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$\mathcal{H}$-STABILITY OF RUNGE-KUTTA METHODS WITH VARIABLE STEPSIZE FOR SYSTEM OF PANTOGRAPH EQUATIONS *

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Abstract

This paper deals with $\mathcal{H}$-stability of Runge-Kutta methods with variable stepsize for the system of pantograph equations. It is shown that both Runge-Kutta methods with nonsingular matrix coefficient A and stiffly accurate Runge-Kutta methods are $\mathcal{H}$-stable if and only if the modulus of stability function at infinity is less than 1.

Mathematics subject classification: 65L06, 65L20.
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1. Introduction

In recent years, much research has been focused on theoretical and numerical solutions of pantograph equations. These systems can be found in variety of scientific and engineering fields such as analytic number theory, nonlinear dynamical systems, collection of current by the pantograph of an electric locomotive and so on, which have a comprehensive list in [7].

As far as we know the delay differential equations can be classified into two cases according to time lag, one is those with finite time lags, and the other is those with infinite time lags. There are remarkable differences, both analytically and numerically, between these two classes. Theoretical study of the second class of equations can be found in [7]. The numerical methods for this class have been studied by [2], in which the grid is uniform. However, this kind of equation has unbounded time lags, it is usually difficult to investigate numerically the long time dynamical behavior of exact solution due to limited computer memory as shown in [10]. There are two kinds of ways to avoid the storage problem. One is to transform the equation into an equation with constant time lag and variable coefficients as shown in [9] and apply a numerical method with constant stepsize $h = \frac{-\ln q}{m}$ to it. As a matter of fact, it seems like applying the numerical method with a grid which is not uniform, $t_n = q^{-\frac{n}{m}}$ and variable stepsizes $h_n = q^{-\frac{n}{m}}(q^{-\frac{1}{m}} - 1)$. Another way is applying a numerical method with variable stepsizes to the equation directly, which are considered in [1,10]. The nonconstant stepsize strategy is considered in [1], which is a special case of that introduced by Liu [10].

In this paper, we focus on the $\mathcal{H}$-stability of Runge-Kutta methods with variable stepsize applied to the system of pantograph equations. Some conclusions about the asymptotical stability of analytical solutions to the system are recalled. Furthermore, variable stepsize scheme is given. Finally, Runge-Kutta methods with nonsingular matrix coefficient A and stiffly accurate Runge-Kutta methods are applied to this system, respectively. The same sufficient and necessary condition such that the methods are $\mathcal{H}$-stable is presented.

2. Runge-Kutta Method with Variable Stepsize

In this paper, we consider the two-dimensional pantograph equations:

$$\begin{cases} x'(t) &= \lambda_1 x(t) + \mu_1 y(q_1 t), \\ y'(t) &= \lambda_2 y(t) + \mu_2 x(q_2 t), \end{cases} \qquad t > 0, \tag{2.1}$$

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where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathcal{C}$, $0 < q_2 \le q_1 < 1$ with the initial conditions $x(0) = x_0$, $y(0) = y_0$.

The existence and uniqueness of solutions to the system (2.1) has been studied by [7]. It is demonstrated in our another recent paper [11] that the solutions tend to zero (algebraically) if

$$|\mu_1| < -\Re\lambda_1, \quad |\mu_2| < -\Re\lambda_2 \quad \text{and} \quad \Re\lambda_1 < 0, \quad \Re\lambda_2 < 0, \tag{2.2}$$

where $\Re\lambda$ is the real part of λ.

Definition 2.1. *System (2.1) is called asymptotically stable if, for any $q_1, q_2 \in (0,1)$ and any initial values, the solutions $x(t), y(t)$ of this system tend to zero as t approaches infinity.*

Thus the analytical stability region can be defined as $\mathbf{S}=\{(\lambda_1,\lambda_2,\mu_1,\mu_2)\in\mathcal{C}^4 | (\lambda_1,\lambda_2,\mu_1,\mu_2)$ satisfies condition (2.2)$\}$.

