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CONVERGENCE OF THE EXPLICIT DIFFERENCE SCHEME AND THE BINOMIAL TREE METHOD FOR AMERICAN OPTIONS *1)

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Abstract

This paper is concerned with numerical methods for American option pricing. We employ numerical analysis and the notion of viscosity solution to show uniform convergence of the explicit difference scheme and the binomial tree method. We also prove the existence and convergence of the optimal exercise boundaries in the above approximations.

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1. Introduction

In the probability theory, the Black-Scholes model for American option pricing belongs to the optimal stopping problems. On the other hand, in the viewpoint of PDE, it is a parabolic variational inequality. Consequently, roughly speaking, there are two kinds of numerical methods for American option pricing based on the probabilistic approach and finite difference respectively.

The binomial tree method, as a discrete time model, is the most common approach for pricing options. Amin and Khanna (1994), using the probabilistic approach, first provided a convergence proof of the binomial tree method for American options [1]. In essence, the binomial tree method belongs to the probabilistic one. However, it can be proved that the binomial tree method is consistent with an explicit difference scheme. By virtue of the notion of viscosity solutions, Barles and Souganidis (1991) presented a framework to prove the convergence of difference schemes for fully nonlinear PDE problems [3]. Jaillet etc. (1990) studied the Brennan-Schwartz algorithm for pricing American put option based on the framework of variational inequalities [9]. Lamberton (1993) showed the convergence of the resulting optimal exercise boundary (critical price) [11]. He also proved the convergence result within the probabilistic approach.

This paper will concentrate on the explicit difference scheme and the binomial tree method for American options. The main purpose is to prove the convergence of the above approximations by using numerical analysis and the notion of viscosity solution, especially in the case of American call option for which the approximate sequence is not uniformly bounded in l^∞-norm.

The remainder of this paper is organized as follows: In section 2, we recall the Black-Scholes model, the explicit difference scheme and the binomial tree method for American options. In section 3 we will concentrate on the explicit difference scheme and show the existence of optimal exercise boundary computed in the approximation of the explicit difference scheme. Section 4 is

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devoted to the convergence proofs of the explicit difference scheme and the approximate optimal exercise boundary. We extend the results of the explicit difference scheme to the binomial tree method in section 5.

2. Black-Scholes Model and Numerical Methods for American Options

The Black-Scholes model for American options with continuous dividend yield is the following:

$$\begin{cases} \min\left(-\frac{\partial V}{\partial t} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV, V-\psi\right) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad in\ [0,T)\times(0,\infty) \\ V(T,S) = \psi(S) \quad in\ (0,\infty), \end{cases} \tag{2.1}$$

where $\psi(S) = (S-E)^+$ (call option) or $\psi(S) = (E-S)^+$ (put option), $r > 0, q$ and σ represent the interest rate, dividend yield and volatility [8].

Using the simple transformations $u(x,t) = V(S,t)$, $S = e^x$, (2.1) is transformed into the following constant-coefficient problem

$$\begin{cases} \min\left(-\frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - (r-q-\frac{\sigma^2}{2})\frac{\partial u}{\partial x} + ru, u-\varphi\right) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad in\ [0,T)\times(-\infty,\infty) \\ u(T,x) = \varphi(x) \quad in\ (-\infty,\infty), \end{cases} \tag{2.2}$$

where $\varphi(x) = (e^x - E)^+$ (call option) or $\varphi(x) = (E - e^x)^+$ (put option).

We now present the explicit difference scheme for (2.2). Given mesh size $\Delta x, \Delta t > 0$, $N\Delta t = T$, let $Q = \{(n\Delta t, j\Delta x) : 0 \le n \le N,\ j \in Z\}$ stand for the lattice. U_j^n represents the value of numerical approximation at $(n\Delta t, j\Delta x)$ and $\varphi_j = \varphi(j\Delta x)$. Taking the explicit difference for time and the conventional difference discretization for space, we have

$$\min\left(-\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{\sigma^2}{2}\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}\right.$$
$$\left. -(r-q-\frac{\sigma^2}{2})\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} + rU_j^n, U_j^n - \varphi_j\right) = 0$$

or

$$\begin{aligned} U_j^n \;=\; & \max\left(\frac{1}{1+r\Delta t}\left((1-\frac{\sigma^2\Delta t}{\Delta x^2})U_j^{n+1} + \frac{\sigma^2\Delta t}{\Delta x^2}(\frac{1}{2} + \frac{\Delta x}{2\sigma^2}(r-q-\frac{\sigma^2}{2}))U_{j+1}^{n+1}\right.\right. \\ & \left.\left. + \frac{\sigma^2\Delta t}{\Delta x^2}(\frac{1}{2} - \frac{\Delta x}{2\sigma^2}(r-q-\frac{\sigma^2}{2}))U_{j-1}^{n+1}\right), \varphi_j\right), \end{aligned}$$

which is denoted by

$$U_j^n \widehat{=} \max\left(\frac{1}{1+r\Delta t}\left((1-\alpha)U_j^{n+1} + \alpha(aU_{j+1}^{n+1} + cU_{j-1}^{n+1})\right), \varphi_j\right), \tag{2.3}$$

where

$$\alpha = \sigma^2\frac{\Delta t}{\Delta x^2},\ a = \frac{1}{2} + \frac{\Delta x}{2\sigma^2}(r-q-\frac{\sigma^2}{2}),\ c = 1-a.$$

