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BOUNDEDNESS AND ASYMPTOTIC STABILITY OF MULTISTEP METHODS FOR GENERALIZED PANTOGRAPH EQUATIONS *1)

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Abstract

In this paper, we deal with the boundedness and the asymptotic stability of linear and one-leg multistep methods for generalized pantograph equations of neutral type, which arise from some fields of engineering. Some criteria of the boundedness and the asymptotic stability for the methods are obtained.

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Key words: Boundedness, Asymptotic stability, Multistep methods, Generalized pantograph equations.

1. Introduction

Consider generalized pantograph equations

$$\begin{cases} Y'(t) = AY(t) + BY(pt) + CY'(pt), & t > 0, \\ Y(0) = Y_0, \end{cases} \tag{1.1}$$

where $A, B, C \in \mathbb{C}^{d\times d}, p \in (0,1)$. The above equations possess numerous applications in some fields of engineering (cf. [1]), and therefore has induced much research (cf. [1]-[9]). In particular, Iserles [1, 2] and Liu [3] proved respectively that

[I] (1.1) *has a unique solution* $Y(t)$ *on space* $\mathbb{C}^{N+1}[0,\infty)$, *provided that* $p^N\|C\| < 1$ *for any given norm* $\|\bullet\|$ *and matrices* $I - p^nC$ $(n = 0, 1, \ldots, N-1)$ *are nonsingular;*

[IL] *the solution* $Y(t)$ *of* (1.1) *is asymptotically stable (i.e.,* $\lim_{t\to+\infty} Y(t) = 0$*), provided*

$$\rho(A^{-1}B) < 1 \quad \text{and} \quad \alpha(A) < 0, \tag{1.2}$$

where $\rho(\bullet)$ *denotes the spectral radius and* $\alpha(\bullet)$ *the spectral abscissa (i.e., the maximal real part of the eigenvalues of the corresponding matrix).*

A remarkable fact is that there exist some differences between equations (1.1) and delay equations of the form

$$\begin{cases} Y'(t) = AY(t) + BY(t-\tau) + CY'(t-\tau), & t > 0, \\ Y(t) = Y_0(t), & -\tau \le t \le 0. \end{cases} \tag{1.3}$$

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These differences are embodied mainly in the smoothness of solutions and the numerical treatment of equations (cf. [1]-[9]). The most significant difference is in storage (cf.[5, 6]). Namely, when solving (1.1) with a numerical method, we first need to resolve the storage problem for the existence of infinite delays in (1.1), while the computation for (1.3) will not suffer such a difficulty, because there is only a constant-delay in it. To overcome the computational storage problem for (1.1), Liu [4] (see also Koto [9]) considered a transformation of the form

$$y(t) = Y(\exp(t)), \quad t \geq t_0 + \ln p \quad (t_0 > 0), \tag{1.4}$$

which converts (1.1) into the equations

$$\begin{cases} y'(t) = \exp(t)Ay(t) + \exp(t)By(t + \ln p) + p^{-1}Cy'(t + \ln p), & t > t_0, \\ y(t) = Y(\exp(t)), \quad t_0 + \ln p \leq t \leq t_0, \end{cases} \tag{1.5}$$

where $Y(t) \ \ (0 < t \leq \exp(t_0))$ can be obtained numerically by the assigned numerical methods to (1.1).

Making use of the above technique, Liu [4] and Koto [9] studied the stability of θ-methods and Runge-Kutta methods for (1.1), respectively. We note that the previous research dealt mainly with one-step methods while multistep methods have not been involved. Hence, in the present paper, we focus on the boundedness and the asymptotic stability of linear and one-leg multistep methods. The corresponding results can be found in section 3 and section 4. In section 5, some examples are given to illustrate the applicability of the obtained theoretical results.

