_3)M_1(t) + \sum_{j=2}^{4} b_{j0}M_j(t),\ u = \frac{u_3+u_1}{2} + (u_1-u_3)M_1(s) + \sum_{j=2}^{4} b_{j1}M_j(s),$$

$$u = \frac{u_2+u_1}{2} + (u_1-u_2)M_1(s) + \sum_{j=2}^{4} b_{j2}M_j(s), \tag{4}$$

respectively, i.e. which can be uniquely defined by the values of u on σ. Whereas b_{33}, b_{43}, b_{44} are defined by the values of u in E, for example, by use of Taylor expansion and orthogonal projection and so on. Here and below, the high order remainder $O(h^5)$ is omitted, which does not influence our analysis and results.

Assume that u_I is a desired cubic superclose function of u, and its remainder is

$$R = u - u_I = L(t,s,R) + \sum_{j=2,3,4}\ \sum_{i=0,1,2} b_{ji}\phi_{ji} + \sum_{3\leq i\leq j\leq 4} b_{ji}\phi_{ji}, \tag{5}$$

where $b_{4i} = O(h^4), 0 \leq i \leq 4$, are the coefficients given, and others are the constants to be defined.

We first require $R \perp \{M_1, M_3\}$ on σ. Noting that Gram matrix of $\{M_1, M_3\}$ is positive, it leads to $b_{30} = b_{31} = b_{32} = 0$ and $R_2 - R_3 = R_1 - R_3 = R_1 - R_2 = 0$, so $R_1 = R_2 = R_3 = d$ is still undefined. This is an interesting phenomenon. Secondly require $R \perp 1$ on σ, we have $b_{20} = b_{02} = b_{22} = 6d$, and then

$$R = d\rho + b\phi + \sum_{i=0}^{4} b_{4i}\phi_{4i}, \tag{6}$$

where d and $b = b_{33}$ are the constants to be defined, and $\rho = 1 + 6(\phi_{20}+\phi_{21}+\phi_{22}) = 1 + 6(s^2+t^2+st-s-t), \phi = st(1-s-t)$. Note that $\rho = l_2(t) = 6t^2-6t+1, l_2(s), l_2(s)$ on three sides are Legendre's polynomials. Thus, we see that $R \perp P_1$ on σ.

Further we require that R satisfies two orthogonality conditions in E:

$$A_E(R,\rho) = 0, \quad A_E(R,\phi) = 0. \tag{7}$$

As $R \perp P_1$ and $\phi = 0$ on σ, by Green formulas $A_E(R,\rho) = \int_\sigma RD_n\rho dl - (R, \triangle\rho)_E = 0$ and $A_E(R,\phi) = \int_\sigma \phi D_n R dl - (\phi, \triangle R)_E = 0$, we know that two conditions mentioned above are equivalent to the following conditions

$$(R,1)_E=0,\quad (\phi,\triangle R)_E=0. \tag{8}$$

By the integral formula of area coordinate, we have, for $\phi=st-s^2t-st^2$,

$$(\phi,1)=1/5!,(\phi,s)=(\phi,t)=2/6!,(\phi,s^2)=(\phi,t^2)=6/7!,(\phi,st)=4/7!,$$

and (8) leads to a linear system of two equations,

$$(R,1)=-\frac{1}{4}d+\frac{1}{5!}b+\frac{1}{4!3}(b_{40}+b_{41}-b_{42})+\frac{2}{6!}(b_{43}+b_{44})=0,$$
$$(\phi,\triangle)=\frac{24}{5!}d-\frac{8}{6!}b-\frac{96}{7!}(b_{40}+b_{41})+\frac{120}{7!}b_{42}-\frac{20}{7!}(b_{43}+b_{44})=0,$$

and their solution is $504d=-2(b_{40}+b_{41})+20b_{42}-(b_{43}+b_{44})$, $28b=-50(b_{40}+b_{41})+80b_{42}-11(b_{43}+b_{44})\}$. Substituting them into R, we get a fine expression

$$R=\sum_{i=0}^{4}b_{4i}g_{4i}(t,s), \tag{9}$$

where

$$g_{40}=M_4(t)-\psi,\quad g_{41}=M_4(s)-\psi,\ \psi=\phi_{42}+\frac{2}{504}\rho+\frac{50}{28}\phi,$$
$$g_{42}=\phi_{42}+\frac{20}{504}\rho+\frac{80}{28}\phi,\ g_{43}=s\phi-Z,\ g_{44}=t\phi-Z,\ Z=\frac{1}{504}\rho+\frac{11}{28}\phi. \tag{10}$$

