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ON THE CONVERGENCE OF WAVEFORM RELAXATION METHODS FOR LINEAR INITIAL VALUE PROBLEMS *1)

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Abstract

We study a class of blockwise waveform relaxation methods, and investigate its convergence properties in both asymptotic and monotone senses. In addition, the monotone convergence rates between different pointwise/blockwise waveform relaxation methods resulted from different matrix splittings, and those between the pointwise and blockwise waveform relaxation methods are discussed in depth.

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1. Introduction

The waveform relaxation method is a basic and efficient iteration technique for solving ordinary differential equations and differential-algebraic equations. It differs from classical iterative techniques in that it is a continuous-time method, iterating with functions in a functional space, and therefore is quite suitable for parallel computation. This kind of waveform relaxation method was first proposed by Lelarasmee, Ruehli and Sangiovanni-Vincentelli[19] in VLSI-simulation, and was further studied and improved by many authors on both method models and convergence properties. For example, Nevanlinna[23, 24] discussed the waveform relaxation method on finite interval in terms of Picard-Lindelöf iteration, Janssen and Vandewalle[18] studied the convolution SOR waveform relaxation methods, and Miekkala[20] studied the applications of the waveform relaxation method to differential-algebraic equations. In addition, Zubik-Kowal and Vandewalle[30] recently extended waveform relaxation technique to functional-differential equations. For further details we refer to [20, 13, 14, 17, 21, 22] and references therein.

However, so far as we know, most of these theoretical convergence results are about the pointwise waveform relaxation method, and there is few about its blockwise alternative.

In this paper, we will consider convergence properties of the blockwise waveform relaxation method for the linear initial value problems on the infinite interval $[0, +\infty)$ in both asymptotic and monotone senses. By making use of the block partition and the accelerated overrelaxation techniques[16], we first set up a kind of blockwise waveform accelerated overrelaxation method. This new method involves three arbitrary parameters, and therefore its convergence properties can be considerably improved by suitable adjustments of these parameters. In addition, a series of applicable and efficient blockwise waveform relaxation methods can be produced by various choices of the parameters. Under suitable conditions, we prove the asymptotic convergence of the blockwise waveform relaxation method for block H-matrix of different types. Moreover, we demonstrate the monotone convergence properties as well as the monotone comparison

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theorems, which reveal the influence of the matrix splitting and the initial approximation upon the convergence rate of this kind of method.

The organization of this paper is as follows. We introduce the definition of block H-matrix and some related properties in Section 2, and establish the blockwise waveform relaxation method in Section 3. The asymptotic and monotone convergence properties of the blockwise waveform relaxation method are discussed in Sections 4 and 5, respectively. In Section 6, we demonstrate the comparison theorem for the waveform relaxation methods. As a consequence, the result of the convergence rates between the pointwise and blockwise waveform accelerated overrelaxation methods is given in the monotone sense. We present numerical results by solving a two-dimensional heat equation in Section 7, and at last, we end this paper with a brief concluding remark in Section 8.

2. Preliminaries

The partial orderings " $\leq$ ", " $<$ " and the absolute value $|\bullet|$ in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ are defined according to the elements. For a matrix $A\in\mathbb{C}^{n\times n}$, let $\ell(\leq n)$ and $n_i(\leq n)(i=1,\ldots,\ell)$ be positive integers satisfying $\sum_{i=1}^{\ell} n_i=n$, and define the blockwise vector and matrix spaces[3]

$$\begin{aligned}\mathbb{V}_n(n_1,\ldots,n_\ell) &= \{\ x\in\mathbb{C}^n\ |\ x=(x_1^T,\ldots,x_\ell^T)^T,\ x_i\in\mathbb{C}^{n_i},\ i=1,\ldots,\ell\ \};\\ \mathbb{L}_n(n_1,\ldots,n_\ell) &= \{\ A\in\mathbb{C}^{n\times n}\ |\ A=(A_{ij}),\ A_{ij}\in\mathbb{C}^{n_i\times n_j},\ i,j=1,\ldots,\ell\ \};\\ \mathbb{L}_{n,I}(n_1,\ldots,n_\ell) &= \{M=(M_{ij})\in\mathbb{L}_n(n_1,\ldots,n_\ell)|M_{ii}\in\mathbb{C}^{n_i\times n_i}\ \text{nonsingular},\ i=1,\ldots,\ell\},\end{aligned}$$

which will be denoted simply by $\mathbb{V}_n,\mathbb{L}_n$ and $\mathbb{L}_{n,I}$, respectively, if there is no ambiguity.

A matrix $G=(g_{ij})\in\mathbb{R}^{n\times n}$ is called an M-matrix if $g_{ij}\leq 0(i\neq j),i,j=1,\ldots,n$, and G^{-1} exists with $G^{-1}\geq 0$; an H-matrix if its comparison matrix $\mathfrak{M}(G)$ is an M-matrix, where $\mathfrak{M}(G)=(\mathfrak{m}_{ij})$ is an $n\times n$ matrix with $\mathfrak{m}_{ii}=|g_{ii}|$ and $\mathfrak{m}_{ij}=-|g_{ij}|(i\neq j)$; and an H_+-matrix if G is an H-matrix satisfying $g_{ii}>0(i=1,\ldots,n)$[8]. Evidently, if we denote

$$D_G=diag(g_{11},\ldots,g_{nn}),\quad B_G=D_G-G,\quad J_G=D_G^{-1}B_G,$$

and

$$\mathcal{L}^{n\times n}=\{M=(m_{ij})\ |\ M\in\mathbb{R}^{n\times n},\ m_{ij}\leq 0,\ i\neq j,\ i,j=1,\ldots,n\ \},$$

then $G\in\mathcal{L}^{n\times n}$ with positive diagonals is an M-matrix if and only if $\rho(J_G)<1$[29], where $\rho(\bullet)$ denotes the spectral radius of a matrix. For $M\in\mathbb{L}_{n,I}$, its type-I (type-II) block comparison matrix $\langle M\rangle=(\langle M\rangle_{ij})\in\mathbb{R}^{\ell\times\ell}$ ($\langle\langle M\rangle\rangle=(\langle\langle M\rangle\rangle_{ij})\in\mathbb{R}^{\ell\times\ell}$) is defined by $\langle M\rangle_{ii}=\|M_{ii}^{-1}\|^{-1}$ ($\langle\langle M\rangle\rangle_{ii}=1$) and $\langle M\rangle_{ij}=-\|M_{ij}\|$ ($\langle\langle M\rangle\rangle_{ij}=-\|M_{ii}^{-1}M_{ij}\|$) for $i\neq j,\ i,j=1,\ldots,\ell$; see [6, 7, 9, 15, 25]. $M\in\mathbb{L}_{n,I}$ is called a type-I (type-II) block H-matrix if $\langle M\rangle(\langle\langle M\rangle\rangle)$ is an M-matrix, and we simply denote it by $M\in H_B^{(I)}(M\in H_B^{(II)})$. Evidently, it holds that $H_B^{(I)}\subseteq H_B^{(II)}$.

For $M\in\mathbb{L}_n$, we use $[M]=(\|M_{ij}\|)\in\mathbb{R}^{\ell\times\ell}$ to represent the block absolute value. The block absolute value of a vector $x\in\mathbb{V}_n$ can be defined in an analogous way.

The following lemmas will be frequently used in what follows.

Lemma 2.1. *Let $L,M\in\mathbb{L}_n,x,y\in\mathbb{V}_n$ and $\gamma\in\mathbb{C}^1$. Then*

(1) $|[L]-[M]|\leq[L+M]\leq[L]+[M]$ ($|[x]-[y]|\leq[x+y]\leq[x]+[y]$); [3]

(2) $[LM]\leq[L][M]$ ($[Mx]\leq[M][x]$); [3]

(3) $[\gamma M]=|\gamma|[M]$ ($[\gamma x]=|\gamma|[x]$); [3]

(4) $\rho(M)\leq\rho([M])$.

Proof. (1)-(3) were proved in [3], and (4) was proved in [3] for $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Here we just need to verify the validity of (4) for any consistent matrix norm.

To this end, we assume that $\lambda \in \mathbb{C}^1$ is an eigenvalue of the matrix M such that $\rho(M) = |\lambda|$, and $x \neq 0$ is the corresponding eigenvector, that is $Mx = \lambda x$. By (2)and (3) we have $|\lambda|[x] = [Mx] \leq [M][x]$, and it follows from Theorem 2.1.11 in [11] that $\rho(M) \leq \rho([M])$.