2. Multistep Methods

For the initial value problems of ordinary differential equations

$$\begin{cases} x'(t) = f(t, x(t)), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

two standard discretization schemes are the linear multistep methods

$$\sum_{j=0}^{k} \alpha_j x_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2.2}$$

and the corresponding one-leg methods

$$\sum_{j=0}^{k} \alpha_j x_{n+j} = hf(\sum_{j=0}^{k} \beta_j t_{n+j}, \sum_{j=0}^{k} \beta_j x_{n+j}). \tag{2.3}$$

They can be characterized by the polynomials

$$P(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \qquad Q(\xi) = \sum_{j=0}^{k} \beta_j \xi^j, \qquad \xi \in \mathbb{C},$$

where $\alpha_j, \ \ \beta_j \ \ (j = 0, 1, \ldots, k)$ are real constants with

$$P(1) = 0, \quad P'(1) = Q(1) = 1. \tag{2.4}$$

Motivated by an idea of Hu and Mitsui [11] (see also [13]), we adapt (2.2) and (2.3) to (1.5), respectively, and thus obtain two computational schemes:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j \exp(t_{n+j})(Ay_{n+j} + By_{n+j-m}) + p^{-1}C \sum_{j=0}^{k} \alpha_j y_{n+j-m}, \tag{2.5}$$

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\exp(\sum_{i=0}^{k}\beta_i t_{n+i})(A\sum_{j=0}^{k}\beta_j y_{n+j} + B\sum_{j=0}^{k}\beta_j y_{n+j-m}) + p^{-1}C\sum_{j=0}^{k}\alpha_j y_{n+j-m}, \quad (2.6)$$

where the stepsize is $h = -\frac{\ln p}{m}$, m is a positive integer and greater than k, $t_n = t_0 + nh$, y_n is an approximation to $y(t_n)$ $(n = 1, 2, \ldots,)$. When $-m < n \leq 0$, we set $y_n = y(t_n)$.

Theorem 2.1. *Suppose that the method* (2.5) *satisfies one of the following conditions:*
(1) $\alpha_k \neq 0, \quad \beta_k = 0;$
(2) $\beta_k \neq 0, \quad \alpha(A) < 0, \quad \frac{\alpha_k}{\beta_k} \geq 0.$
Then this method has a unique solution

$$y_{n+k} = \sum_{j=0}^{k-1} R_{n,j} y_{n+j} + \sum_{j=0}^{k} S_{n,j} y_{n+j-m}, \quad (2.7)$$

where

$$R_{n,j} = -[\alpha_k I_d - h\beta_k \exp(t_{n+k})A]^{-1}[\alpha_j I_d - h\beta_j \exp(t_{n+j})A],$$
$$S_{n,j} = [\alpha_k I_d - h\beta_k \exp(t_{n+k})A]^{-1}[h\beta_j \exp(t_{n+j})B + \alpha_j p^{-1}C]$$

and I_d *is the* $d \times d$ *identity matrix.*

Proof. We only need to prove that the matrices $[\alpha_k I_d - h\beta_k \exp(t_{n+k})A]$ $(n \geq 0)$ are nonsingular. Otherwise, we have

$$\prod_{i=1}^{d}[\alpha_k - h\beta_k \exp(t_{n+k})\lambda_i^A] = \det[\alpha_k I_d - h\beta_k \exp(t_{n+k})A] = 0, \quad (2.8)$$

where λ_i^A $(i = 1, 2, \ldots, d)$ are the eigenvalues of the matrix A. When (1) holds, (2.8) leads obviously to a contradiction. When (2) holds, (2.8) will yield another contradiction. In fact, it holds by (2.8) that there exists some i_0 $(1 \leq i_0 \leq d)$ such that

$$\alpha_k = h\beta_k \exp(t_{n+k})Re(\lambda_{i_0}^A).$$

Hence, it follows that

$$\frac{\alpha_k}{\beta_k \exp(t_{n+k})} = hRe(\lambda_{i_0}^A) \leq h\alpha(A) < 0,$$

which implies that $\frac{\alpha_k}{\beta_k} < 0$. Therefore, this proof is completed.