In the next two sections, we focus our attention on the numerically asymptotical stability of Runge-Kutta method. Now we recall the method presented in [3] and the variable stepsize schemes introduced in [11].

For the general pantograph equation:

$$z'(t) = f(t, z(t), z(qt)),$$

where $0 < q < 1$.

The Runge-Kutta method, presented by [3], gives out the recurrence relation:

$$\begin{cases} z_{n+1} &= z_n + h_{n+1} \sum\limits_{i=1}^{s} b_i f(t_n + c_i h_{n+1}, Z_{n,i}, z^h(t_n + c_i h_{n+1})), \\ Z_{n,i} &= z_n + h_{n+1} \sum\limits_{j=1}^{s} a_{ij} f(t_n + c_j h_{n+1}, Z_{n,j}, z^h(t_n + c_j h_{n+1})), \end{cases} \tag{2.3}$$

where (A, b, c) denotes Runge-Kutta method, with matrix $A = (a_{ij})_{s\times s}$, vectors $b = (b_1, b_2, \ldots, b_s)^T$, $c = (c_1, c_2, \ldots, c_s)^T$. And $Z_{n,i}$, $z^h(t_n + c_i h_{n+1})$ can be interpreted as the approximation to $z(t_n + c_i h_{n+1})$ and $z(q(t_n + c_i h_{n+1}))$ respectively, for $i = 1, \ldots, s$.

Here, $z^h(t)$ is defined by the piecewise linear interpolation for $t > 0$, i.e.,

$$z^h(t) = \frac{t - t_i}{t_j - t_i} z_j + \frac{t_j - t}{t_j - t_i} z_i, \qquad t_i < t \le t_j.$$

Here, variable stepsize introduced in [11] is recalled.

For simplicity, without loss of generality, we assume that $t_0 = 1$ and the numerical solution is available till some point $t_0 > 0$.

Let $T_l = \frac{1}{q^l} t_0 = \frac{1}{q^l}$, $T_{l+1} = \frac{1}{q^{l+1}} t_0 = \frac{1}{q^{l+1}}$, where $l = 0, 1, 2, \ldots$.

If we are interested in the values of $z(t)$ at points $t^{(1)}, t^{(2)}, \ldots, t^{(m-1)}$ with $t^{(1)} < t^{(2)} < \ldots < t^{(m-1)}$, then there must exist integers $k_1, k_2, \ldots, k_{m-1}$ such that $q^{k_i} \cdot t^{(i)} \in [T_0, T_1)$, for $i = 1, 2, \ldots, m-1$ and $q \in (0,1)$.

We define the grid points and variable stepsizes as follows,

$$t_0 = T_0, \quad t_m = T_1, \quad t_i = q^{k_i} t^{(i)},$$

and

$$t_{km+i} = q^{-k} t_i, \quad h_{n+1} = t_{n+1} - t_n,$$

for $k = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, m-1$.

It is easy to see that for any n, we have

$$qt_n = t_{n-m}, \quad qh_n = h_{n-m} \quad \text{and} \quad \lim_{n\to\infty} h_n = \infty, \tag{2.4}$$

and the stepsizes increase geometrically.

If $h_1 = h_2 = \ldots = h_{m-1}$, then the above definition about grid points and variable stepsizes turns out to be that described in [1]. If $t_i = q^{-\frac{i}{m}}, i = 0, 1, 2, \ldots, m-1$, then the above definition is shown in [10] and appears to be the constant step in [9], by a change of the variable.

Then we formulate the condition of numerically asymptotical stability. The condition will depend on parameters λ_1, λ_2 and μ_1, μ_2, but turns out to be independent of q_1, q_2 and of the constant m that defines the mesh $H = \{t_0, t_1, \ldots, t_n, \ldots\}$ in (2.4) (see also Section 3 in [1]).