By putting $\alpha = 1$ in (2.3), namely $\sigma^2\Delta t/\Delta x^2 = 1$, we get

$$U_j^n = \max\left(\frac{1}{1+r\Delta t}\left(aU_{j+1}^{n+1} + cU_{j-1}^{n+1}\right), \varphi_j\right). \tag{2.4}$$

The final values are given as follows:

$$U_j^N = \varphi_j, \ j \in Z.$$

Now we recall the binomial tree method, which is the most common approach for option pricing. It can be described as follows:

$$V_j^n = \max\left(\frac{1}{\rho}(pV_{j+1}^{n+1} + (1-p)V_{j-1}^{n+1}), \left(S_0u^j - E\right)^+\right), \ j = n, n-2, \cdots, -n, \tag{2.5}$$

$$V_j^N = \begin{cases} \left(S_0u^j - E\right)^+ & \textit{call option} \\ \left(E - S_0u^j\right)^+ & \textit{put option} \end{cases}, \ j = N, N-2, \cdots, -N \tag{2.6}$$

where

$$p = \frac{\rho/\exp(q\Delta t) - d}{u-d}, \rho = \exp(r\Delta t), \ u = \exp(\sigma\sqrt{\Delta t}), \ d = \exp(-\sigma\sqrt{\Delta t}).$$

Set $\Delta x = \sigma\sqrt{\Delta t}$. In view of

$$\rho = 1 + r\Delta t + O\left(\Delta t^2\right)$$

and

$$p = \frac{1}{2}(1 + \frac{\Delta x}{\sigma^2}(r - q - \frac{\sigma^2}{2})) + O(\Delta t^{3/2}),$$

we deduce that the binomial tree method is equivalent to explicit difference scheme (2.4) in the sense of neglecting a higher order of Δt.

Remark 1. The trinomial tree method is equivalent to scheme (2.3).

3. Existence of Optimal Exercise Boundary in the Explicit Difference Approximation

This section is to show the existence of optimal exercise boundary in explicit difference approximation (2.3). Later we will concentrate on American call option. It's easy to generalize all results to American put option.

We will always assume

$$0 < \alpha \leq 1 \tag{3.1}$$

and

$$\left|\frac{\Delta x}{\sigma^2}(r - q - \frac{\sigma^2}{2})\right| < 1. \tag{3.2}$$

The second assumption always holds for Δx small and implies $a, c > 0$. In addition, we will assume $q > 0$ because when $q = 0$ the American call option is reduced to the European one.

Now we investigate the properties of scheme (2.3).

Lemma 1. *The explicit scheme (2.3) has the following properties:*

(1) $U_j^n \leq U_{j+1}^n$ *for all* j, n.

(2) $U_j^{n+1} \leq U_j^n$ *for all* $n < N$, j.

(3) $U_j^n \leq e^{j\Delta x}$ *for all* j, n.

Proof. To prove property (1), we use the induction. Clearly $U_j^N = \varphi_j \leq \varphi_{j+1} \leq U_{j+1}^N$. If $U_j^{k+1} \leq U_{j+1}^{k+1}$ for all j,

$$\begin{aligned} U_j^k &= \max\left(\frac{1}{1+r\Delta t}\left((1-\alpha)U_j^{k+1} + \alpha(aU_{j+1}^{k+1} + cU_{j-1}^{k+1})\right), \varphi_j\right) \\ &\leq \max\left(\frac{1}{1+r\Delta t}\left((1-\alpha)U_{j+1}^{k+1} + \alpha(aU_{j+2}^{k+1} + cU_j^{k+1})\right), \varphi_{j+1}\right) \\ &= U_{j+1}^k, \end{aligned}$$

which is the desired result. Similarly we can prove property (2). Next we will prove property (3). Let U^k denote $\{U_j^k\}_{j\in Z}$. Since $U^k \notin l^\infty(Z)$, we introduce a weighted norm $\|\cdot\|_{l_*^\infty(Z)}$ as follows:

$$\left\|U^k\right\|_{l_*^\infty(Z)} = \sup_{j\in Z}\left|e^{-j\Delta x}U_j^k\right|. \tag{3.3}$$

It suffices to show that

$$\left\|U^k\right\|_{l_*^\infty(Z)} = 1 \text{ for all } k. \tag{3.4}$$

Clearly

$$\begin{aligned} e^{-j\Delta x}U_j^k &= \max\left(\frac{1}{1+r\Delta t}\left((1-\alpha)e^{-j\Delta x}U_j^{k+1}\right.\right. \\ &\quad \left.\left.+\alpha(ae^{\Delta x}e^{-(j+1)\Delta x}U_{j+1}^{k+1}+ce^{-\Delta x}e^{-(j-1)\Delta x}U_{j-1}^{k+1})\right), e^{-j\Delta x}\varphi_j\right). \end{aligned}$$