Along the similar line to the proof of Theorem 2.1, we can obtain a counterpart for the method (2.6).

Theorem 2.2. *Suppose that the method* (2.6) *satisfies condition (1) or (2) in Theorem 2.1. Then this method yields a unique solution,*

$$y_{n+k} = \sum_{j=0}^{k-1} \hat{R}_{n,j} y_{n+j} + \sum_{j=0}^{k} \hat{S}_{n,j} y_{n+j-m}, \quad (2.9)$$

where

$$\hat{R}_{n,j} = -[\alpha_k I_d - h\beta_k \exp(\sum_{i=0}^{k}\beta_i t_{n+i})A]^{-1}[\alpha_j I_d - h\beta_j \exp(\sum_{i=0}^{k}\beta_i t_{n+i})A],$$

and

$$\hat{S}_{n,j} = [\alpha_k I_d - h\beta_k \exp(\sum_{i=0}^{k}\beta_i t_{n+i})A]^{-1}[h\beta_j \exp(\sum_{i=0}^{k}\beta_i t_{n+i})B + \alpha_j p^{-1}C].$$

3. Boundedness

In the subsequent discussion, it will be a remarkable fact when $\beta_k \neq 0$ that

$$\begin{cases} R_j := \lim\limits_{n\to\infty} R_{n,j} = -\frac{\beta_j}{\beta_k \exp[(k-j)h]} I_d, \quad j = 0,1,\dots,k-1, \\ S_j := \lim\limits_{n\to\infty} S_{n,j} = -\frac{\beta_j}{\beta_k \exp[(k-j)h]} A^{-1}B, \quad j = 0,1,\dots,k, \\ \hat{R}_j := \lim\limits_{n\to\infty} \hat{R}_{n,j} = -\frac{\beta_j}{\beta_k} I_d, \quad j = 0,1,\dots,k-1, \\ \hat{S}_j := \lim\limits_{n\to\infty} \hat{S}_{n,j} = -\frac{\beta_j}{\beta_k} A^{-1}B, \quad j = 0,1,\dots,k. \end{cases} \tag{3.1}$$

Moreover, we also introduce the following notations:

$$R_k = \hat{R}_k = -I_d.$$

With (3.1), equations (2.7) and (2.9) can be viewed as the perturbation of the difference equations

$$T(\{u_i\}_{i=0}^{\infty}, n) := u_{n+k} - \sum_{j=0}^{k-1} R_j u_{n+j} - \sum_{j=0}^{k} S_j u_{n+j-m} = 0 \tag{3.2}$$

and

$$\hat{T}(\{v_i\}_{i=0}^{\infty}, n) := v_{n+k} - \sum_{j=0}^{k-1} \hat{R}_j v_{n+j} - \sum_{j=0}^{k} \hat{S}_j v_{n+j-m} = 0, \tag{3.3}$$

respectively.

Theorem 3.1. *Suppose that* $\beta_k \neq 0$, $\frac{\alpha_k}{\beta_k} \geq 0$ *and* $\frac{1}{|\beta_k|}\sum\limits_{j=0}^{k-1}|\beta_j| \leq 1$. *Then the solution* $\{y_i\}_{i=0}^{\infty}$ *of the method* (2.5) *is bounded whenever* (1.2) *holds.*