Definition 2.2. *Let $q_1, q_2 \in (0, 1)$ and $H = \{t_0, t_1, \ldots, t_n, \ldots\}$ be an assigned mesh. The asymptotical stability region of a numerical method for system (2.1) is the set $\boldsymbol{S}(\boldsymbol{H})$ of $(\lambda_1, \lambda_2, \mu_1, \mu_2)$, such that the discrete numerical solutions x_n, y_n tend to zero as n approaches infinity.*

Definition 2.3. *The numerical method applied to system (2.1) is called $\mathcal{H}$-stable, if $\boldsymbol{S}(\boldsymbol{H}) \supseteq \boldsymbol{S}$, for any $q_1, q_2 \in (0, 1)$ and $H = \{t_0, t_1, \ldots, t_n, \ldots\}$.*

Moreover, for system (2.1), since $0 < q_2 \le q_1 < 1$, there must exist a positive integer l such that $q_1^{l+1} \le q_2 \le q_1^l$.

Here, we choose $q = q_1$ in (2.4) for system (2.1), then $t_{n-(l+1)m} + c_i h_{n+1-(l+1)m} \le q_2(t_n + c_i h_{n+1}) \le t_{n-lm} + c_i h_{n+1-lm}$.

Let $\delta_1 = \frac{q_2}{q_1^{l+1}} \ge 1$ and $\delta = \frac{q_2(t_n+c_ih_{n+1})-(t_{n-(l+1)m}+c_ih_{n+1-(l+1)m})}{(t_{n-lm}+c_ih_{n+1-lm})-(t_{n-(l+1)m}+c_ih_{n+1-(l+1)m})} \in [0, 1]$, then by (2.4), we have

$$\delta = \frac{(\delta_1 - 1)q_1}{1 - q_1},$$

which implies that δ is independent of m and n.

3. $\mathcal{H}$-stability of Runge-Kutta Method with Nonsingular A

This section deals with Runge-Kutta method with nonsingular coefficient A. Using the above results of variable stepsize, we find for system (2.1), the Runge-Kutta method (2.3), gives out the recurrence relation:

$$\begin{cases} x_{n+1} &= x_n + h_{n+1} \sum\limits_{i=1}^{s} b_i(\lambda_1 X_{n,i} + \mu_1 Y_{n-m,i}), \\ X_{n,i} &= x_n + h_{n+1} \sum\limits_{j=1}^{s} a_{ij}(\lambda_1 X_{n,j} + \mu_1 Y_{n-m,j}), \\ y_{n+1} &= y_n + h_{n+1} \sum\limits_{i=1}^{s} b_i[\lambda_2 Y_{n,i} + \mu_2((1-\delta)X_{n-(l+1)m,i} + \delta X_{n-lm,i})], \\ Y_{n,i} &= y_n + h_{n+1} \sum\limits_{j=1}^{s} a_{ij}[\lambda_2 Y_{n,j} + \mu_2((1-\delta)X_{n-(l+1)m,j} + \delta X_{n-lm,j})], \end{cases} \tag{3.1}$$

where $X_{n,i}$, $Y_{n,i}$ is interpreted as the approximation to $x(t_n + c_i h_{n+1})$, $y(t_n + c_i h_{n+1})$ for $i = 1, 2, \ldots, s$, respectively.

Let

$$U_{n+1} = (x_{n+1}, X_{n,1}, X_{n,2}, \ldots, X_{n,s}, y_{n+1}, Y_{n,1}, Y_{n,2}, \ldots, Y_{n,s})^T,$$

then

$$U_{n+1} = G_1^{(n)} U_n + G_2^{(n)} U_{n-m+1} + \delta G_3^{(n)} U_{n-lm+1} + (1-\delta) G_3^{(n)} U_{n-(l+1)m+1},$$