By the Taylor expansions, we have

$$ae^{\Delta x}+ce^{-\Delta x} = 1+(r-q)\frac{\Delta x^2}{\sigma^2}+O\left(\Delta x^4\right). \tag{3.5}$$

Then

$$\begin{aligned} \left\|U^k\right\|_{l_*^\infty(Z)} &\le \max\left(\frac{1}{1+r\Delta t}((1-\alpha)+\alpha(ae^{\Delta x}+ce^{-\Delta x}))\left\|U^{k+1}\right\|_{l_*^\infty(Z)}, 1\right) \\ &\le \max\left(\frac{1+\alpha(r-q)\frac{\Delta x^2}{\sigma^2}+\alpha O\left(\Delta x^4\right)}{1+r\Delta t}\left\|U^{k+1}\right\|_{l_*^\infty(Z)}, 1\right) \\ &\le \max\left(\left\|U^{k+1}\right\|_{l_*^\infty(Z)}, 1\right), \text{ for small } \Delta x, \end{aligned}$$

which yields the desired result by combining with $\left\|U^N\right\|_{l_*^\infty(Z)} = 1$.

In order to prove the existence of approximate optimal exercise boundary, it suffices to show that

Lemma 2. *Let Δt be sufficiently small. For each $n < N$, there exists an integer j_n such that*

$$\begin{cases} U_j^n = \varphi_j > \dfrac{1}{1+r\Delta t}\left((1-\alpha)U_j^{n+1}+\alpha(aU_{j+1}^{n+1}+cU_{j-1}^{n+1})\right), \ j \ge j_n \\ U_j^n = \dfrac{1}{1+r\Delta t}\left((1-\alpha)U_j^{n+1}+\alpha(aU_{j+1}^{n+1}+cU_{j-1}^{n+1})\right) \ge \varphi_j, \ j < j_n. \end{cases} \tag{3.6}$$

Furthermore, we have

$$j_n \le j_{n-1} \le j_n + 1.$$

Proof. Let $k_1 = \inf\left\{j : e^{j\Delta x}-E \ge 0\right\}$. If $j \ge k_1+1$, in terms of (3.5) we can get

$$\begin{aligned} &\frac{1}{1+r\Delta t}\left((1-\alpha)\varphi_j+\alpha(a\varphi_{j+1}+c\varphi_{j-1})\right)-\varphi_j \\ =\ &\frac{\alpha}{1+r\Delta t}\left((rE-qe^{j\Delta x})\frac{\Delta x^2}{\sigma^2}+e^{j\Delta x}O\left(\Delta x^4\right)\right). \end{aligned} \tag{3.7}$$

(i) In the case of $q \ge r$. When $j \ge k_1+1$, we deduce from (3.7)

$$U_j^{N-1} = \varphi_j > \frac{1}{1+r\Delta t}\left((1-\alpha)\varphi_j+\alpha(a\varphi_{j+1}+c\varphi_{j-1})\right) \text{ for } \Delta t \text{ small.}$$

On the other hand, $U_j^{N-1} = \frac{1}{1+r\Delta t}((1-\alpha)\varphi_j+\alpha(a\varphi_{j+1}+c\varphi_{j-1})) \ge 0 = \varphi_j$ if $j < k_1$. Therefore we choose $j_{N-1} = k_1$ if $\varphi_{k_1} > \frac{1}{1+r\Delta t}((1-\alpha)\varphi_{k_1}+\alpha(a\varphi_{k_1+1}+c\varphi_{k_1-1}))$. Otherwise we choose $j_{N-1} = k_1+1$.

(ii) In the case of $q < r$. Take $k_2 = \inf\left\{j : qe^{j\Delta x} - rE \geq 0\right\}$. When $j \geq k_2 + 1$, due to (3.7), we get $U_j^{N-1} = \varphi_j > \frac{1}{1+r\Delta t}((1-\alpha)\varphi_j + \alpha(a\varphi_{j+1} + c\varphi_{j-1}))$; When $j < k_1, U_j^{N-1} = \frac{1}{1+r\Delta t}((1-\alpha)\varphi_j + \alpha(a\varphi_{j+1} + c\varphi_{j-1})) \geq 0 = \varphi_j$; When $k_1 \leq j < k_2 - 1$, in view of (3.7) and $rE - qe^{(j+1)\Delta x} > 0$, we have $\frac{1}{1+r\Delta t}\left((1-\alpha)\varphi_j + \alpha(a\varphi_{j+1} + c\varphi_{j-1})\right) - \varphi_j \geq 0$. As for $U_{k_2}^{N-1}$ and $U_{k_2-1}^{N-1}$, we may always assume $k_2 - 2 \geq k_1$ for Δx small. Hence for $j = k_2 - 1,\, k_2$,

$$U_j^{N-1} - \varphi_j = \frac{\alpha}{1 + r\Delta t}\left((rE - qe^{j\Delta x})\frac{\Delta x^2}{\sigma^2} + e^{j\Delta x}O\left(\Delta x^4\right)\right)$$

is monotonically decreasing with respect to $j\Delta x$. Thus there exists $j_{N-1} \in [k_2 - 1, k_2 + 1]$ such that (3.6) holds.