Lemma 2.2. [3] *Let* $M \in H_B^{(I)}$ $(H_B^{(II)})$. *Then*

(1) M *is nonsingular;*

(2) $[M^{-1}] \leq \langle M\rangle^{-1}$ $([M^{-1}] \leq \langle\langle M\rangle\rangle^{-1}[D(M)^{-1}])$;

(3) $\rho(J_{\langle M\rangle}) < 1$ $(\rho(J_{\langle\langle M\rangle\rangle}) < 1)$,

where $D(M) = Diag(M_{11}, \ldots, M_{\ell\ell})$.

Lemma 2.3. *Let* $D \in \mathbb{C}^{m\times m}$ *be a Hermitian positive definite matrix. Then*

$$\|(\xi I + D)^{-1}D\|_2 \leq 1 \quad \textit{and} \quad \|(\xi I + D)^{-1}\|_2 \leq \|D^{-1}\|_2$$

hold for any $\xi \in \mathbb{C}^1$ *satisfying* $Re(\xi) \geq 0$.

Proof. Because D is a Hermitian positive definite matrix, there exist a unitary matrix $V \in \mathbb{C}^{m\times m}$ and a diagonal matrix $\Lambda = diag(\lambda_1, \ldots, \lambda_m)$ such that $D = V^*\Lambda V$, where V^* is the conjugate transpose of V and $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of D. Evidently, all $\lambda_i (i = 1, \ldots, m)$ are positive reals.

Because $\|\cdot\|_2$ is a unitarily invariant matrix norm, it follows that

$$\begin{aligned}
\|(\xi I + D)^{-1}D\|_2 &= \|(V^*(\xi I + \Lambda)V)^{-1}V^*\Lambda V\|_2 \\
&= \|(\xi I + \Lambda)^{-1}\Lambda\|_2 \\
&= \left\| diag\left(\frac{\lambda_1}{\xi + \lambda_1}, \ldots, \frac{\lambda_m}{\xi + \lambda_m}\right)\right\|_2 \\
&= \max_{1\leq i\leq m} \frac{\lambda_i}{|\xi + \lambda_i|} \leq 1,
\end{aligned}$$

and

$$\|(\xi I + D)^{-1}\|_2 = \|V^*(\xi I + \Lambda)^{-1}V\|_2 = \max_{1\leq i\leq m} \frac{1}{|\xi + \lambda_i|} \leq \max_{1\leq i\leq m} |\lambda_i|^{-1} = \|D^{-1}\|_2$$

when $Re(\xi) \geq 0$.

3. The Blockwise Waveform Relaxation Methods

Consider the linear initial value problem(IVP)

$$\begin{cases} \dot{x}(t) + Ax(t) = f(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a given matrix, $f(t) \in \mathbb{C}^n$ is a known vector-valued function and $x(t) \in \mathbb{C}^n$ is the unknown vector. By the famous Picard-Lindelöf Theorem, we know that for any given initial vector $x_0 \in \mathbb{C}^n$, there exists a unique solution of IVP (3.1) in any interval $[0, T](0 < T < +\infty)$, provided that $f(t)$ is continuous in $[0, T]$. Therefore, in the following, we assume that $f(t) \in C[0, \infty)$.

If we have a splitting $A = M - N$, then IVP (3.1) can be successively approximated by the pointwise waveform relaxation(PWR) iteration

$$\begin{cases} \dot{y}^{(k)}(t) + My^{(k)}(t) = Nx^{(k-1)}(t) + f(t), \\ x^{(k)}(t) = \beta y^{(k)}(t) + (1-\beta)x^{(k-1)}(t), \\ x^{(k)}(0) = y^{(k)}(0) = x_0, \end{cases} \tag{3.2}$$

where $x^{(0)}(t)$ is a given initial guess, and $\beta \in (0, \infty)$ a relaxation parameter. Immediately, we know that the solution of (3.2) has the brief expression

$$x^{(k)}(t) = \mathcal{K}_P x^{(k-1)}(t) + \Phi_P(t), \tag{3.3}$$

where

$$\begin{aligned} \mathcal{K}_P x(t) &= \beta \int_0^t e^{(s-t)M} N x(s) ds + (1-\beta)x(t), \\ \Phi_P(t) &= \beta\Big(e^{-tM}x(0) + \int_0^t e^{(s-t)M} f(s) ds\Big). \end{aligned} \tag{3.4}$$

Evidently, when $\beta = 1$, the PWR iteration reduces to the waveform relaxation iteration discussed in [19].

If we let $A = D - L - U$, where $D, -L, -U$ are the diagonal, strictly lower and upper triangular parts of the matrix A, respectively, then we can obtain a specific class of practical and efficient waveform relaxation iteration, called the pointwise waveform accelerated overrelaxation(PWAOR) method, with

$$\begin{aligned} \mathcal{K}_{PWAOR} x(t) &= \beta \int_0^t e^{(s-t)M(\gamma,\omega)} N(\gamma,\omega) x(s) ds + (1-\beta)x(t), \\ \Phi_{PWAOR}(t) &= \beta\Big(e^{-tM(\gamma,\omega)}x_0 + \int_0^t e^{(s-t)M(\gamma,\omega)} f(s) ds\Big), \end{aligned} \tag{3.5}$$

where

$$M(\gamma,\omega) = \frac{1}{\omega}(D - \gamma L), \quad N(\gamma,\omega) = \frac{1}{\omega}((1-\omega)D + (\omega-\gamma)L + \omega U),$$

and $\gamma \in [0, \infty)$ is a relaxation parameter, $\omega, \beta \in (0, \infty)$ the acceleration factors.

Noticing that there are three arbitrary parameters γ, ω and β in the PWAOR method, we can reasonably adjust these parameters so that this method possesses better convergence behavior. Moreover, suitable choices of these three parameters (γ, ω, β) can result in a series of applicable and efficient waveform relaxation methods, such as the Jacobi ($\gamma = 0, \omega = \beta = 1$), Gauss-Seidel ($\gamma = \omega = \beta = 1$), SOR ($\gamma = \omega > 0, \beta = 1$) waveform iteration methods, and extrapolated Jacobi ($\gamma = 0, \omega = 1$), Gauss-Seidel ($\gamma = \omega = 1$), SOR ($\gamma = \omega > 0$) waveform iteration methods, etc..

It is well known that the PWR iteration (3.3) converges if and only if the spectral radius of the kernel operator $\mathcal{K}_P$ is less than one, *i.e.*, $\rho(\mathcal{K}_P) < 1$. Since one always have $\rho(\mathcal{K}_P) = |1-\beta|$ on any finite time interval[23], we would learn nothing about the effect of the matrix splitting on the actual rate of convergence. Hence, in the following, we will concentrate on the convergence properties of the waveform relaxation method on the infinite interval $[0, +\infty)$.

Let $L^p(\mathbb{R}_+, \mathbb{C}^n)(1 \le p \le \infty)$ denote the usual L^p-space of $\mathbb{C}^n$-valued functions defined on $[0, +\infty)$. Then we can prove the following theorems, as those in [21].

Theorem 3.1. *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{C}^{n\times n}$, and the operator $\mathcal{K}_P$ in (3.4) be defined in $L^p(\mathbb{R}_+, \mathbb{C}^n)$ with $1 \le p \le \infty$. If all eigenvalues of the matrix A have* *positive real parts, then $\mathcal{K}_P$ is a bounded operator if and only if the eigenvalues of the matrix M have positive real parts. Furthermore, if all eigenvalues of the matrix M have positive real parts, then*

$$\begin{aligned}\rho(\mathcal{K}_P) &= \sup_{Re(z)\geq 0} \rho(\beta(zI+M)^{-1}N+(1-\beta)I)\\ &\leq \beta \sup_{Re(z)\geq 0} \rho((zI+M)^{-1}N)+|1-\beta|\\ &= \beta \sup_{Re(z)\geq 0} \rho(K_P(z))+|1-\beta|,\end{aligned}$$

where $K_P(z)=(zI+M)^{-1}N$ is the Laplace transform of the kernel of $\mathcal{K}$.