where $G_1^{(n)} = \begin{pmatrix} M_1^{(n)} & 0 \\ 0 & M_2^{(n)} \end{pmatrix}$, $G_2^{(n)} = \begin{pmatrix} 0 & N_1^{(n)} \\ 0 & 0 \end{pmatrix}$, $G_3^{(n)} = \begin{pmatrix} 0 & 0 \\ N_2^{(n)} & 0 \end{pmatrix}$ with $M_i^{(n)} = \begin{pmatrix} \Delta_{n,i} & 0 \\ \Gamma_{n,i}e & 0 \end{pmatrix}$, $N_i^{(n)} = \begin{pmatrix} 0 & \mu_{n,i}b^T\Gamma_{n,i} \\ 0 & \mu_{n,i}\Gamma_{n,i}A \end{pmatrix}$, and $\lambda_{n,i} = \lambda_i h_{n+1}$, $\mu_{n,i} = \mu_i h_{n+1}$, $\Gamma_{n,i} = (I - \lambda_{n,i}A)^{-1}$, $\Delta_{n,i} = 1 + \lambda_{n,i}b^T\Gamma_{n,i}e$ for $i = 1, 2$, and 0 is the matrix with appropriate dimension .

Let $V_n = (U_n^T, \ldots, U_{n-(l-1)m}^T, \ldots, U_{n-lm}^T)^T$, then $V_{n+1} = Q_1^{(n)}V_n + Q_2^{(n)}V_{n-m+1}$, where

$$Q_1^{(n)} = \begin{pmatrix} G_1^{(n)} & 0 & \ldots & 0 & 0 \\ I & 0 & \ldots & 0 & 0 \\ \ldots & & \ldots & & \ldots \\ 0 & 0 & \ldots & I & 0 \end{pmatrix},$$

$$Q_2^{(n)} = \begin{pmatrix} G_2^{(n)} & 0 & \ldots & \delta G_3^{(n)} & \ldots & 0 & (1-\delta)G_3^{(n)} \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \ldots & & \ldots & & \ldots & & \ldots \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Let $W_n = (V_n^T, V_{n-1}^T, \ldots, V_{n-m+1}^T)^T$, then $W_{n+1} = K_n W_n$, where

$$K_n = \begin{pmatrix} Q_1^{(n)} & 0 & \ldots & 0 & 0 & Q_2^{(n)} \\ I & 0 & \ldots & 0 & 0 & 0 \\ \ldots & & \ldots & & & \ldots \\ 0 & 0 & \ldots & 0 & I & 0 \end{pmatrix}.$$

Let $K = \lim_{n\to\infty} K_n$, $Q_i = \lim_{n\to\infty} Q_i^{(n)}$, $G_j = \lim_{n\to\infty} G_j^{(n)}$, for $i = 1, 2$, $j = 1, 2, 3$, then

$$G_1 = \begin{pmatrix} 1 - b^TA^{-1}e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - b^TA^{-1}e & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & -\frac{\mu_1}{\lambda_1}b^TA^{-1} \\ 0 & 0 & 0 & -\frac{\mu_1}{\lambda_1}I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu_2}{\lambda_2}b^TA^{-1} & 0 & 0 \\ 0 & -\frac{\mu_2}{\lambda_2}I & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} G_1 & 0 & \ldots & 0 & 0 \\ I & 0 & \ldots & 0 & 0 \\ \ldots & & \ldots & & \ldots \\ 0 & 0 & \ldots & I & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} G_2 & 0 & \ldots & \delta G_3 & \ldots & 0 & (1-\delta)G_3 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \ldots & & \ldots & & \ldots & & \ldots \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} Q_1 & 0 & \ldots & 0 & Q_2 \\ I & 0 & \ldots & 0 & 0 \\ \ldots & & \ldots & & \ldots \\ 0 & 0 & \ldots & I & 0 \end{pmatrix}.$$

Thus, the characteristic polynomial of K could be read

$$\det[C_K(\zeta)] = \det[\zeta I - K] = \det[(\zeta I - Q_1)\zeta^{m-1} - Q_2], \tag{3.2}$$

The following lemma is an immediate consequence of Corollary 1.2 in [5] (see also Lemma 2.1 in [8]).