We have shown that (3.6) holds when $n = N - 1$. Suppose (3.6) is true when $n = k + 1$. When $j < j_{k+1}$, due to Lemma 1 (2), $\frac{1}{1+r\Delta t}\left((1-\alpha)U_j^{k+1} + \alpha(aU_{j+1}^{k+1} + cU_{j-1}^{k+1})\right) \geq \frac{1}{1+r\Delta t}\left((1-\alpha)U_j^{k+2} + \alpha(aU_{j+1}^{k+2} + cU_{j-1}^{k+2})\right) \geq \varphi_j$. When $j > j_{k+1}$, since $U_i^{k+2} = U_i^{k+1} = \varphi_i$, $i = j-1, j, j+1$, we have

$$\begin{aligned} U_j^k &\geq U_j^{k+1} > \frac{1}{1+r\Delta t}\left((1-\alpha)U_j^{k+2} + \alpha(aU_{j+1}^{k+2} + cU_{j-1}^{k+2})\right) \\ &= \frac{1}{1+r\Delta t}\left((1-\alpha)U_j^{k+1} + \alpha(aU_{j+1}^{k+1} + cU_{j-1}^{k+1})\right) \end{aligned}$$

and thus $U_j^k = \varphi_j$. Therefore we take $j_k = j_{k+1}$ or $j_{k+1} + 1$. The proof is completed.

Now we can define the approximate optimal exercise boundary.

Definition 3. *For fixed Δt, define the approximate optimal exercise boundary $x = \rho_{\Delta t}(t)$ as follows: for $t \in [(n-1)\Delta t, n\Delta t]$, $1 \leq n < N$,*

$$\rho_{\Delta t}(t) = \frac{t - (n-1)\Delta t}{\Delta t} j_n \Delta x + \frac{n\Delta t - t}{\Delta t} j_{n-1}\Delta x,$$

By definition $\rho_{\Delta t}(t)$ is monotonically decreasing.

The proof of Lemma 2 implies

Corollary 4.

$$\rho_{\Delta t}(T - \Delta t) \in \left[\max\left\{\ln E, \ln\frac{rE}{q}\right\} - \Delta x, \max\left\{\ln E, \ln\frac{rE}{q}\right\} + 2\Delta x\right].$$

4. Convergence of the Explicit Difference Scheme

This section is devoted to the convergence of the explicit difference scheme. Meanwhile we will show that $\rho_{\Delta t}(t)$ converges uniformly to the true optimal exercise boundary $\rho(t)$. To simplify notation, (2.3) will also be written as

$$U^n = \overrightarrow{F\varphi}(\Delta t)\left(U^{n+1}\right), \tag{4.1}$$

where $U^k = \left\{U_j^k\right\}_{j\in Z}$. It is a simple fact that (4.1) is monotone under the assumptions (3.1) and (3.2), that is,

$$\overrightarrow{F\varphi}(\Delta t)U \leq \overrightarrow{F\varphi}(\Delta t)V \quad if \quad U \leq V. \tag{4.2}$$

It is easy to check that

$$\overrightarrow{F\varphi}(\Delta t)(U + K) \leq \overrightarrow{F\varphi}(\Delta t)U + K\ ,\ K \geq 0, \tag{4.3}$$

where we identify K with the non-negative constant function on Z.

Define the extension function $u_{\Delta t}(t,x)$ as follows: for $x \in [(j-\frac{1}{2})\Delta x, (j+\frac{1}{2})\Delta x)$, $t \in [(n-\frac{1}{2})\Delta t, (n+\frac{1}{2})\Delta t)$,

$$u_{\Delta t}(t,x) = U_j^n.$$

By definition and Lemma 1 (3) we have

$$u_{\Delta t}(t,x) = \left(\overrightarrow{F\varphi}(\Delta t) u_{\Delta t}(t+\Delta t, \cdot)\right)(x), \text{ for all } (t,x) \in [0, T-\Delta t] \times R \tag{4.4}$$

and

$$0 \le u_{\Delta t}(t,x) \le e^{x+\Delta x/2}, \text{ for small } \Delta t. \tag{4.5}$$

We will show that

Theorem 5. *Suppose that $u(t,x)$ is the solution to the problem (2.2). Under the assumption (3.1), as $\Delta t \to 0$, we have*

(1) $u_{\Delta t}(t,x)$ converges to $u(t,x)$.

(2) $\rho_{\Delta t}(t)$ converges to $\rho(t)$.

The proof relies on the notion of viscosity solution ([3],[4]). Before the proof of Theorem 5, we review the notion of viscosity solution as follows.