Theorem 3.2. *For any Hermitian positive definite matrix $A \in \mathbb{R}^{n\times n}$ and for the splitting $A = M - N$ that M and N are both Hermitian with M positive definite, the PWR method converges, provided that the matrix $2M - A$ is positive definite.*

Theorem 3.3. *For any H_+-matrix $A \in \mathbb{R}^{n\times n}$, the PWAOR method converges, provided the relaxation parameters satisfy*

$$0\leq\gamma\leq\omega, \quad 0<\omega<\frac{2}{1+\rho(|J_A|)} \quad \textit{and} \quad 0<\beta<\frac{1}{1+\sup\limits_{Re(z)\geq 0}\rho(K_{PWAOR}(z))},$$

where $K_{PWAOR}(z)=((z/\omega)I+M(\gamma,\omega))^{-1}N(\gamma,\omega)$.

In addition, analogously to [4], we have the following result for the PWAOR method with $\beta = 1$, which is an extension of the result of Stein and Rosenberg[26].

Theorem 3.4. *Let $A \in \mathcal{L}^{n\times n}$ have positive diagonals and $\beta = 1$. Then for all γ and ω satisfying $0 \leq \gamma \leq \omega$ and $0 < \omega \leq 1$, it holds that*

(1) $\rho(J_A) < 1$ if and only if $\rho(\mathcal{K}_{PWAOR}) < 1$;

(2) $\rho(J_A) < 1$ ($\rho(\mathcal{K}_{PWAOR}) < 1$) if and only if A is an M-matrix;

(3) if $\rho(J_A) < 1$, then $\rho(\mathcal{K}_{PWAOR}) \leq 1-\omega+\omega\rho(J_A)$;

(4) if $\rho(J_A) \geq 1$, then $\rho(\mathcal{K}_{PWAOR}) \geq 1-\omega+\omega\rho(J_A)$;

(5) $\rho(\mathcal{K}_{PWAOR}) \geq 1-\omega$;

(6) $\rho(J_A) = 0$ if and only if $\rho(\mathcal{K}_{PWAOR}) = 1-\omega$.

Proof. Because $\beta = 1$, we have $\rho(\mathcal{K}_{PWAOR}) = \sup_{Re(z)\geq 0}\rho(K_{PWAOR}(z))$, where

$$K_{PWAOR}(z)=(\frac{z}{\omega}I+M(\gamma,\omega))^{-1}N(\gamma,\omega).$$

Now, we first prove the following statement:

$$\rho(\mathcal{K}_{PWAOR})=\rho(\mathcal{K}(\gamma,\omega)), \tag{3.6}$$

where $\mathcal{K}(\gamma,\omega)=M(\gamma,\omega)^{-1}N(\gamma,\omega)$.

Obviously, $\rho(\mathcal{K}_{PWAOR}) \geq \rho(\mathcal{K}(\gamma,\omega))$. On the other hand, we have

$$\begin{aligned}|K_{PWAOR}(z)| &= |(\frac{z}{\omega}I+M(\gamma,\omega))^{-1}N(\gamma,\omega)|\\ &= |(zI+D-\gamma L)^{-1}((1-\omega)D+(\omega-\gamma)L+\omega U)|\\ &= |(I-\gamma(zI+D)^{-1}L)^{-1}\cdot(zI+D)^{-1}\cdot((1-\omega)D+(\omega-\gamma)L+\omega U)|.\end{aligned} \tag{3.7}$$

Since $D \geq 0$, for $z \in \mathbb{C}^1$ satisfying $Re(z) \geq 0$, it holds that $|(zI+D)^{-1}| \leq D^{-1}$, and

$$\begin{aligned} |(I-\gamma(zI+D)^{-1}L)^{-1}| &= \left|\sum_{i=0}^{n-1}(\gamma(zI+D)^{-1}L)^i\right| \\ &\leq \sum_{i=0}^{n-1}\left|\gamma(zI+D)^{-1}L\right|^i \\ &\leq \sum_{i=0}^{n-1}(\gamma D^{-1}L)^i \\ &= (I-\gamma D^{-1}L)^{-1}, \end{aligned}$$

which, together with (3.7), lead to

$$|K_{PWAOR}(z)| \leq (I-\gamma D^{-1}L)^{-1}((1-\omega)I+(\omega-\gamma)D^{-1}L+\omega D^{-1}U) = \mathcal{K}(\gamma,\omega).$$

Hence, we have $\rho(\mathcal{K}_{PWAOR}) \leq \rho(\mathcal{K}(\gamma,\omega))$, which shows the validity of (3.6).

For $\varepsilon > 0$, if we denote

$$J_\varepsilon = D^{-1}(L+U)+\varepsilon ee^T = J_A + \varepsilon ee^T, \quad e=(1,1,\dots,1)^T \in \mathbb{R}^n,$$

then J_ε is a nonnegative and irreducible matrix, and $\rho(J_A) \leq \rho(J_\varepsilon) < 1$ for any small ε, provided $\rho(J_A) < 1$. By applying the Perron-Frobenious Theorem[27] to the matrix J_ε, we see that there exists a positive vector $u \in \mathbb{R}^n$ such that

$$J_\varepsilon u = \rho(J_\varepsilon)u.$$

Thus, it holds that

$$\begin{aligned} \mathcal{K}(\gamma,\omega)u &= (I-\gamma D^{-1}L)^{-1}((1-\omega)I+(\omega-\gamma)D^{-1}L+\omega D^{-1}U)u \\ &= (I-\omega(I-\gamma D^{-1}L)^{-1}(I-D^{-1}(L+U)))u \\ &\leq (I-\omega(I-\gamma D^{-1}L)^{-1}(I-J_\varepsilon))u \\ &= (I-\omega(1-\rho(J_\varepsilon))(I-\gamma D^{-1}L)^{-1})u \\ &\leq (1-\omega+\omega\rho(J_\varepsilon))u. \end{aligned}$$

It follows from Theorem 2.1.11 in [11] that $\rho(\mathcal{K}(\gamma,\omega)) \leq 1-\omega+\omega\rho(J_\varepsilon)$. Since $\lim_{\varepsilon\to 0}\rho(J_\varepsilon) = \rho(J_A)$, we have $\rho(\mathcal{K}(\gamma,\omega)) \leq 1-\omega+\omega\rho(J_A)$ as $\varepsilon \to 0$. Therefore we have (3).

In addition, since $J_A \geq 0$, $\rho(J_A)$ is an eigenvalue of J_A, that is, there exists a vector $v \in \mathbb{R}^n$ such that

$$J(\gamma,\omega)v = (1-\omega+\omega\rho(J_A))v,$$

where

$$\begin{aligned} J(\gamma,\omega) &= (I-\alpha D^{-1}L)^{-1}((1-\omega)I+(\omega-\gamma)D^{-1}L+\omega D^{-1}U), \\ \alpha &= \frac{\gamma}{1-\omega+\omega\rho(J_A)}. \end{aligned}$$

Hence, $1-\omega+\omega\rho(J_A)$ is an eigenvalue of the matrix $J(\gamma,\omega)$, and therefore, $1-\omega+\omega\rho(J_A) \leq \rho(J(\gamma,\omega))$. If $\rho(J_A) \geq 1$, then $\alpha \leq \gamma$ and $(I-\alpha D^{-1}L)^{-1} \leq (I-\gamma D^{-1}L)^{-1}$, which implies that $J(\gamma,\omega) \leq \mathcal{K}(\gamma,\omega)$, and thus

$$\rho(J(\gamma,\omega)) \leq \rho(\mathcal{K}(\gamma,\omega)). \tag{3.8}$$

The relationships (3.6) and (3.8) show that

$$\rho(\mathcal{K}_{PWAOR}) \geq \rho(J(\gamma,\omega)) \geq 1-\omega+\omega\rho(J_A),$$

provided $\rho(J_A) \geq 1$, that is to say, the statement (4) is true.

(1) can be got directly from (3) and (4), and (2) is an obvious result.

Noticing that

$$\mathcal{K}(\gamma,\omega) = (I-\gamma D^{-1}L)^{-1}((1-\omega)I + (\omega-\gamma)D^{-1}L + \omega D^{-1}U) \geq (1-\omega)I \geq 0$$

when $0 < \omega \leq 1$ and $0 \leq \gamma \leq \omega$, we have $\rho(\mathcal{K}(\gamma,\omega)) \geq 1-\omega$. By (3.6), we know that (5) is true.

