Journal of Computational Mathematics, Vol.23, No.2, 2005, 165–176.

FINITE VOLUME NUMERICAL ANALYSIS FOR PARABOLIC EQUATION WITH ROBIN BOUNDARY CONDITION *1)

Xia Cui
(*Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009-26, Beijing 100088, China*)

Abstract

In this paper, finite volume method on unstructured meshes is studied for a parabolic convection-diffusion problem on an open bounded set of R^d ($d = 2$ or 3) with Robin boundary condition. Upwinding approximations are adapted to treat both the convection term and Robin boundary condition. By directly getting start from the formulation of the finite volume scheme, numerical analysis is done. By using several discrete functional analysis techniques such as summation by parts, discrete norm inequality, et al, the stability and error estimates on the approximate solution are established, existence and uniqueness of the approximate solution and the 1st order temporal norm and L^2 and H^1 spacial norm convergence properties are obtained.

Mathematics subject classification: 65M12, 76M12.
Key words: Finite volume, Parabolic convection diffusion equations, Numerical analysis.

1. Introduction

Finite Volume Methods are known to be well applicable to the numerical simulation of many problems, particularly in the presence of convection terms, with irregular geometry domain or unstructed meshes partition. Many works have been done on their construction and application, as well as some theoretical studies [1]-[3]. Two main directions are usually followed to obtain their convergence properties. One is to write the finite volume as a finite element or mixed finite element method by some numerical integration, and follow the general finite element framework to prove the convergence (see, for instance, [3], where they come forward as generalized difference methods, and the citations of [1]). The second (see, for example, [1][2][4]) is to establish the convergence by using the direct formulation of the finite volume scheme together with appropriate discrete functional analysis techniques; following which, for elliptic equation, general boundary condition problems are studied in [1][2]; for parabolic equation, L^2 and H^1 error estimate only for Dirichlet boundary problem is considered respectively in [1][2] and [4].

In this paper, the finite volume discrete method on unstructured meshes including Voronoï or triangular meshes for parabolic convection-diffusion problem with a general Robin boundary condition is studied. The second approach, which is natural and direct to the original problem, is applied for numerical analysis. An "s" points (where s is the number of sides of each cell) finite volume scheme and an upstream scheme is adapted for the diffusion and the convection term respectively. An artificial upwinding is introduced in the treatment of the Robin boundary condition in order for the scheme to be well defined with no additional restriction on the mesh. But it brings difficulties and requires additional work for numerical analysis compared to that of the Dirichlet or Neumann case, which appears more evident for time involved parabolic

* Received July 7, 2003.
1) The project is supported by China National Key Program for Developing Basic (G1999032801), Mathematical Tianyuan Foundation (10226026), the National Natural Science Foundation of China (No.19932010) and the Foundation of CAEP (20040653).

problem compared with stationary elliptic problem. To solve this question, several discrete functional analysis techniques including summation-by-parts formula, discrete norm inequality, et al, are used. The stability and error estimates on the approximate solution are established. The existence and uniqueness of the approximate solution are shown. If the exact solution is at least in $L^\infty(0,T;H^2(\Omega))$, then the 1st order $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;H^1(\Omega))$ norm convergence of the scheme are obtained.

Consider the parabolic problem with a Robin boundary condition:

$$\begin{aligned} u_t + \nabla\cdot(a\nabla u) + div(vu) + bu = f, &\quad x\in\Omega,\ t\in J,\\ a\nabla u\cdot\eta + \lambda u = g, &\quad x\in\partial\Omega,\ t\in J,\\ u(x,0) = u_0(x), &\quad x\in\Omega, \end{aligned} \tag{1.1}$$

where Ω is an open bounded subset of R^d ($d=2$, or 3) which is a polygonal for $d=2$ and polyhedral for $d=3$ with $\partial\Omega$ its boundary, η is the unit normal to $\partial\Omega$ outward to Ω. $J=[0,T]$, with T a positive constant. $v=v(x,t)$ is a given vector function, $a=a(x,t), b=b(x,t), f=f(x,t), \lambda=\lambda(x,t), g=g(x,t)$ are given functions. Herein we study problem (1.1) with the following assumptions.