Definition 6. *A function $u \in USC([0,T]\times R)$ (resp. $LSC([0,T]\times R)$) is a viscosity subsolution (resp. supersolution) of the problem (2.2) if $u(T,x) \le \varphi(x)$ (resp. $u(T,x) \ge \varphi(x)$), and whenever $\phi \in C^{1,2}([0,T) \times R)$ and $u - \phi$ attains its local maximum (resp. local minimum) at $(t,x) \in [0,T) \times R$ we have*

$$\min\left(-\frac{\partial \phi}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 \phi}{\partial x^2} - (r-q-\frac{\sigma^2}{2})\frac{\partial \phi}{\partial x} + ru, u - \varphi\right)_{(t,x)} \le 0$$

(resp.

$$\min\left(-\frac{\partial \phi}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 \phi}{\partial x^2} - (r-q-\frac{\sigma^2}{2})\frac{\partial \phi}{\partial x} + ru, u - \varphi\right)_{(t,x)} \ge 0).$$

We call $u \in C([0,T] \times R)$ is a viscosity solution of (2.2) if it is both a viscosity subsolution and supersolution of (2.2).

Lemma 7. *Suppose u and v are viscosity subsolution and supersolution of problem (2.2) respectively and*

$$|u(t,x)|, |v(t,x)| \le e^x,$$

then $u \le v$. (see [4])

Theorem 8. *The problem (2.2) has a unique viscosity solution. (see [4])*

Proof of Theorem 5. Suppose $u(t,x)$ is the viscosity solution of the problem (2.2). Denote

$$\begin{aligned} u^*(t,x) &= \limsup_{\Delta t \to 0, (s,y)\to(t,x)} u_{\Delta t}(s,y), \\ u_*(t,x) &= \liminf_{\Delta t \to 0, (s,y)\to(t,x)} u_{\Delta t}(s,y). \end{aligned}$$

Owing to (4.5), u^* and u_* are well defined and

$$0 \le u_*(t,x) \le u^*(t,x) \le e^x.$$

Obviously $u^* \in USC([0,T]\times R)$ and $u_* \in LSC([0,T]\times R)$. If we show u^* and u_* are subsolution and supersolution of (2.2) respectively, then in terms of Lemma (7) we deduce $u^* \le u_*$ and thus $u^* = u_* = u(t,x)$, which guarantees that the whole sequence converges to the viscosity solution $u(t,x)$.

We only need to show that u^* is a subsolution of (2.2). It can be shown that $u^*(T,x) = \varphi(x)$. Suppose that for $\phi \in C^{1,2}([0,T)\times R)$, $u^* - \phi$ attains a local maximum at $(t_0,x_0) \in [0,T)\times R$ and $(u^*-\phi)(t_0,x_0) = 0$. We might as well assume that (t_0,x_0) is a strict local maximum on $B_r = \{t_0 \le t \le t_0 + r, |x - x_0| \le r\}$, $r > 0$. Set $\Phi = \phi - \epsilon$, $\epsilon > 0$, then $u^* - \Phi$ attains a strict local maximum at (t_0,x_0) and

$$(u^* - \Phi)(t_0,x_0) > 0. \tag{4.6}$$

By the definition of u^*, there exists a sequence $u_{\Delta t_k}(s_k,y_k)$ such that $\Delta t_k \to 0$, $(s_k,y_k) \to (t_0,x_0)$, $u_{\Delta t_k}(s_k,y_k) \to u^*(t_0,x_0)$ when $k \to \infty$. Assuming that $(\hat{s}_k,\hat{y}_k)$ is a global maximum point of $u_{\Delta t_k} - \Phi$ on B_r, we can deduce that there is a subsequence $u_{\Delta t_{k_i}}(\hat{s}_{k_i},\hat{y}_{k_i})$, such that

$$\Delta t_{k_i} \quad\to\quad 0,\ (\hat{s}_{k_i},\hat{y}_{k_i}) \to (t_0,x_0),\ (u_{\Delta t_{k_i}} - \Phi)(\hat{s}_{k_i},\hat{y}_{k_i}) \to (u^*-\Phi)(t_0,x_0) \quad \mathit{as}\ k_i \to \infty. \tag{4.7}$$

Indeed, suppose $(\hat{s}_{k_i},\hat{y}_{k_i}) \to (\hat{s},\hat{y})$, then

$$\begin{aligned}(u^*-\Phi)(t_0,x_0) &= \lim_{k_i\to\infty}(u_{\Delta t_{k_i}} - \Phi)(s_{k_i},y_{k_i}) \\ &\le \lim_{k_i\to\infty}(u_{\Delta t_{k_i}} - \Phi)(\widehat{s}_{k_i},\widehat{y}_{k_i}) \le (u^*-\Phi)(\hat{s},\hat{y}),\end{aligned}$$

which forces $(\hat{s},\hat{y}) = (t_0,x_0)$ since (t_0,x_0) is a local strict maximum point of $u^* - \Phi$. Therefore

$$(u_{\Delta t_{k_i}} - \Phi)(\hat{s}_{k_i} + \Delta t_{k_i}, \bullet) \le (u_{\Delta t_{k_i}} - \Phi)(\hat{s}_{k_i},\hat{y}_{k_i}) \quad \mathit{in}\ B_r,$$

that is

$$u_{\Delta t_{k_i}}(\hat{s}_{k_i} + \Delta t_{k_i}, \bullet) \le \Phi(\hat{s}_{k_i} + \Delta t_{k_i}, \bullet) + (u_{\Delta t_{k_i}} - \Phi)(\hat{s}_{k_i},\hat{y}_{k_i}) \quad \mathit{in}\ B_r.$$