If $\rho(J_A) = 0$, it follows from (3) and (5) that $\rho(\mathcal{K}_{PWAOR}) = 1-\omega$. Meanwhile, under the assumption $\rho(\mathcal{K}_{PWAOR}) = 1-\omega$, we know that $\rho(\mathcal{K}_{PWAOR})$ is independent of the parameter γ, and hence it holds that

$$\rho(\mathcal{K}_{PWAOR}) = \rho(\mathcal{K}(\gamma,\omega)) = \rho(\mathcal{K}(0,\omega)) = 1-\omega+\omega\rho(J_A) = 1-\omega,$$

which implies that $\rho(J_A) = 0$. Therefore we have (6).

The proof of Theorem 3.4 is completed now.

If $A, M, N, L, U \in \mathbb{L}_n$ and $D \in \mathbb{L}_{n,I}$, then we can obtain the following blockwise waveform relaxation(BWR) method

$$x^{(k)}(t) = \mathcal{K}_B x^{(k-1)}(t) + \Phi_B(t),$$

and the blockwise waveform accelerated overrelaxation(BWAOR) method

$$x^{(k)}(t) = \mathcal{K}_{BWAOR} x^{(k-1)}(t) + \Phi_{BWAOR}(t),$$

respectively, which are block variants of the PWR and PWAOR methods, respectively. Here $\mathcal{K}_B, \Phi_B$ and $\mathcal{K}_{BWAOR}, \Phi_{BWAOR}$ have the same expressions as $\mathcal{K}_P, \Phi_P$ and $\mathcal{K}_{PWAOR}, \Phi_{PWAOR}$ in (3.4) and (3.5), respectively. However, the philosophies behind these expressions are quite different, because the former is understood in blockwise sense, while the latter in pointwise one.

Evidently, Theorem 3.1 is also true for the BWR method, that is,

$$\rho(\mathcal{K}_B) = \sup_{Re(z)\geq 0} \rho(\beta(zI+M)^{-1}N + (1-\beta)I) \leq \sup_{Re(z)\geq 0} \rho(K_B(z)) + |1-\beta|, \tag{3.9}$$

where $K_B(z) = (zI+M)^{-1}N$, provided all eigenvalues of the matrix M have positive real parts.

4. Asymptotic Convergence Analysis for the Blockwise Methods

In this section, we discuss the asymptotic convergence properties of the BWR and BWAOR methods.

Theorem 4.1. *For any $A \in H_B^{(I)}$, let $D(A) = Diag(A_{11},\ldots,A_{\ell\ell})$ be a symmetric positive definite matrix, and the block splitting $A = M - N$ satisfy*

$$D(M) = D(A), \quad \langle A\rangle = \langle M\rangle - [N] = D_{\langle A\rangle} - B_{\langle A\rangle}. \tag{4.1}$$

Then the BWR method converges, provided the relaxation parameter β satisfies

$$0 < \beta < \frac{2}{1+\rho_0},$$

where $\rho_0 = \sup_{Re(z)\geq 0} \rho(K_B(z))$ and $\|\cdot\|$ is $\|\cdot\|_2$.

Proof. Because all block diagonals $A_{ii} (i = 1,\ldots,\ell)$ of the block matrix $A \in \mathbb{L}_n$ are symmetric positive definite, by making use of Lemma 2.3 and (4.1), for $z \in \mathbb{C}^1$ satisfying $Re(z) \geq 0$, we have

$$\|(zI+M_{ii})^{-1}\|_2 = \|(zI+A_{ii})^{-1}\|_2 \leq \|A_{ii}^{-1}\|_2 = \|M_{ii}^{-1}\|_2.$$

It straightforwardly follows that

$$\langle A\rangle \le \langle M\rangle \le \langle zI+M\rangle \le D_{\langle A\rangle}.$$

Since both $\langle A\rangle$ and $D_{\langle A\rangle}$ are M-matrices, we know that $\langle M\rangle$ and $\langle zI+M\rangle$ are M-matrices. Therefore, we have $zI+M\in H_B^{(I)}$ and $\langle zI+M\rangle^{-1}\le\langle M\rangle^{-1}$. By Lemma 2.2, it holds that

$$[(zI+M)^{-1}] \le \langle zI+M\rangle^{-1}.$$

Hence

$$\begin{aligned}[K_B(z)] &= [(zI+M)^{-1}N]\\ &\le [(zI+M)^{-1}][N]\\ &\le \langle zI+M\rangle^{-1}[N]\\ &\le \langle M\rangle^{-1}(\langle M\rangle - D_{\langle A\rangle}+B_{\langle A\rangle})\\ &= I-\langle M\rangle^{-1}D_{\langle A\rangle}^{-1}(I-J_{\langle A\rangle}) \ \triangleq G_1.\end{aligned}$$

Because $A\in H_B^{(I)}$, it follows from Lemma 2.2 that $\rho(J_{\langle A\rangle})<1$, which implies that $(I-J_{\langle A\rangle})^{-1}$ exists and

$$(I-J_{\langle A\rangle})^{-1}=\sum_{i=0}^{\infty}(J_{\langle A\rangle})^i\ge 0.$$

Let $v=(I-J_{\langle A\rangle})^{-1}e$, where $e=(1,1,\ldots,1)^T\in\mathbb{R}^\ell$. Then $v>0$ and

$$G_1v = v-\langle M\rangle^{-1}D_{\langle A\rangle}^{-1}(I-J_{\langle A\rangle})v<v,$$

which yields that $\rho(G_1)<1$. Thus, from Theorem 2.8 in [27] we can obtain

$$\rho_0=\sup_{Re(z)\ge 0}\rho(K_B(z))\le \sup_{Re(z)\ge 0}\rho([K_B(z)])\le\rho(G_1)<1. \tag{4.2}$$

Moreover, by (3.9) we have $\rho(\mathcal{K}_B)<1$ when $0<\beta<2/(1+\rho_0)$.

Theorem 4.2. *For any $A\in H_B^{(II)}$, let $D(A)=Diag(A_{11},\ldots,A_{\ell\ell})$ be a symmetric positive definite matrix, and the block splitting $A=M-N$ satisfy*

$$D(M)=D(A),\quad \langle\langle A\rangle\rangle=\langle\langle M\rangle\rangle-[D(A)^{-1}N]=I-B_{\langle\langle A\rangle\rangle}. \tag{4.3}$$

Then the BWR method converges, provided the relaxation parameter β satisfies

$$0<\beta<\frac{2}{1+\rho_0},$$

where $\rho_0=\sup_{Re(z)\ge 0}\rho(K_B(z))$ and $\|\cdot\|$ is $\|\cdot\|_2$.

Proof. Because all block diagonals $A_{ii}(i=1,\ldots,\ell)$ are symmetric positive definite, from Lemma 2.3 and (4.3), we have

$$\begin{aligned}[D(zI+M)^{-1}D(M)] &= [D(zI+A)^{-1}D(A)]\\ &= diag(\|(zI+A_{11})^{-1}A_{11}\|_2,\ldots,\|(zI+A_{\ell\ell})^{-1}A_{\ell\ell}\|_2)\\ &\le I\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\|(zI+M_{ii})^{-1}M_{ij}\|_2 &= \|(zI+A_{ii})^{-1}M_{ij}\|_2 \\ &\le \|(zI+A_{ii})^{-1}A_{ii}\|_2\cdot\|A_{ii}^{-1}M_{ij}\|_2 \\ &\le \|A_{ii}^{-1}M_{ij}\|_2 \\ &= \|M_{ii}^{-1}M_{ij}\|_2,\end{aligned}$$

or equivalently,

$$\langle\langle zI+M\rangle\rangle_{ij}\ge\langle\langle M\rangle\rangle_{ij}$$

for $z\in\mathbb{C}^1$ satisfying $Re(z)\ge 0$. Therefore

$$\langle\langle A\rangle\rangle\le\langle\langle M\rangle\rangle\le\langle\langle zI+M\rangle\rangle\le I. \tag{4.5}$$

Since both $\langle\langle M\rangle\rangle$ and I are M-matrices, and

$$\langle\langle D(zI+M)^{-1}(zI+M)\rangle\rangle=\langle\langle zI+M\rangle\rangle,$$

it follows from (4.5) that $\langle\langle M\rangle\rangle$ and $\langle\langle D(zI+M)^{-1}(zI+M)\rangle\rangle$ are M-matrices. Therefore, it holds that $D(zI+M)^{-1}(zI+M)\in H_B^{(II)}$ and $\langle\langle D(zI+M)^{-1}(zI+M)\rangle\rangle^{-1}\le\langle\langle M\rangle\rangle^{-1}$.