Proof. Write

$$F_n = \sum_{j=0}^{k-1}(R_{n,j} - R_j)y_{n+j} + \sum_{j=0}^{k}(S_{n,j} - S_j)y_{n+j-m}, \tag{3.4}$$

by which, (2.7) (or (2.5)) can be read as

$$T(\{y_i\}_{i=0}^{\infty}, n) = F_n. \tag{3.5}$$

Moreover, let $\omega_n = -\sum\limits_{j=0}^{k} R_j y_{n+j}$. Then (3.5) becomes

$$\omega_n = -(A^{-1}B)\omega_{n-m} + F_n. \tag{3.6}$$

Induction in (3.6) yields

$$\omega_n = (-A^{-1}B)^{q_n}\omega_{n-q_n m} + \sum_{j=0}^{q_n-1}(-A^{-1}B)^j F_{n-jm}, \tag{3.7}$$

where q_n is a positive integer with

$$\frac{n}{m} < q_n \leq \frac{n}{m} + 1. \tag{3.8}$$

It follows from $\rho(A^{-1}B) < 1$ that there exists a matrix norm $\|\bullet\|$, induced by some vectorial norm $\|\bullet\|$ on $\mathbb{C}^d$, such that

$$\|A^{-1}B\| < 1. \tag{3.9}$$

Also, since (3.1) implies that

$$\lim_{l\to\infty}\|R_{l,j} - R_j\| = 0 \ (j = 1, 2, \ldots, k-1) \ \text{ and } \ \lim_{l\to\infty}\|S_{l,j} - S_j\| = 0 \ \ (j = 1, 2, \ldots, k), \tag{3.10}$$

we deduce that there exists a positive integer N_0, which depends only on m, such that when $l > N_0$,

$$\|R_{l,j} - R_j\| < \exp(-lh) \ \ (j = 1, 2, \ldots, k-1), \quad \|S_{l,j} - S_j\| < \exp(-lh) \ \ (j = 1, 2, \ldots, k), \tag{3.11}$$

where it is remarkable that $h = -\frac{\ln p}{m}$. Moreover, (3.10) also suggests that there is a positive constant M_0 such that

$$\|R_{l,j} - R_j\| \le M_0, \quad \|S_{l,j} - S_j\| \le M_0 \tag{3.12}$$

for all l, j. It follows from (3.4) and (3.11) that

$$\begin{aligned} \|F_l\| &\le \exp(-lh)(k \max_{0\le j\le k-1}\|y_{l+j}\| + (k+1)\max_{0\le j\le k}\|y_{l+j-m}\|) \\ &\le (2k+1)\exp(-lh) \max_{l-m\le j\le l+k-1}\|y_j\| \\ &\le (\tfrac{2k+1}{m})m\exp(-lh) \max_{-m\le j\le l+k-1}\|y_j\|, \quad \forall l > N_0. \end{aligned} \tag{3.13}$$

When $0 \le l \le N_0$, it is derived from (3.4) and (3.12) that

$$\begin{aligned} \|F_l\| &\le M_0(k \max_{0\le j\le k-1}\|y_{l+j}\| + (k+1)\max_{0\le j\le k}\|y_{l+j-m}\|) \\ &\le M_0(2k+1) \max_{l-m\le j\le l+k-1}\|y_j\| \\ &= [\tfrac{M_0(2k+1)}{m}\exp(lh)]m\exp(-lh) \max_{l-m\le j\le l+k-1}\|y_j\| \\ &\le [\tfrac{M_0(2k+1)}{m}\exp(N_0 h)]m\exp(-lh) \max_{-m\le j\le l+k-1}\|y_j\| \\ &= [\tfrac{M_0(2k+1)}{m}\exp(-\tfrac{N_0\ln p}{m})]m\exp(-lh) \max_{-m\le j\le l+k-1}\|y_j\|. \end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14) yields

$$\|F_l\| \le Mm\exp(-lh) \max_{-m\le j\le l+k-1}\|y_j\|, \quad \forall l \ge 0, \tag{3.15}$$

where $M = \max\{\frac{2k+1}{m}, \ \ \frac{M_0(2k+1)}{m}\exp(-\frac{N_0\ln p}{m})\}$. A combination of (3.7),(3.8) (3.9) and (3.15) yields

$$\begin{aligned} \|y_{n+k}\| \le \ & \{\sum_{j=0}^{k-1}\|R_j\| + M(n+m)\exp[-n(1-\tfrac{1}{m})h] \\ & + \|A^{-1}B\|^{\frac{n}{m}}\sum_{j=0}^{k}\|R_j\|\} \max_{-m\le l\le n+k-1}\|y_l\|, \quad n \ge 0. \end{aligned} \tag{3.16}$$