Without loss of generality, we may assume that the above inequality holds in $(t_0, t_0 + r)\times R$. From (4.6) and (4.7), we observe an important fact

$$(u_{\Delta t_{k_i}} - \Phi)(\hat{s}_{k_i},\hat{y}_{k_i}) > 0, \quad \mathit{when}\ k_i\ \mathit{is\ large\ enough}. \tag{4.8}$$

By (4.4), one gets

$$\begin{aligned} u_{\Delta t_{k_i}}(\hat{s}_{k_i},\hat{y}_{k_i}) &= \left(\overrightarrow{F\varphi}(\Delta t_{k_i}) u_{\Delta t_{k_i}}(\hat{s}_{k_i} + \Delta t_{k_i}, \bullet)\right)(\hat{y}_{k_i}) \\ &\le \left(\overrightarrow{F\varphi}(\Delta t_{k_i})\left(\Phi(\hat{s}_{k_i} + \Delta t_{k_i}, \bullet) + (u_{\Delta t_{k_i}} - \Phi)(\hat{s}_{k_i},\hat{y}_{k_i})\right)\right)(\hat{y}_{k_i}) \\ &\le \left(\overrightarrow{F\varphi}(\Delta t_{k_i})\Phi(\hat{s}_{k_i} + \Delta t_{k_i}, \bullet)\right)(\hat{y}_{k_i}) + (u_{\Delta t_{k_i}} - \Phi)(\hat{s}_{k_i},\hat{y}_{k_i}),\end{aligned}$$

where the last two inequalities are due to (4.2), (4.3) and (4.8). Thus

$$\Phi(\hat{s}_{k_i},\hat{y}_{k_i}) - \left(\overrightarrow{F\varphi}(\Delta t_{k_i})\Phi(\hat{s}_{k_i} + \Delta t_{k_i}, \bullet)\right)(\hat{y}_{k_i}) \le 0,$$

namely

$$\begin{aligned} &\min\left(\frac{\Delta t_{k_i}}{1 + r\Delta t_{k_i}}\left(\frac{\Phi(\hat{s}_{k_i},\hat{y}_{k_i}) - \Phi(\hat{s}_{k_i} + \Delta t_{k_i},\hat{y}_{k_i})}{\Delta t_{k_i}}\right.\right. \\ &-\frac{\sigma^2}{2}\frac{\Phi(\hat{s}_{k_i} + \Delta t_{k_i},\hat{y}_{k_i} + \Delta x_{k_i}) - 2\Phi(\hat{s}_{k_i} + \Delta t_{k_i},\hat{y}_{k_i}) + \Phi(\hat{s}_{k_i} + \Delta t_{k_i},\hat{y}_{k_i} - \Delta x_{k_i})}{\Delta x_{k_i}^2} \\ &-(r - q - \frac{\sigma^2}{2})\frac{\Phi(\hat{s}_{k_i} + \Delta t_{k_i},\hat{y}_{k_i} + \Delta x_{k_i}) - \Phi(\hat{s}_{k_i} + \Delta t_{k_i},\hat{y}_{k_i} - \Delta x_{k_i})}{2\Delta x_{k_i}} \\ &\left.\left.+ r\Phi(\hat{s}_{k_i},\hat{y}_{k_i})\right)\right), \Phi(\hat{s}_{k_i},\hat{y}_{k_i}) - \overline{\varphi}(\hat{s}_{k_i},\hat{y}_{k_i})) \le 0,\end{aligned}$$

Dividing the first term by $\Delta t_{k_i}/(1+r\Delta t_{k_i})$ and letting k_i go to infinity, we get

$$\min\left(-\frac{\partial \Phi}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2\Phi}{\partial x^2}-(r-q-\frac{\sigma^2}{2})\frac{\partial \Phi}{\partial x}+r\Phi,\Phi-\varphi\right)_{(t_0,x_0)}\leq 0.$$

Letting ϵ tend to 0, we have

$$\min\left(-\frac{\partial \phi}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2\phi}{\partial x^2}-(r-q-\frac{\sigma^2}{2})\frac{\partial \phi}{\partial x}+r\phi,\phi-\varphi\right)_{(t_0,x_0)}\leq 0,$$

which yields the desired result because of $u(t_0,x_0)=\phi(t_0,x_0)$.