Lemma 3.1. *Assume that* $|\zeta-(1-b^TA^{-1}e)|\neq 0$, *and consider the following three statements:*
(a) $det[\zeta I-Q_1]\neq 0$ *for any* $|\zeta|\geq 1$ *and* $sup_{|\zeta|=1}\rho[(\zeta I-Q_1)^{-1}Q_2]<1$;
(b) all roots of characteristic polynomial (3.2) lie inside the unit circle for any $m\geq 0$;
(c) $det[\zeta I-Q_1]\neq 0$ *for any* $|\zeta|\geq 1$ *and* $sup_{|\zeta|=1}\rho[(\zeta I-Q_1)^{-1}Q_2]\leq 1$,
where $\rho(\cdot)$ *denotes the spectral radius of a matrix. Then (a) implies (b) and (b) implies (c).*

Theorem 3.1. *Assume the matrix A is nonsingular, then Runge-Kutta method* (A,b,c), *applied to the system (2.1) with condition (2.2), is* $\mathcal{H}$*-stable if and only if* $|r_\infty|<1$, *where* $r_\infty=1-b^TA^{-1}e$ *is the limit of stability function of Runge-Kutta method.*

Proof. Assume that $|r_\infty|<1$, it is easily seen that $\det(\zeta I-G_1)\neq 0$ for any $|\zeta|\geq 1$. This implies that the matrix $(\zeta I-Q_1)$ is invertible, whenever $|\zeta|\geq 1$.

Moreover, after some calculations, we obtain

$$(\zeta I-Q_1)^{-1}Q_2=$$
$$\begin{pmatrix} (\zeta I-G_1)^{-1}G_2 & 0 & \dots & 0 & \delta(\zeta I-G_1)^{-1}G_3 & 0 & \dots & 0 & (1-\delta)(\zeta I-G_1)^{-1}G_3 \\ \zeta^{-1}(\zeta I-G_1)^{-1}G_2 & 0 & \dots & 0 & \delta\zeta^{-1}(\zeta I-G_1)^{-1}G_3 & 0 & \dots & 0 & (1-\delta)\zeta^{-1}(\zeta I-G_1)^{-1}G_3 \\ \dots & & \dots & & \dots & & \dots & & \dots \\ \zeta^{-lm}(\zeta I-G_1)^{-1}G_2 & 0 & \dots & 0 & \delta\zeta^{-lm}(\zeta I-G_1)^{-1}G_3 & 0 & \dots & 0 & (1-\delta)\zeta^{-lm}(\zeta I-G_1)^{-1}G_3 \end{pmatrix}.$$

where

$$(\zeta I-G_1)^{-1}G_2=\begin{pmatrix} 0 & 0 & 0 & \Lambda_1 \\ 0 & 0 & 0 & -\zeta^{-1}\frac{\mu_1}{\lambda_1}I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (\zeta I-G_1)^{-1}G_3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \Lambda_2 & 0 & 0 \\ 0 & -\zeta^{-1}\frac{\mu_2}{\lambda_2}I & 0 & 0 \end{pmatrix}$$

with $\Lambda_i=-[\zeta-(1-b^TA^{-1}e)]^{-1}\cdot\frac{\mu_i}{\lambda_i}b^TA^{-1}$ for $i=1,2$.

To study eigenvalues of matrix $(\zeta I-Q_1)^{-1}Q_2$, we just consider the characteristic polynomial

$$\begin{aligned} & \det[\xi I_{2(lm+1)(s+1)}-(\zeta I-Q_1)^{-1}\cdot Q_2] \\ = \; & \det[\xi I_{2(s+1)}-(\zeta I-G_1)^{-1}G_2]\cdot\det[\xi I_{2(s+1)}-\delta\zeta^{-(l-1)m}(\zeta I-G_1)^{-1}G_3] \\ \cdot \; & \det[\xi I_{2(s+1)}-(1-\delta)\zeta^{-lm}(\zeta I-G_1)^{-1}G_3]\cdot(\det[\xi I_{2(s+1)}])^{lm-2}. \end{aligned}$$

where I is the unit matrix with appropriate dimension.