By $\frac{1}{|\beta_k|}\sum_{j=0}^{k-1}|\beta_j| \le 1$, we have

$$\sum_{j=0}^{k-1}\|R_j\| \le 1, \qquad \sum_{j=0}^{k}\|R_j\| \le 2. \tag{3.17}$$

Inserting (3.17) into (3.16) generates

$$\|y_{n+k}\| \le \{1 + M(n+m)\exp[-n(1-\frac{1}{m})h] + 2\|A^{-1}B\|^{\frac{n}{m}}\} \max_{-m\le l\le n+k-1} \|y_l\|, \quad n \ge 0, \quad (3.18)$$

which implies that

$$\|y_{n+k}\| \le \prod_{i=0}^{n}[1 + M(i+m)\exp[-i(1-\frac{1}{m})h] + 2\|A^{-1}B\|^{\frac{i}{m}}] \max_{-m\le l\le k-1} \|y_l\|, \quad n \ge 0. \quad (3.19)$$

On the other hand, one readily check that the sequence

$$\prod_{i=0}^{n}[1 + M(i+m)\exp[-i(1-\frac{1}{m})h] + 2\|A^{-1}B\|^{\frac{i}{m}}]$$

is convergent as $n \to \infty$. Therefore, this conclues the proof.

For the method (2.6), we also have an analogous result.

Theorem 3.2. *Suppose that the method* (2.6) *satisfies the same conditions as in Theorem 3.1. Then the solution* $\{y_i\}_{i=0}^{\infty}$ *of this method is bounded whenever* (1.2) *holds.*

4. Asymptotic Stability

A natural expectation is that the solutions of the methods (2.5) and (2.6) should possess a similar long time dynamical behavior as the analytic solution of system (1.1) under the condition (1.2). For this , in what follows, we first examine the asymptotic behavior of the difference equation (3.2). This will be based on its characteristic polynomial

$$G(\lambda) := \det[\lambda^m(\lambda^k I_d - \sum_{j=0}^{k-1}\lambda^j R_j) - \sum_{j=0}^{k}\lambda^j S_j]. \quad (4.1)$$

Applying the Corollary 1.2 in in't Hout [10] to (4.1) yields.

Lemma 4.1. $G(\lambda)$ *is a Schur polynomial if* $\rho(A^{-1}B) < 1$, $\beta_k \ne 0$ *and* $\sum_{j=0}^{k}\beta_j[\exp(h)\lambda]^j \ne 0$ *for* $|\lambda| \ge 1$.

Moreover, the following Lemma will also be quite useful in the context.

Lemma 4.2 *(cf. [12, 13]). Given matrix* $L \in \mathbb{C}^{N\times N}$ *and vectorial sequence* $W_n \in \mathbb{C}^N$. *Then the solution sequence of linear difference equation*

$$U_{n+1} = LU_n + W_n$$

satisfies

$$\lim_{n\to\infty} U_n = 0$$

iff $\rho(L) < 1$ *and* $\lim_{n\to\infty} W_n = 0$.

Theorem 4.1. *Suppose that*

$$\beta_k \neq 0, \qquad \frac{\alpha_k}{\beta_k} \geq 0, \qquad \frac{1}{|\beta_k|}\sum_{j=0}^{k-1}|\beta_j| \leq 1$$

and

$$\sum_{j=0}^{k}\beta_j[\exp(h)\lambda]^j \neq 0 \quad \textit{for} \quad |\lambda| \geq 1.$$

Then the method (2.5) *is asymptotically stable (i.e.,* $\lim_{n\to\infty} y_n = 0$*) whenever* (1.2) *holds.*