By Lemma 2.2 we have

$$\begin{aligned}[(D(zI+M)^{-1}(zI+M))^{-1}] &\le \langle\langle D(zI+M)^{-1}(zI+M)\rangle\rangle^{-1}[D(D(zI+M)^{-1}(zI+M))^{-1}] \\ &= \langle\langle D(zI+M)^{-1}(zI+M)\rangle\rangle^{-1} \\ &\le \langle\langle M\rangle\rangle^{-1},\end{aligned}$$

which, together with (4.3) and (4.4), yield

$$\begin{aligned}[K_{BWR}(z)] &= [(zI+M)^{-1}N] \\ &\le [(zI+M)^{-1}D(zI+M)]\cdot[D(zI+M)^{-1}D(M)]\cdot[D(M)^{-1}N] \\ &\le [(D(zI+M)^{-1}(zI+M))^{-1}]\cdot[D(M)^{-1}N] \\ &\le \langle\langle M\rangle\rangle^{-1}(\langle\langle M\rangle\rangle-I+B_{\langle\langle A\rangle\rangle}) \\ &= I-\langle\langle M\rangle\rangle^{-1}(I-B_{\langle\langle A\rangle\rangle})\ \stackrel{\triangle}{=} G_2.\end{aligned}$$

Noting that $A\in H_B^{(II)}$, by Lemma 2.2, we know that

$$\rho(B_{\langle\langle A\rangle\rangle})=\rho(J_{\langle\langle A\rangle\rangle})<1.$$

Similarly to (4.2), we have

$$\rho_0=\sup_{Re(z)\ge 0}\rho(K_B(z))\le\sup_{Re(z)\ge 0}\rho([K_B(z)])\le\rho(G_2)<1,$$

Now, (3.9) immediately yields that $\rho(\mathcal{K}_B)<1$ when $0<\beta<2/(1+\rho_0)$.

Next, we turn to the asymptotic convergence of the BWAOR method.

Theorem 4.3. *For any $A\in H_B^{(I)}$, let $A=D-L-U$ be a block splitting of the matrix A, where $D\in\mathbb{L}_{n,I}$ is symmetric positive definite. Then the BWAOR method converges, provided the relaxation parameters γ,ω and β satisfy*

$$0\le\gamma\le\omega,\quad 0<\omega<\frac{2}{1+\rho(J_{\langle A\rangle})}\quad \textit{and}\quad 0<\beta<\frac{2}{1+\rho_1},$$

where $\rho_1=\sup_{Re(z)\ge 0}\rho(K_{BWAOR}(z))$ and $\|\cdot\|$ is $\|\cdot\|_2$.

Proof. Because the parameters γ and ω satisfy $0 \le \gamma \le \omega$ and the complex $z \in \mathbb{C}^1$ satisfies $Re(z) \ge 0$, by applying Lemma 2.3 we have

$$\begin{aligned}\big[(I-\gamma(zI+D)^{-1}L)^{-1}\big] &= \Big[\sum_{i=0}^{\ell-1}(\gamma(zI+D)^{-1}L)^i\Big] \\ &\le \sum_{i=0}^{\ell-1}\Big(\gamma\big[(zI+D)^{-1}D\big]\big[D^{-1}L\big]\Big)^i \\ &\le \sum_{i=0}^{\ell-1}[\gamma D^{-1}L]^i \\ &= (I-\gamma[D^{-1}L])^{-1}\end{aligned}$$

and

$$[(zI+D)^{-1}((1-\omega)D+(\omega-\gamma)L+\omega U)] \le |1-\omega|I+(\omega-\gamma)[D^{-1}L]+\omega[D^{-1}U].$$

It follows from these estimates that

$$\begin{aligned}[K_{BWAOR}(z)] &= [(\frac{z}{\omega}I+M(\gamma,\omega))^{-1}N(\gamma,\omega)] \\ &= [((zI+D)-\gamma L)^{-1}((1-\omega)D+(\omega-\gamma)L+\omega U)] \\ &\le [(I-\gamma(zI+D)^{-1}L)^{-1}[(zI+D)^{-1}((1-\omega)D+(\omega-\gamma)L+\omega U)] \\ &\le (I-\gamma[D^{-1}L])^{-1}(|1-\omega|I+(\omega-\gamma)[D^{-1}L]+\omega[D^{-1}U]) \\ &= I-(I-\gamma[D^{-1}L])^{-1}(I-H(\omega)) \ \stackrel{\triangle}{=} G_3,\end{aligned}$$

where $H(\omega)=|1-\omega|I+\omega([D^{-1}L]+[D^{-1}U])$.

Since $A \in H_B^{(I)}$, by Lemma 2.2 and Theorem 2.8 [27] we can obtain that

$$\rho([D^{-1}L]+[D^{-1}U]) \le \rho([D^{-1}]([L]+[U])) = \rho(J_{\langle A\rangle}) < 1.$$

Therefore, it holds that $\rho(H(\omega)) < 1$ when $0 < \omega < 2/(1+\rho(J_{\langle A\rangle}))$.

Similarly to (4.2), we know that $\rho(G_3) < 1$. Hence

$$\rho_1 = \sup_{Re(z)\ge 0}\rho(K_{BWAOR}(z)) \le \sup_{Re(z)\ge 0}\rho([K_{BWAOR}(z)]) \le \rho(G_3) < 1,$$

and $\rho(\mathcal{K}_{BWAOR}) < 1$ holds when $0 < \beta < 2/(1+\rho_1)$.

Theorem 4.4. *For any $A \in H_B^{(II)}$, let $A = D - L - U$ be a block splitting of the matrix A, where $D \in \mathbb{L}_{n,I}$ is symmetric positive definite. Then the BWAOR method converges, provided the relaxation parameters γ, ω and β satisfy*

$$0 \le \gamma \le \omega, \quad 0 < \omega < \frac{2}{1+\rho(J_{\langle\langle A\rangle\rangle})} \quad \text{and} \quad 0 < \beta < \frac{2}{1+\rho_1},$$

where $\rho_1 = \sup_{Re(z)\ge 0}\rho(K_{BWAOR}(z))$ and $\|\cdot\|$ is $\|\cdot\|_2$.

Proof. Because $A \in H_B^{(II)}$, it holds that $\rho([D^{-1}L]+[D^{-1}U]) = \rho(J_{\langle\langle A\rangle\rangle}) < 1$. Similarly to the proof of Theorem 4.3, we can immediately fulfill the proof of this theorem.

5. Monotone Convergence Analysis

To discuss the monotone convergence properties of the waveform relaxation methods, we first introduce several elementary lemmas.

Lemma 5.1. [27] *If A, B are both $n\times n$ matrices, then*

$$\exp(tA+tB)=\exp(tA)\cdot\exp(tB),\quad t>0,$$

if and only if $AB=BA$.

Lemma 5.2. *For $A\in\mathbb{C}^{n\times n}$, suppose that the splitting $A=M-N$ satisfies $N\geq 0$. Then the operator $\mathcal{K}_P$ defined by (3.4) is isotone on $L^p(\mathbb{R}_+,\mathbb{C}^n)$ if and only if $M\in\mathcal{L}^{n\times n}$.*

Proof. Because $M\in\mathcal{L}^{n\times n}$, there exists a sufficiently large $\alpha\in\mathbb{R}$ such that $\alpha I-M\geq 0$. Then for $s\leq t$, we have

$$e^{(s-t)(M-\alpha I)}=e^{(t-s)(\alpha I-M)}=\sum_{i=0}^{\infty}\frac{(\alpha I-M)^i(t-s)^i}{i!}\geq 0,$$

which, together with Lemma 5.1, show that

$$e^{(s-t)M}=e^{(s-t)(\alpha I+M-\alpha I)}=e^{(s-t)\alpha I}e^{(s-t)(M-\alpha I)}=e^{\alpha(s-t)}e^{(s-t)(M-\alpha I)}\geq 0.$$

Let $x(t),y(t)\in L^p(\mathbb{R}_+,\mathbb{C}^n)$ satisfy $y(t)\geq x(t)$ for $t>0$. Then

$$\begin{aligned}\mathcal{K}_P y(t)-\mathcal{K}_P x(t) &= \int_0^t e^{(s-t)M}Ny(s)ds-\int_0^t e^{(s-t)M}Nx(s)ds\\ &= \int_0^t e^{(s-t)M}N(y(s)-x(s))ds \ \geq 0,\end{aligned}$$

which implies that the operator $\mathcal{K}_P$ is isotone on $L^p(\mathbb{R}_+,\mathbb{C}^n)$.