It can be proved that $\sup\rho[(\zeta I-Q_1)^{-1}Q_2]<1$, whenever $|\zeta|=1$.

By Lemma 3.1, the above two conclusions imply that the roots ζ of characteristic polynomial (3.2) satisfy $|\zeta|<1$.

According to Lemma 5.6.10 in [4], there exists a norm $\|\cdot\|$ such that $\|K\|<1$. In view of the definition of limit, for any fixed ε with $1-\|K\|>\varepsilon>0$, there exists $N_1>0$ such that, for $n\geq N_1$, $\|K_n-K\|<1-\|K\|-\varepsilon$, and $\|W_{n+1}\|\leq(\|K\|+\|K_n-K\|)\cdot\|W_n\|<(1-\varepsilon)\|W_n\|$, which means $\lim_{n\to\infty}x_n=0$, $\lim_{n\to\infty}y_n=0$.

Condition (2.2) ensures that the analytical solutions of system (2.1) can be asymptotically stable. However, it is not used here in proving the $\mathcal{H}$-stability of numerical solutions.

Conversely, assume that $|r_\infty|\geq 1$. Then we choose $\mu_1=\mu_2=0$ and $\Re\lambda_1,\Re\lambda_2<0$, which satisfies condition (2.2) and the system (3.1) reduces to $x_{n+1}=r(\lambda_{n,1})x_n$, $y_{n+1}=r(\lambda_{n,2})y_n$, where $r(\lambda_{n,i})=1+\lambda_{n,i}b^T(I-\lambda_{n,i}A)^{-1}e$ and $\lim_{n\to\infty}r(\lambda_{n,i})=r_\infty$.

If $|r_\infty|>1$, then for any ε with $|r_\infty|-1>\varepsilon>0$, there exists N_2 such that for $n\geq N_2$,

$$\begin{aligned} |x_n| &= |(r_\infty+r(\lambda_{n,1})-r_\infty)x_n| \\ &\geq (|r_\infty|-|r(\lambda_{n,1})-r_\infty|)\cdot|x_n| \\ &\geq (|r_\infty|-\varepsilon)\cdot|x_n|, \end{aligned}$$

which implies x_n does not vanish, and y_n also does not vanish.

If $|r_\infty| = 1$, it is obviously seen that x_n and y_n are bounded, but does not vanish.

Corollary 3.1. *The Gauss-Legendre methods are not $\mathcal{H}$-stable.*

Corollary 3.2. *The RadauIA methods, RadauIIA methods, LobattoIIIC methods are $\mathcal{H}$-stable.*

Corollary 3.3. *The one-leg θ-method is $\mathcal{H}$-stable if and only if $1/2 < \theta \le 1$.*

4. $\mathcal{H}$-stability of Stiffly Accurate Method

This part is concerned with the Runge-Kutta method with singular A. Here we just consider the stiffly accurate Runge-Kutta method, i.e., $a_{sj} = b_j$, $1 \le j \le s$.

Since A is singular, without loss of generality, we assume that $a_{1j} = 0$, $1 \le j \le s$, and write $A = \begin{pmatrix} 0 & 0 \\ \bar{a} & \bar{A} \end{pmatrix}$, where $\bar{a} = (a_{21}, a_{31}, \dots, a_{s1})^T$, $\bar{A} = (a_{ij})_{i,j=2}^{s}$ which is nonsingular.

In fact, LobattoIIIA methods and linear θ-method are typical examples of such RK method.

In [9], some properties of stiffly accurate method are given.

Lemma 4.1. *If the Runge-Kutta method is stiffly accurate and $\bar{A}$ is nonsingular, then $\lim_{z\to\infty} r(z) = -e_{s-1}^T \bar{A}^{-1}\bar{a}$, with $e_{s-1} = (0, 0, \dots, 0, 1)^T \in \mathcal{R}^{s-1}$ and $r(z) = 1 + zb^T(I - zA)^{-1}e$.*

Applying this special method to system (2.1), if the RK method is stiffly accurate, then from proof of Theorem 4.1 in [9], we have $x_{n+1} = X_{n,s}$, $y_{n+1} = Y_{n,s}$.