Proof. Write

$$Y_n = \begin{pmatrix} y_{n-m} \\ y_{n-m+1} \\ \vdots \\ y_{n+k-1} \end{pmatrix}, \qquad W_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ F_n \end{pmatrix} \in \mathbb{C}^{(m+k)d},$$

$$J = \begin{pmatrix} 0 & I_d & & & & & & & & \\ 0 & & I_d & & & & & & & \\ \vdots & & & \ddots & & & & & & \\ \vdots & & & & \ddots & & & & & \\ \vdots & & & & & \ddots & & & & \\ \vdots & & & & & & \ddots & & & \\ \vdots & & & & & & & \ddots & & \\ \vdots & & & & & & & & \ddots & \\ 0 & & & & & & & & & I_d \\ S_0 & S_1 & \dots & S_k & 0 & \dots & 0 & R_0 & \dots & R_{k-1} \end{pmatrix} \in \mathbb{C}^{(m+k)d\times(m+k)d},$$

by which the equation (3.5) can be read as

$$Y_{n+1} = JY_n + W_n, \tag{4.2}$$

where $\lim_{n\to\infty} W_n = 0$, since by (3.1) and Theorem 3.1 it holds that $\lim_{n\to\infty} F_n = 0$. A direct computation follows that

$$\det[\lambda I_{(m+k)d} - J] = G(\lambda). \tag{4.3}$$

Furthermore, by Lemma 4.1, $G(\lambda)$ is a Schur polynomial. Hence (4.3) shows that $\rho(J) < 1$. With Lemma (4.2) we conclude that $\lim_{n\to\infty} Y_n = 0$, which yields $\lim_{n\to\infty} y_n = 0$.

Again, we use the Corollary 1.2 of in't Hout [10] to get

Lemma 4.3. *The polynomial*

$$H(\lambda) := \det[\lambda^m(\lambda^k I_d - \sum_{j=0}^{k-1}\lambda^j\hat{R}_j) - \sum_{j=0}^{k}\lambda^j\hat{S}_j]$$

is of Schur type whenever $\rho(A^{-1}B) < 1, \beta_k \neq 0$ *and* $\sum_{j=0}^{k}\beta_j\lambda^j \neq 0 \quad \textit{for} \quad |\lambda| \geq 1.$

With the help of Lemma 4.2, 4.3 and a proof similar as that of Theorem **??**, we can get the asymptotic stability results for the method (2.6).

Theorem 4.2. *Suppose that*

$$\beta_k \neq 0, \qquad \frac{\alpha_k}{\beta_k} \geq 0, \qquad \frac{1}{|\beta_k|}\sum_{j=0}^{k-1}|\beta_j| \leq 1$$

and

$$\sum_{j=0}^{k}\beta_j\lambda^j \neq 0 \quad \textit{for} \quad |\lambda| \geq 1.$$

Then the method (2.6) *is asymptotically stable whenever* (1.2) *holds.*

In view of Dahlquist [15], it holds that an A-stable method (2.2) (or (2.3)) possesses the following properties:

$$[D_1] \qquad \beta_k \neq 0;$$

$$[D_2] \qquad Q(\xi) = 0 \Longrightarrow |\xi| \leq 1;$$

$$[D_3] \qquad |\xi| \geq 1 \Longrightarrow Re[\tfrac{P(\xi)}{Q(\xi)}] \geq 0.$$

Hence, by $[D_1] - [D_3]$ we conclude that

$$[D_4] \qquad \sum_{j=0}^{k}\beta_j[\exp(h)\lambda]^j = 0 \Longrightarrow |\lambda| < 1;$$

$$[D_5] \qquad \sum_{j=0}^{k}\beta_j[\lambda]^j = 0 \Longrightarrow |\lambda| \leq 1;$$

$$[D_6] \qquad \frac{\alpha_k}{\beta_k} = \lim_{|\xi|\to\infty} Re[\tfrac{P(\xi)}{Q(\xi)}] \geq 0.$$

Combining the above properties with both Theorem **??** and Theorem 4.2 yields the following results.