In addition, assume that there exists an index pair (i,j) satisfying $i\neq j, 1\leq i,j\leq n$ such that $m_{ij}>0$. If we define $F(t)=e^{-tM}$ for $t\geq 0$, then it holds that $F'(t)=-Me^{-tM}$. Obviously, $F(0)=I$ and $F'(0)=-M$. Therefore, we have $(F(0))_{ij}=0$ and $(F'(0))_{ij}=-m_{ij}<0$, where $(F(0))_{ij}$ denotes the (i,j)-th element of the matrix $F(0)$. By continuity of the matrix function $F(t)$, $(F(t))_{ij}<0$ holds for any $t\in(0,\delta)$, provided δ is sufficiently small, or in other words, $(e^{(s-t)M})_{ij}<0$ for $0<s<t<\delta$.

Let $x(t),y(t)\in L^p(\mathbb{R}_+,\mathbb{C}^n)$, $y(t)\geq x(t)$ satisfy $N(y(t)-x(t))=(0,\ldots,0,\beta(t),0,\ldots,0)^T$, where $\beta(t)$ is the j-th element of the vector $N(x(t)-y(t))$ and satisfies $\beta(t)>0$ for $t>0$. Then we have

$$\begin{aligned}(\mathcal{K}_P y(t)-\mathcal{K}_P x(t))_i &= \left(\int_0^t e^{(s-t)M}N(y(s)-x(s))ds\right)_i\\ &= \left(\int_0^t e^{(s-t)M}(0,\ldots,0,\beta(s),0,\ldots,0)^T ds\right)_i\\ &= \int_0^t \left(e^{(s-t)M}\right)_{ij}\beta(s)ds \ <0\end{aligned}$$

for $0<t<\delta$, which obviously contradicts the assumption that $\mathcal{K}_P$ is isotone on $L^p(\mathbb{R}_+,\mathbb{C}^n)$. Up to now, we have completed the proof.

Theorem 5.1. *For $A\in\mathbb{R}^{n\times n}$, let the splitting $A=M-N$ satisfy $M\in\mathcal{L}^{n\times n}$ and $N\geq 0$. Assume that $x^{(0)},y^{(0)}\in\mathbb{R}^n$ are initial guesses obeying $x^{(0)}\leq y^{(0)}$, and $\{x^{(p)}\}$, $\{y^{(p)}\}$ are iterative sequences, starting from $x^{(0)},y^{(0)}$, respectively, and generated by the PWR method. If $0<\beta\leq 1$ and $x^{(0)}\leq x^{(1)}$, $y^{(0)}\geq y^{(1)}$, then*

(1) $x^{(p)}\leq x^{(p+1)}\leq y^{(p+1)}\leq y^{(p)}$, $p=0,1,2,\ldots$;

(2) $\lim_{p\to\infty} x^{(p)} = x^* = y^* = \lim_{p\to\infty} y^{(p)}$, *and* $x^* = y^*$ *is the unique solution of the IVP (3.1);*

(3) for any $z^{(0)} \in \mathbb{R}^n$ *obeying* $x^{(0)} \le z^{(0)} \le y^{(0)}$, *the iterative sequence* $\{z^{(p)}\}$, *starting from* $z^{(0)}$ *and generated by the PWR method, satisfies* $x^{(p)} \le z^{(p)} \le y^{(p)} (p = 0, 1, 2, \ldots)$. *Hence* $\lim_{p\to\infty} z^{(p)} = x^* = y^*$.

Proof. It follows from Lemma 5.2 that the operator $\mathcal{K}_P$ is isotone under the assumptions. We can demonstrate (1) by directly using induction. From (1), we know that $\lim_{p\to\infty} x^{(p)} = x^* \le y^* = \lim_{p\to\infty} y^{(p)}$, and x^*, y^* are solutions of the IVP (3.1). On the other hand, since the IVP (3.1) has a unique solution (see Section 3), we immediately obtain that $x^* = y^*$. Finally, the proof of (3), analogous to that of (1), can be done by direct and technical manipulations.

The monotone convergence result of the blockwise waveform relaxation method can be got in an analogous way to Theorem 5.1.

Theorem 5.2. *For* $A \in \mathbb{L}_n$, *let the splitting* $A = M - N$ *satisfy* $M \in \mathcal{L}^{n\times n}$ *and* $N \ge 0$. *Assume that* $x^{(0)}, y^{(0)} \in \mathbb{V}_n$ *are initial guesses obeying* $x^{(0)} \le y^{(0)}$, *and* $\{x^{(p)}\}$, $\{y^{(p)}\}$ *are iterative sequences, starting from* $x^{(0)}, y^{(0)}$, *respectively, and generated by the BWR method. If* $0 < \beta \le 1$, *and* $x^{(0)} \le x^{(1)}$, $y^{(0)} \ge y^{(1)}$, *then*

(1) $x^{(p)} \le x^{(p+1)} \le y^{(p+1)} \le y^{(p)}$, $p = 0, 1, 2, \ldots$;

(2) $\lim_{p\to\infty} x^{(p)} = x^* = y^* = \lim_{p\to\infty} y^{(p)}$, *and* $x^* = y^*$ *is the unique solution of the IVP (3.1);*

(3) for any $z^{(0)} \in \mathbb{V}_n$ *obeying* $x^{(0)} \le z^{(0)} \le y^{(0)}$, *the iterative sequence* $\{z^{(p)}\}$, *starting from* $z^{(0)}$ *and generated by the BWR method, satisfies* $x^{(p)} \le z^{(p)} \le y^{(p)} (p = 0, 1, 2, \ldots)$. *Hence,* $\lim_{p\to\infty} z^{(p)} = x^* = y^*$.

From Theorem 5.1 and Theorem 5.2, we can obtain the following result for the PWAOR and BWAOR methods.

Corollary 5.1. *For* $A \in \mathbb{L}_n$, *let* $A = D - L - U$ *satisfy* $D \in \mathcal{L}^{n\times n}$ *and* $L \ge 0$, $U \ge 0$. *Assume that* $x^{(0)}, y^{(0)} \in \mathbb{R}^n$ *are initial guesses obeying* $x^{(0)} \le y^{(0)}$, *and* $\{x^{(p)}\}$, $\{y^{(p)}\}$ *are iterative sequences, starting from* $x^{(0)}, y^{(0)}$, *respectively, and generated by the PWAOR (BWAOR) method. If* $x^{(0)} \le x^{(1)}$, $y^{(0)} \ge y^{(1)}$, *and the parameters* γ, ω *and* β *satisfy* $0 \le \gamma \le \omega$, $0 < \omega \le 1$ *and* $0 < \beta \le 1$, *then*

(1) $x^{(p)} \le x^{(p+1)} \le y^{(p+1)} \le y^{(p)}$, $p = 0, 1, 2, \ldots$;

(2) $\lim_{p\to\infty} x^{(p)} = x^* = y^* = \lim_{p\to\infty} y^{(p)}$, *and* $x^* = y^*$ *is the unique solution of the IVP (3.1);*

(3) for any $z^{(0)} \in \mathbb{R}^n$ *obeying* $x^{(0)} \le z^{(0)} \le y^{(0)}$, *the iterative sequence* $\{z^{(p)}\}$, *starting from* $z^{(0)}$ *and generated by the PWAOR (BWAOR) method, satisfies* $x^{(p)} \le z^{(p)} \le y^{(p)} (p = 0, 1, 2, \ldots)$. *Hence,* $\lim_{p\to\infty} z^{(p)} = x^* = y^*$.

6. Comparison Results

In this section, based on the spirits of [1, 2, 10, 28], we will study the convergence rates of the waveform relaxation methods resulted from different matrix splittings in the sense of monotonicity. Meanwhile, we will investigate the influence of the relaxation parameters on the convergence behaviors of these methods.

For this purpose, we need the differential comparison theorem in [12].