Let $\bar{U}_{n+1} = (X_{n,1}, X_{n,2}, \dots, X_{n,s-1}, x_{n+1}, Y_{n,1}, Y_{n,2}, \dots, Y_{n,s-1}, y_{n+1})^T$, then

$$\bar{U}_{n+1} = \bar{G}_1^{(n)}\bar{U}_n + \bar{G}_2^{(n)}\bar{U}_{n-m+1} + \delta\bar{G}_3^{(n)}\bar{U}_{n-lm+1} + (1-\delta)\bar{G}_3^{(n)}\bar{U}_{n-(l+1)m+1}, \tag{4.1}$$

where $\bar{G}_1^{(n)} = \begin{pmatrix} R_1^{(n)} & 0 \\ 0 & R_2^{(n)} \end{pmatrix}$, $\bar{G}_2^{(n)} = \begin{pmatrix} 0 & S_1^{(n)} \\ 0 & 0 \end{pmatrix}$, $\bar{G}_3^{(n)} = \begin{pmatrix} 0 & 0 \\ S_2^{(n)} & 0 \end{pmatrix}$ with $R_i^{(n)} = \begin{pmatrix} 0 & 1 \\ 0 & \bar{\Delta}_{n,i} \end{pmatrix}$, $S_i^{(n)} = \begin{pmatrix} 0 & 0 \\ \mu_{n,i}\bar{\Gamma}_{n,i}\bar{a} & \mu_{n,i}\bar{\Gamma}_{n,i}\bar{A} \end{pmatrix}$, and $\bar{\Gamma}_{n,i} = (I - \lambda_{n,i}\bar{A})^{-1}$, $\bar{\Delta}_{n,i} = \lambda_{n,i}\bar{\Gamma}_{n,i}\bar{a} + \bar{\Gamma}_{n,i}e$ for $i = 1, 2$, and 0 is the matrix with appropriate dimension.

Let $\bar{V}_n = (\bar{U}_n^T, \dots, \bar{U}_{n-(l-1)m}^T, \dots, \bar{U}_{n-lm}^T)^T$, then $\bar{V}_{n+1} = \bar{Q}_1^{(n)}\bar{V}_n + \bar{Q}_2^{(n)}\bar{V}_{n-m+1}$, where

$$\bar{Q}_1^{(n)} = \begin{pmatrix} \bar{G}_1^{(n)} & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & \dots & I & 0 \end{pmatrix},$$

$$\bar{Q}_2^{(n)} = \begin{pmatrix} \bar{G}_2^{(n)} & 0 & \dots & \delta\bar{G}_3^{(n)} & \dots & 0 & (1-\delta)\bar{G}_3^{(n)} \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ \dots & & \dots & & \dots & & \dots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Let $\bar{W}_n = (\bar{V}_n^T, \bar{V}_{n-1}^T, \dots, \bar{V}_{n-m+1}^T)^T$, then $\bar{W}_{n+1} = J_n\bar{W}_n$, where

$$J_n = \begin{pmatrix} \bar{Q}_1^{(n)} & 0 & \dots & 0 & 0 & \bar{Q}_2^{(n)} \\ I & 0 & \dots & 0 & 0 & 0 \\ \dots & & \dots & & & \dots \\ 0 & 0 & \dots & 0 & I & 0 \end{pmatrix}.$$

Denoting $J = \lim_{n\to\infty} J_n$, $\bar{Q}_i = \lim_{n\to\infty} \bar{Q}_i^{(n)}$, $\bar{G}_j = \lim_{n\to\infty} \bar{G}_j^{(n)}$, $R = R_i = \lim_{n\to\infty} R_i^{(n)}$, $S_i = \lim_{n\to\infty} S_i^{(n)}$, for $i = 1, 2$, $j = 1, 2, 3$, then $R = \begin{pmatrix} 0 & 1 \\ 0 & -\bar{A}^{-1}\bar{a} \end{pmatrix}$, $S_i = \begin{pmatrix} 0 & 0 \\ -\frac{\mu_i}{\lambda_i}\bar{A}^{-1}\bar{a} & -\frac{\mu_i}{\lambda_i}I \end{pmatrix}$.