Next we will show $\rho_{\Delta t}(t)$ converges to $\rho(t)$. The basic idea is from [11]. Let $t_0\in[0,T)$. Assume $x>\liminf\rho_{\Delta t}(t_0)$. Then there are subsequences $\Delta t_n\to 0$, $t_n\to t_0$ and $x_n\to x$, such that $u_{\Delta t_n}(t_n,x_n)=\varphi(x_n)$. Letting $\Delta t_n\to 0$, we have $u(t_0,x)=\varphi(x)$, and $x\geq\rho(t_0)$. Hence $\liminf\rho_{\Delta t}(t_0)\geq\rho(t_0)$. To prove $\limsup\rho_{\Delta t}(t_0)\leq\rho(t_0)$, we use the reduction to absurdity. If there is $\varepsilon>0$ such that $\limsup\rho_{\Delta t}(t_0)>\rho(t_0)+\varepsilon$. Since $\rho(t)$ is continuous, then there exists $\delta>0$ such that

$$\limsup\rho_{\Delta t}(t_0)>\rho(t)+\varepsilon,\ \forall\ t\in[t_0-\delta,t_0+\delta].$$

Now we can assume $\rho_{\Delta t_n}(t_0)>\rho(t)+\varepsilon$, $\forall\ t\in[t_0-\delta,t_0+\delta]$ for some subsequence $\Delta t_n\to 0$. Denote $Q_0=\{(t,x):t\in[t_0-\delta,t_0],x\in[\rho(t_0-\delta),\rho(t_0-\delta)+\varepsilon]$. Because $\rho_{\Delta t_n}(t)$ is monotonically decreasing, we infer

$$\rho_{\Delta t_n}(t)\geq x>\rho(t)+\varepsilon \text{ in } Q_0.$$

Since $u_{\Delta t_n}(t,x)$ converges to the viscosity solution $u(t,x)$ to (2.2), due to the regularity of viscosity solution [4] and $\rho_{\Delta t_n}(t)\geq x$, we deduce that

$$\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}+\left(r-q-\frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x}-ru=0 \text{ in } Q_0.$$

On the other hand, from $x>\rho(t)+\varepsilon$ we get $u(t,x)=e^x-E$ satisfying

$$\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}+\left(r-q-\frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x}-ru=Er-qe^x \text{ in } Q_0.$$

One has reached a contradiction. Due to the uniqueness of viscosity solution to (2.2), the whole sequence converges to $\rho(t)$.

Remark 2. The proof of Theorem 5 implies the convergence of any uniformly bounded, monotone and consistent explicit scheme satisfying assumption (4.3).

Now we introduce the following lemma:

Lemma 9. *Let $\Omega\subset R^m$ and $f_n(x_1,x_2,\cdots,x_m)$ be pointwise convergent to the continuous function $f(x_1,x_2,\cdots,x_m)$ on Ω. Assume $f_n(x_1,x_2,\cdots,x_m)$ and $f(x_1,x_2,\cdots,x_m)$ are monotone on Ω. Then $f_n(x_1,x_2,\cdots,x_m)$ converges uniformly to $f(x_1,x_2,\cdots,x_m)$ on $\overline{\Omega}_0\subset\subset\Omega$.*

Proof. Without loss of generality, we assume $n=2$ and $f_n(x_1,x_2)$ is monotonically increasing with respect to x_1 and decreasing x_2. Since $f(x_1,x_2)$ is continuous, then $\forall\varepsilon>0$, $P_0=(x_1^0,x_2^0)$, there exists $\delta(P_0)>0$ such that for all $(x_1,x_2)\in B\delta^0=\{|x_1-x_1^0|\leq\delta,|x_2-x_2^0|\leq\delta\}$,

$$\left|f(x_1,x_2)-f(x_1^0,x_2^0)\right|<\varepsilon/2. \tag{4.9}$$

Due to the monotonicity of $f(x_1,x_2)$ and $f_n(x_1,x_2)$, we have

$$\begin{aligned}&\max_{B\delta^0}|f(x_1,x_2)-f_n(x_1,x_2)|\\ \leq\ &\max\{|f(x_1^0+\delta(P_0),x_2^0-\delta(P_0))-f_n(x_1^0-\delta(P_0),x_2^0+\delta(P_0))|,\\ &|f(x_1^0-\delta(P_0),x_2^0+\delta(P_0))-f_n(x_1^0+\delta(P_0),x_2^0-\delta(P_0))|\}.\end{aligned} \tag{4.10}$$

Since $f_n(x_1,x_2)$ converges to $f(x_1,x_2)$, there exists $N(P_0)$ such that when $n > N(P_0)$

$$|f(x_1^0 \mp \delta(P_0), x_2^0 \pm \delta(P_0)) - f_n(x_1^0 \mp \delta(P_0), x_2^0 \pm \delta(P_0))| \le \varepsilon/2. \tag{4.11}$$

Combination of (4.9), (4.10) and (4.11) gives

$$\max_{B\delta^0} |f(x_1,x_2) - f_n(x_1,x_2)| < \varepsilon.$$

Then for $\overline{\Omega}_0 \subset\subset \Omega$, there are finite $\left\{B\delta^i(P_i,\delta(P_i)\right\}_{i=1}^N$ such that $\overline{\Omega}_0 \subset \cup_{i=1}^N B\delta^i(P_i,\delta(P_i))$. Therefore when $n > \max_i N(P_i)$,

$$|f(x_1,x_2) - f_n(x_1,x_2)| < \varepsilon, \text{ for all } (x_1,x_2) \in \overline{\Omega}_0,$$

which comes to the conclusion.

Theorem 10. *As Δt tends to zero,*

(1) $u_{\Delta t}(t,x)$ converges uniformly to $u(t,x)$ on any semibounded domain $[0,T]\times(-\infty,M)$.