Lemma 6.1. [12] *Let $x_1(t)$ and $x_2(t)$ be solutions of the ordinary differential equations*

$$\dot{x} = f_1(t,x) \quad and \quad \dot{x} = f_2(t,x),$$

respectively, where

$$f_1(t,x) \leq f_2(t,x), \quad for \quad t \in [a,b],\ -\infty \leq a \leq b \leq +\infty.$$

Assume that either f_1 or f_2 satisfies the Lipschitz condition. If $x_1(a) = x_2(a)$, then

$$x_1(t) \leq x_2(t) \quad for\ all \quad t \in [a,b].$$

For $A \in \mathbb{R}^{n\times n}$, let $A = M_1 - N_1 = M_2 - N_2$ be two different splittings. Corresponding to these two matrix splittings, we compare the monotone convergence rates between two different iterative sequences $\{x^{(p)}\}$ and $\{y^{(p)}\}$ defined by

$$\dot{x}^{(p)}(t) + M_1 x^{(p)}(t) = N_1 x^{(p-1)}(t) + f(t), \quad x^{(p)}(0) = x_0 \tag{6.1}$$

and

$$\dot{y}^{(p)}(t) + M_2 y^{(p)}(t) = N_2 y^{(p-1)}(t) + f(t), \quad y^{(p)}(0) = x_0, \tag{6.2}$$

respectively.

Theorem 6.1. *For $A \in \mathbb{R}^{n\times n}$, let $A = M_i - N_i (i = 1,2)$ be two different splittings. Assume that $x^{(0)} = y^{(0)} \in \mathbb{R}^n$ is an initial guess, and the iterative sequences $\{x^{(p)}\}$ and $\{y^{(p)}\}$ are defined by (6.1) and (6.2), respectively. If $M_i \in \mathcal{L}^{n\times n}$ and $N_i \geq 0$ for $i = 1,2$, then*

(1) $x^{(p)} \geq y^{(p)} (p = 0,1,2,\ldots)$, as $x^{(0)} \leq x^{(1)}$ or $y^{(0)} \leq y^{(1)}$;

(2) $x^{(p)} \leq y^{(p)} (p = 0,1,2,\ldots)$, as $x^{(0)} \geq x^{(1)}$ or $y^{(0)} \geq y^{(1)}$,

provided $N_2 \geq N_1$.

Proof. We first prove (1) by induction.

For $p = 0$, (1) is obviously true. Suppose that (1) has been verified for all $p \leq k$. Without loss of generality, we assume that $x^{(0)} \leq x^{(1)}$. Then, according to Theorem 5.1, $\{x^{(p)}\}$ is monotonously increasing. By subtracting (6.2) from (6.1) we obtain

$$\dot{z}^{(k+1)} + M_2 z^{(k+1)} = N_2 z^{(k)} + u^{(k)}, \tag{6.3}$$

where we have used the notations

$$z^{(i)} = x^{(i)} - y^{(i)} (i = k, k+1) \quad \text{and} \quad u^{(k)} = (N_2 - N_1)(x^{(k+1)} - x^{(k)}) \geq 0.$$

We next show that

$$z^{(k+1)} = x^{(k+1)} - y^{(k+1)} \geq 0.$$

To this end, we construct an initial value problem

$$\dot{\tilde{z}}^{(k+1)} + M_2 \tilde{z}^{(k+1)} = N_2 z^{(k)}, \quad \tilde{z}^{(k+1)}(0) = 0. \tag{6.4}$$

Let $z^{(k+1)}(t)$ and $\tilde{z}^{(k+1)}(t)$ be solutions of (6.3) and (6.4), respectively. Since

$$z^{(k+1)}(0) = x^{(k+1)}(0) - y^{(k+1)}(0) = x_0 - x_0 = 0 = \tilde{z}^{(k+1)}(0)$$

and $u^{(k)} \geq 0$, it follows from Lemma 6.1 that

$$z^{(k+1)}(t) \geq \widetilde{z}^{(k+1)}(t) \quad \text{for} \quad t \geq 0.$$

Therefore it is sufficient to demonstrate $\widetilde{z}^{(k+1)}(t) \geq 0$ for $t \geq 0$.

By the solution formula of ODE $\dot{x} = ax + b$, we have

$$\widetilde{z}^{(k+1)}(t) = \int_0^t e^{(s-t)M_2} N_2 z^{(k)}(s)ds + e^{-tM_2}\widetilde{z}^{(k+1)}(0) = \int_0^t e^{(s-t)M_2} N_2 z^{(k)}(s)ds.$$

Here we have used the condition $\widetilde{z}^{(k+1)}(0) = 0$. Because $M_2 \in \mathcal{L}^{n\times n}$ and $N_2 \geq 0$, from the proof of Lemma 5.2, we know that

$$e^{(s-t)M_2} N_2 \geq 0 \quad \text{for} \quad s \leq t.$$

Considering the induction assumption $z^{(k)} = x^{(k)} - y^{(k)} \geq 0$, we straightforwardly obtain

$$\widetilde{z}^{(k+1)}(t) \geq 0 \quad \text{for} \quad t \geq 0.$$

Hence $z^{(k+1)}(t) \geq 0$, which implies that $x^{(k+1)}(t) \geq y^{(k+1)}(t)$ for $t \geq 0$.

By induction, we demonstrate the validity of (1).

Similarly, we can demonstrate (2).

Analogous to [5], we can immediately obtain the following results from Theorem 6.1.

Corollary 6.1. *For $A \in \mathbb{R}^{n\times n}(\mathbb{L}_n)$, let $A = M_i - N_i (i = 1, 2)$ be two different pointwise (blockwise) splittings. Assume that $x^{(0)} = y^{(0)} \in \mathbb{R}^n$ is an initial guess, and the iterative sequences $\{x^{(p)}\}$ and $\{y^{(p)}\}$ are generated by the PWR(BWR) method corresponding to the above two matrix splittings, respectively. If $0 < \beta \leq 1$ and $M_i \in \mathcal{L}^{n\times n}$, $N_i \geq 0$ for $i = 1, 2$, then*

(1) $x^{(p)} \geq y^{(p)} (p = 0, 1, 2, \ldots)$, as $x^{(0)} \leq x^{(1)}$ or $y^{(0)} \leq y^{(1)}$;

(2) $x^{(p)} \leq y^{(p)} (p = 0, 1, 2, \ldots)$, as $x^{(0)} \geq x^{(1)}$ or $y^{(0)} \geq y^{(1)}$,

provided $N_2 \geq N_1$.

Corollary 6.2. *For $A \in \mathbb{R}^{n\times n}(\mathbb{L}_n)$, let $A = M(\gamma_i, \omega_i) - N(\gamma_i, \omega_i)(i = 1, 2)$ be two different pointwise(blockwise) splittings. Assume that $x^{(0)} = y^{(0)} \in \mathbb{R}^n$ is an initial guess, and the iterative sequences $\{x^{(p)}\}$ and $\{y^{(p)}\}$ are generated by the PWAOR(BWAOR) method corresponding to the above two matrix splittings, respectively. If $0 \leq \gamma_i \leq \omega_i$, $0 < \omega_i \leq 1$ and $0 < \beta_i \leq 1$ for $i = 1, 2$, then*

(1) $x^{(p)} \geq y^{(p)} (p = 0, 1, 2, \ldots)$, as $x^{(0)} \leq x^{(1)}$ or $y^{(0)} \leq y^{(1)}$;

(2) $x^{(p)} \leq y^{(p)} (p = 0, 1, 2, \ldots)$, as $x^{(0)} \geq x^{(1)}$ or $y^{(0)} \geq y^{(1)}$,

provided $\omega_2 \leq \omega_1$, $\omega_1\gamma_2 \leq \omega_2\gamma_1$ and $\beta_2 \leq \beta_1$.

Corollary 6.3. *For $A \in \mathcal{L}^{n\times n}$, Let $x^{(0)} = y^{(0)} \in \mathbb{R}^n$ be an initial guess, and the iterative sequences $\{x^{(p)}\}$ and $\{y^{(p)}\}$ be generated by the BWAOR and PWAOR method, respectively. If $0 < \omega \leq 1$, $0 \leq \gamma \leq \omega$ and $0 < \beta \leq 1$, then*

(1) $x^{(p)} \geq y^{(p)} (p = 0, 1, 2, \ldots)$, as $x^{(0)} \leq x^{(1)}$ or $y^{(0)} \leq y^{(1)}$;

(2) $x^{(p)} \leq y^{(p)} (p = 0, 1, 2, \ldots)$, as $x^{(0)} \geq x^{(1)}$ or $y^{(0)} \geq y^{(1)}$;

This shows that the BWAOR method converges faster than the PWAOR method in the monotone sense.

7. Numerical Results

We consider the heat equation in two space valuables which, through the finite difference discretization, leads to a linear initial value problem of the following form (see [14])

$$\begin{cases} \dot{x}(t) + Ax(t) = 0, \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{7.1}$$

where

$$A = \begin{bmatrix} T & -I & & & \\ -I & T & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & T & -I \\ & & & -I & T \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad T = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{bmatrix} \in \mathbb{R}^{\ell\times\ell},$$

$n = \ell^2$, and I is the identity matrix of dimension ℓ.