Thus, the characteristic polynomial of J could be read

$$\det[C_J(\zeta)] = \det[(\zeta I - \bar{Q}_1)\zeta^{m-1} - \bar{Q}_2], \tag{4.2}$$

Theorem 4.1. *Assume the matrix $\bar{A}$ is nonsingular, then the stiffly accurate Runge-Kutta method (A, b, c), applied to the system (2.1) with condition (2.2), is $\mathcal{H}$-stable if and only if $|\bar{r}_\infty| < 1$, where $\bar{r}_\infty = -e_{s-1}^T \bar{A}^{-1}\bar{a}$.*

Proof. Assume that $|\bar{r}_\infty| < 1$, we denote $\omega = (-1, r_1, \ldots, r_{s-2})^T$ with $r_i = e_i^T \bar{A}^{-1}\bar{a}$, and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathcal{R}^{s-1}$, then $\det(\zeta I - R) = det \begin{pmatrix} \zeta I & \omega \\ 0 & \zeta + r_{s-1} \end{pmatrix} \neq 0$, which means that $(\zeta I - \bar{Q}_1)$ is invertible whenever $|\zeta| \geq 1$.

After some calculations, we have

$$(\zeta I - \bar{Q}_1)^{-1}\bar{Q}_2 =$$
$$\begin{pmatrix} (\zeta I - \bar{G}_1)^{-1}\bar{G}_2 & 0 & \ldots & 0 & \delta(\zeta I - \bar{G}_1)^{-1}\bar{G}_3 & 0 & \ldots & 0 & (1-\delta)(\zeta I - \bar{G}_1)^{-1}\bar{G}_3 \\ \zeta^{-1}(\zeta I - \bar{G}_1)^{-1}\bar{G}_2 & 0 & \ldots & 0 & \delta\zeta^{-1}(\zeta I - \bar{G}_1)^{-1}\bar{G}_3 & 0 & \ldots & 0 & (1-\delta)\zeta^{-1}(\zeta I - \bar{G}_1)^{-1}\bar{G}_3 \\ \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\ \zeta^{-lm}(\zeta I - \bar{G}_1)^{-1}\bar{G}_2 & 0 & \ldots & 0 & \delta\zeta^{-lm}(\zeta I - \bar{G}_1)^{-1}\bar{G}_3 & 0 & \ldots & 0 & (1-\delta)\zeta^{-lm}(\zeta I - \bar{G}_1)^{-1}\bar{G}_3 \end{pmatrix}.$$

Similar to the analysis of Theorem 3.1, we have $\sup\rho[(\zeta I - \bar{Q}_1)^{-1}\bar{Q}_2] < 1$, if $|\zeta| = 1$, and $\rho(\cdot)$ denotes the spectral radius.

According to Lemma 3.1, it can be proved that stiffly accurate Runge-Kutta method is $\mathcal{H}$-stable.

Conversely, assume that $|\bar{r}_\infty| \geq 1$, then we choose $\mu_1 = \mu_2 = 0$ and $\Re\lambda_1, \Re\lambda_2 < 0$, which satisfies condition (2.2), and the system (4.1) reduces to

$$x_{n+1} = r(\lambda_{n,1})x_n, \qquad y_{n+1} = r(\lambda_{n,2})y_n,$$

Similar to the analysis of Theorem 3.1, it can be easily proved that $|\bar{r}_\infty| < 1$.

Corollary 4.1. *The LobattoIIIA methods are not $\mathcal{H}$-stable.*

Corollary 4.2. *The linear θ-method is $\mathcal{H}$-stable if and only if $1/2 < \theta \leq 1$.*

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