(2) $\rho_{\Delta t}(t)$ converges uniformly to $\rho(t)$.

Proof. Due to Lemma 3.1 and Lemma 5.1 of [10], $u(t,x)$ is continuous and monotone with respect to t and x. From Lemma 1, $u_{\Delta t}(t,x)$ is a monotone function of t and x. Then Lemma 9, together with Theorem 5, implies conclusion (1). Similarly, due to the monotonicity of $\rho_{\Delta t}(t)$ and $\rho(t)$, we get conclusion (2).

5. Convergence of the Binomial Tree Method

This section is to extend the results of the explicit difference scheme to the binomial tree method. Due to (2.5), the binomial tree method can be regarded as an explicit scheme defined on the lattice Q with $\sigma^2 \frac{\Delta t}{\Delta x^2} = 1$. Assume $0 < p < 1$, then scheme (2.5) is monotonic.

Let V_j^n and U_j^n represent the values computed by scheme (2.5) and (2.4) at $(n\Delta t, j\Delta x)$, respectively. Without loss of generality we assume $S_0 = 1$ in (2.5). Then

$$e^{-j\Delta x}V_j^n = \max\left\{\frac{1}{\rho}\left(pe^{\Delta x}e^{-(j+1)\Delta x}V_{j+1}^{n+1} + (1-p)e^{-\Delta x}e^{-(j-1)\Delta x}V_{j-1}^{n+1}\right), e^{-j\Delta x}\varphi_j\right\},$$

$$e^{-j\Delta x}U_j^n = \max\left\{\frac{1}{1+r\Delta t}\left(ae^{\Delta x}e^{-(j+1)\Delta x}U_{j+1}^{n+1} + ce^{-\Delta x}e^{-(j-1)\Delta x}U_{j-1}^{n+1}\right), e^{-j\Delta x}\varphi_j\right\}.$$

Therefore

$$\begin{aligned}
& e^{-j\Delta x}\left|V_j^n - U_j^n\right| \\
\le\; & \left|\frac{1}{\rho}\left(pe^{\Delta x}e^{-(j+1)}V_{j+1}^{n+1} + (1-p)e^{-\Delta x}e^{-(j-1)}V_{j-1}^{n+1}\right)\right. \\
& \left. -\frac{1}{1+r\Delta t}\left(ae^{\Delta x}e^{-(j+1)}U_{j+1}^{n+1} + ce^{-\Delta x}e^{-(j-1)}U_{j-1}^{n+1}\right)\right| \\
=\; & \left|\frac{1}{\rho}\left(pe^{\Delta x}e^{-(j+1)}(V_{j+1}^{n+1} - U_{j+1}^{n+1}) + (1-p)e^{-\Delta x}e^{-(j-1)}(V_{j-1}^{n+1} - U_{j-1}^{n+1})\right)\right| \\
& + \left|(\frac{p}{\rho} - \frac{a}{1+r\Delta t})e^{\Delta x}e^{-(j+1)}U_{j+1}^{n+1} + (\frac{1-p}{\rho} - \frac{c}{1+r\Delta t})e^{-\Delta x}e^{-(j-1)}U_{j-1}^{n+1}\right| \\
\le\; & \frac{1}{\rho}(pe^{\Delta x} + (1-p)e^{-\Delta x})\left\|V^{n+1} - U^{n+1}\right\|_{l_*^\infty(Z)} + C\Delta t^{3/2}\left\|U^{n+1}\right\|_{l_*^\infty(Z)} \\
\le\; & e^{-q\Delta t}\left\|V^{n+1} - U^{n+1}\right\|_{l_*^\infty(Z)} + C\Delta t^{3/2} \textit{ for small } \Delta t.
\end{aligned}$$

Here $\|\cdot\|_{l_*^\infty(Z)}$ is defined by (3.3), the last inequality is due to (3.4) and (5.2), C stands for a positive constant independent of Δt. Since $V^N = U^N$, then

$$e^{-j\Delta x}\left|V_j^n - U_j^n\right| \le C\Delta t^{3/2}\sum_{i=0}^{N-n-1} e^{-iq\Delta t} \le \frac{C\Delta t^{3/2}}{1-e^{-q\Delta t}} = C\Delta t^{1/2},$$

namely

$$\left\|V^{n+1} - U^{n+1}\right\|_{l_*^\infty(Z)} \le C\Delta t^{1/2}. \tag{5.1}$$

Owing to Theorem 5, Theorem 10 and (5.1), we have

Theorem 11. *The binomial tree method is locally uniformly convergent for American call option if* $0 < p < 1$.

For the binomial tree method, we have the following equality similar to (3.5),

$$pe^{\Delta x} + (1-p)\,e^{-\Delta x} = e^{(r-q)\Delta t}. \tag{5.2}$$

By using (5.2) and the same arguments as in section 3, we can show that

Lemma 12. *For the binomial tree method, if* $0 < p < 1$, *there exists an optimal exercise boundary.*

Theorem 13. *The optimal exercise boundary in the approximation of the binomial tree method converges uniformly to the true optimal exercise boundary.*

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