We solve the IVP (7.1) on interval $[0, 2]$ with $\ell = 10$ and $x_0 = (1, \ldots, 1)^T \in \mathbb{R}^n$. Here we assume that the integrals in the resulting equations are discretized by the implicit trapezoidal rule with stepsize $\Delta t = 0.01$. We start with an initial vector having all components equal to 1, *i.e.*, $x^{(0)}(t) = (1, \ldots, 1)^T \in \mathbb{R}^n$ for $t \geq 0$, and terminated once the current iteration $x^{(k+1)}(t)$ satisfies $\max_{0\leq t\leq 2} |x^{(k+1)}(t) - x^{(k)}(t)| < 10^{-8}$. We list the numbers of iterations of PWSOR and BWSOR methods in Table 7.1, and those of PWAOR and BWAOR methods in Table 7.2, respectively.

Table 7.1 Iteration numbers of PWSOR and BWSOR methods

ω	0.80	0.90	1.00	1.10	1.15	1.20	1.25	1.30	1.40	1.50	1.80
PWSOR	23	21	19	18	18	18	19	20	21	22	25
BWSOR	18	15	14	13	14	15	16	17	18	19	23

Table 7.2 Iteration numbers of PWAOR and BWAOR methods

ω	1.00				1.07	1.10				1.25			
γ	1.0	1.05	1.1	1.2	1.063	1.0	1.1	1.2	1.35	1.0	1.2	1.35	1.6
PWAOR	19	19	18	19	18	18	18	18	19	20	19	19	20
BWAOR	14	13	14	15	12	14	13	14	15	17	16	15	16

From Tables 7.1 and 7.2, we can see that the blockwise waveform relaxation method is more effective than the pointwise one.

In order to show the monotone convergence behaviors of pointwise/blockwise Gauss-Seidel waveform relaxation(PWGS/BWGS) methods applied to IVP (7.1), and to demonstrate the convergence rates between PWGS and BWGS methods in terms of monotonicity, we further investigated the first ten iterations of $x(t)$ on the interval $[0, 2]$. For simplicity, we only plot in Figure 7.1 the values of $x_1(t), x_{25}(t), x_{45}(t)$ and $x_{100}(t)$ at $t = 2$, where the solid line and the dashed line denote the numerical solution starting from the initial vector $\hat{x}^{(0)}(t) = (1 + 2t, \ldots, 1 + 2t)^T (0 \leq t \leq 2)$ and $\bar{x}^{(0)}(t) = (1 - 2t, \ldots, 1 - 2t)^T (0 \leq t \leq 2)$, respectively, and generated by BWGS method.

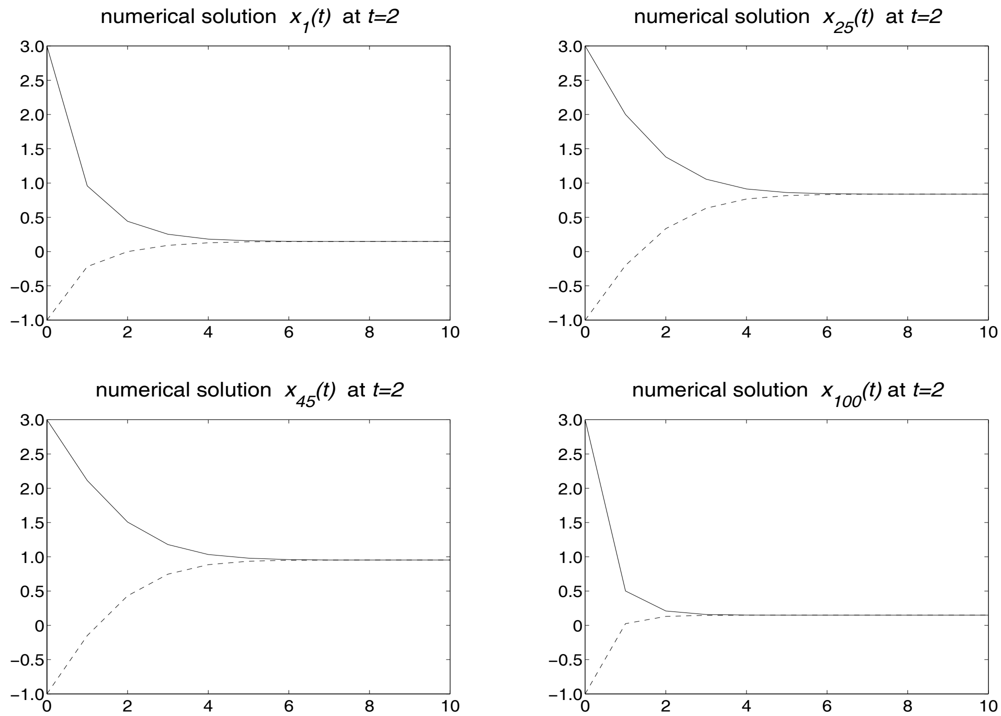

Fig. 7.1. The first ten iterations of $x(t)$ obtained by BWGS method, corresponding to the different initial guesses, $\hat{x}^{(0)}(t)$ and $\bar{x}^{(0)}(t)$, respectively.

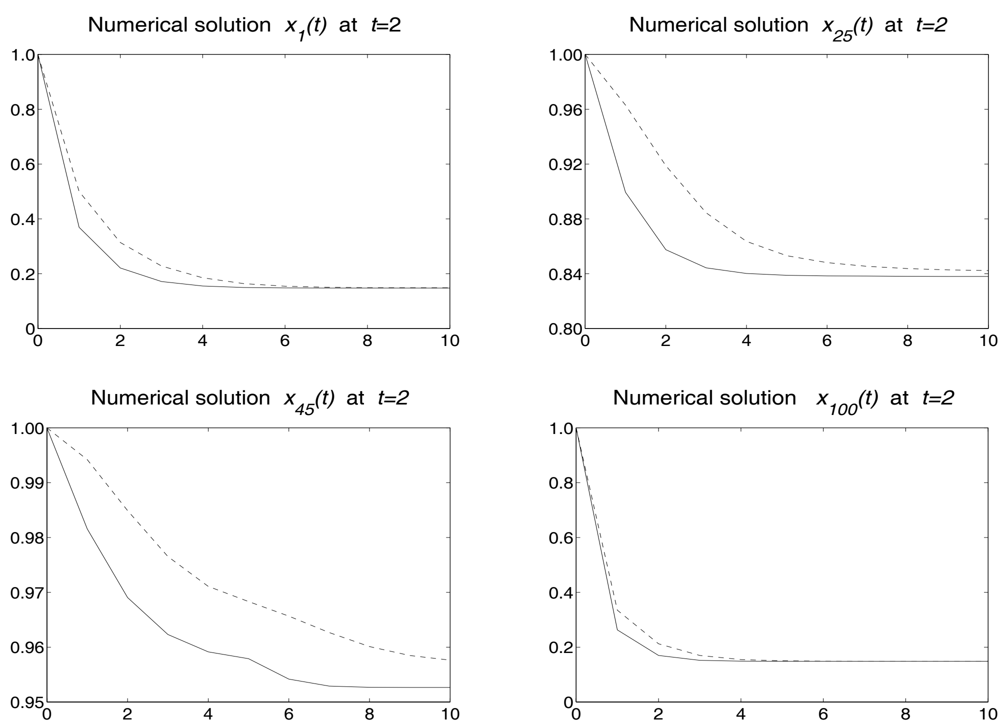

Fig. 7.2. The first ten iterations of $x(t)$ obtained by BWGS and PWGS methods, respectively.

In Figure 7.2, we plot $x_1(t), x_{25}(t), x_{45}(t)$ and $x_{100}(t)$ at $t = 2$, where the solid line and the dashed line denote the numerical solution obtained by PWGS abd BWGS methods, respectively.

Figure 7.1 shows the monotone convergence property of the BWGS method, and Figure 7.2 suggests that BWGS method converges faster than PWGS method. These numerical results well coincide with the theoretical results in Sections 5 and 6.

8. Conclusions

We have studied the asymptotic and monotone convergence properties of the pointwise/ blockwise waveform relaxation methods under suitable conditions. Both theoretical and experimental analyses show that the blockwise waveform relaxation methods converge faster than their pointwise alternatives in the monotone sense.

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