Theorem 4.3. *Suppose that the method 2.2 is A-stable and satisfies* $\frac{1}{|\beta_k|}\sum_{j=0}^{k-1}|\beta_j| \leq 1$. *Then the corresponding method* (2.5) *is asymptotically stable.*

Theorem 4.4. *Suppose that the method* (2.3) *is A-stable and satisfies* $\frac{1}{|\beta_k|}\sum_{j=0}^{k-1}|\beta_j| < 1$. *Then the corresponding method* (2.6) *is asymptotically stable.*

Remark 4.1. In Theorem 4.4, we have deleted the unnecessary condition $\sum_{j=0}^{k}\beta_j\lambda^j \neq 0$ for $|\lambda| = 1$, since $\sum_{j=0}^{k}\beta_j[\pm 1]^j = 0$ leads to a contradictory inequality $\frac{1}{|\beta_k|}\sum_{j=0}^{k-1}|\beta_j| \geq 1$.

5. Some Examples

As applications of the previous results, we consider the following examples.

Example 5.1. Using A-stable second-order methods (for ODEs)

$$P(E)x_n = hQ(E)f_n \tag{5.1}$$

and

$$P(E)x_n = hf(Q(E)t_n, Q(E)x_n), \tag{5.2}$$

where E is the shift operator, $P(\xi) = \frac{3}{2}\xi^2 - 2\xi + \frac{1}{2}$ and $Q(\xi) = \frac{11}{12}\xi^2 + \frac{1}{6}\xi - \frac{1}{12}$, we can derive the following two methods (for (1.5)):

$$P(E)y_n = hQ(E)[\exp(t_n)(Ay_n + By_{n-m})] + p^{-1}CP(E)y_{n-m} \tag{5.3}$$

and

$$P(E)y_n = h\exp[Q(E)(t_n)]A[Q(E)y_n] + h\exp[Q(E)(t_n)]B[Q(E)y_{n-m}] + p^{-1}CP(E)y_{n-m} \tag{5.4}$$

with $h = -\frac{\ln p}{m}$ (m is a positive integer), respectively. Since

$$\beta_2 = \frac{11}{12} \neq 0, \quad \frac{\alpha_2}{\beta_2} = \frac{18}{11} > 0, \quad \frac{1}{|\beta_2|}\sum_{i=0}^{1}|\beta_i| = \frac{3}{11} < 1,$$

we infer by Theorem 3.1, 3.2, 4.3 and 4.4 that the methods (5.3) and (5.4) are bounded and asymptotically stable whenever (1.2) holds.

Example 5.2. Consider the methods

$$\begin{aligned} & y_{n+1} \\ = \ & y_n + h[\theta\exp(t_{n+1})(Ay_{n+1} + By_{n+1-m}) + (1-\theta)\exp(t_n)(Ay_n + By_{n-m})] \\ & +p^{-1}C(y_{n+1-m} - y_{n-m}) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} & y_{n+1} \\ = \ & y_n + h\exp[\theta t_{n+1} + (1-\theta)t_n]\{A[\theta y_{n+1} + (1-\theta)y_n] + B[\theta y_{n+1-m} + (1-\theta)y_{n-m}]\} \\ & +p^{-1}C(y_{n+1-m} - y_{n-m}), \end{aligned} \tag{5.6}$$

which are induced by linear θ-methods and one-leg θ-methods ($0 \le \theta \le 1$) for ODEs, respectively. It is well known that the both methods are A-stable iff $\frac{1}{2} \le \theta \le 1$. Hence, we conclude from Theorem 3.1, 3.2, 4.3 and 4.4 that the method 5.5 is bounded and asymptotically stable whenever $\frac{1}{2} \le \theta \le 1$; and the method (5.6) is bounded and asymptotically stable whenever $\frac{1}{2} < \theta \le 1$ whenever (1.2) holds.

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