We see that R on σ is a linear combinations of M_4 and l_2, which have no common roots, and $R=d=O(h^4)$ at three vertexes in E (i.e. the vertex is not superapproximation point). Below we show that these g_{4i} have no common real roots in E. In fact, if $g_{43}=g_{44}=0$, then their difference $(s-t)\phi=0$ leads to $s=t$. On the line $s=t$, cancelling ρ from $g_{42}=g_{43}=0$ (i.e. by $g_{42}+20g_{43}=0$) and reducing a factor s^2, we get the first equation $35s^2-30s+6=0$. And cancelling s^4 form $g_{42}=g_{43}=0$ (i.e. by $2g_{42}+5g_{43}=0$), we have $5s^3-2s^2+\frac{35}{504}\rho+\frac{105}{28}\phi=0$. Cancelling ρ by $g_{41}=0$ and reducing a factor s, we get the second equation $230s^2-186s+35=0$. Obviously, two quadratic equations above have no common real roots. Thus, the remainder R in E has no superapproximation points independent of u. Besides, we can also show that the gradient DR or average gradient $\bar{D}R$ have no superapproximation points in $\bar{E}$, besides symmetric points. Note that if take the fourth order derivatives in (9), these coefficients b_{4i} can be expressed by $D_s^\alpha D_t^\beta u,\alpha+\beta=4$. Thus we get another error expression

$$R=\frac{1}{120}\{(g_{40}+g_{42})\partial_t^4u+(g_{41}+g_{42})\partial_s^4u+6g_{42}\partial_s^2\partial_t^2u$$
$$-4(5g_{43}+g_{42})\partial_s^3\partial_tu-4(5g_{44}+g_{42})\partial_s\partial_t^3u\}. \tag{11}$$

Now, denoting by $v=L(s,t,v)+\sum_{i=0,1,2}(\beta_{2i}\phi_{2i}+\beta_{3i}\phi_{3i})+\beta_{33}\phi$ any test function and noting that $R\perp P_1$ on σ, $A_\tau(R,P_2)=0$ and $A_\tau(R,\phi)=0$, and we have a fine expression of integral along the contour $\partial\tau$,

$$A_\tau(R,v)=\sum_{i=0}^{2}\beta_{3i}(\tau)\{\int_{\partial\tau}RD_n\phi_{3i}dl-\int_\tau R\triangle\phi_{3i}dxdy\}=ch^6\int_{\partial\tau}F(D^4u)D_l^3vdl, \tag{12}$$

where $\beta_{3i}=Ch^2\int_{\sigma_i}D_l^3v\varphi(t)dl$ are used. Note that (12) is also valid for any smooth support function u, but should add a high order remainder $O(h^5)||v||^*_{2,1,\tau}$. Summing these integrals over all $\tau\subset J^h$ and cancelling the integrals along all common sides, we shall get a desired estimate $A(R,v)=O(h^5)||v||_{2,1,\Omega}$.

Finally, should point out that when u is any smooth function, u_I defined above is discontinuous between the elements, as the constant $d(\tau)=O(h^4)$ are different in the adjacent elements

τ and τ'. But their difference $d(\tau) - d(\tau') = O(h^5)$ are of high order, we can construct a piecewise linear function $L_h \in S^h$ such that $w = L_h - d(\tau) = O(h^5)$ and a new $u_I^* = u_I - w \in S^h$ is continuous in Ω. Obviously, this correction does not damage all estimates above. This means $A(u_h - u_I^*, v) = O(h^5)||v||_{2,1,\Omega}$. Thus we have still $u_h - u_I^* = O(h^5 \ln h)$ by use of the discrete Green's function. Now, from $u - u_h = u - u_I + O(h^5 \ln h)$ we can directly derive the desired conclusions. About the details of proof, see our papers [6,7,8]. Especially, from this we know that $u - u_h = d(\tau) + O(h^5 \ln h)$ at each vertex is not supeconvergent, i.e. Zhu's proposition is invalid.

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