Assumption 1. For $t\in J$, $f(\cdot,t)\in L^2(\Omega)$, $b(\cdot,t)\in L^\infty(\Omega)$ and $v(\cdot,t)\in C^1(\bar\Omega)$ such that $div(v)/2+b\geq 0$ almost everywhere (a.e.) on Ω.

Assumption 2. For $t\in J$, $g(\cdot,t)\in H^{\frac12}(\partial\Omega)$, $\lambda(\cdot,t)\in L^\infty(\partial\Omega)$ such that $v\cdot\eta/2+\lambda\geq 0\, a.e.\, on\, \partial\Omega$. Furthermore, if $v\cdot\eta/2+\lambda=0\, a.e.\, on\, \partial\Omega$, then one assumes the existence of $\mathcal{O}\subset\bar\Omega$ such that its d-dimensional measure $m(\mathcal{O})\neq 0$ and such that $div(v)/2+b\neq 0\, a.e.\, on\, \mathcal{O}$.

Assumption 3. For $t\in J$, $a(\cdot,t)$ is a piecewise C^1 function from $\bar\Omega$ to R, and there exists positive constant a_* such that $a(x,t)\geq a_*$ for a.e. $(x,t)\in\Omega\times J$.

Assumption 4. The functions a,b,λ,v and $div(v)$ are Lipschitz continuous with respect to t.

The outline of the paper is as follows. Section 2 introduces the restricted admissible meshes needed for the discretization, formulates the finite volume approximation and gives the definition of related spacial norms. Section 3 demonstrates corresponding numerical analysis, which includes the statement of stability and convergence properties and necessary reasoning procedure.

2. Finite Volume Discretization

2.1 Mesh Partition

Define the restricted admissible meshes as in [1], which includes meshes made with triangles and rectangles in two space dimensions, and Voronoï meshes.

Definition 1 (Restricted Admissible Meshes). *A finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of "control volumes", which are open polygonal (if $d=2$) or polyhedral (if $d=3$) convex subsets of Ω (with positive measure), a family of subsets of $\bar\Omega$ contained in hyperplanes of R^d, denoted by ε (these are the edges (if $d=2$) or sides (if $d=3$) of the control volumes), with strictly positive $(d-1)$-dimensional measure and a family of points of $\bar\Omega$ denoted by P. The finite volume mesh is called to be restricted admissible, if the properties (i) to (v) are satisfied.*

(i) The closure of the union of all the control volume is $\bar\Omega$.

(ii) For any $K\in\mathcal{T}$, there exists a subset ε_K of ε such that $\partial K=\bar K/K=\cup_{\sigma\in\varepsilon_K}\bar\sigma$. Let $\varepsilon=\cup_{K\in\mathcal{T}}\varepsilon_K$.

(iii) For any $(K,L)\in\mathcal{T}^2$ with $K\neq L$, either the $(d-1)$-dimensional Lebesgue measure of $\bar K\cap\bar L$ is 0 or $\bar K\cap\bar L=\bar\sigma$ for some $\sigma\in\varepsilon$, which will then be denoted by $K|L$.

(iv) The family $P=(x_K)_{K\in\mathcal{T}}$ is such that $x_K\in\bar K$ (for all $K\in\mathcal{T}$) and, if $\sigma=K|L$, it is assumed that $x_K\neq x_L$, and that the straight line $D_{K,L}$ going through x_K and x_L is orthogonal to $K|L$.

(v) For any $\sigma \in \varepsilon$ such that $\sigma \subset \partial\Omega$, let K be the control volume such that $\sigma \in \varepsilon_K$. If $x_K \notin \sigma$ (let $D_{K,\sigma}$ be the straight line going through x_K and orthogonal to σ), then the condition $D_{K,\sigma} \cap \sigma \neq \emptyset$ is assumed; let $y_\sigma = D_{K,\sigma} \cap \sigma$.

Define the mesh size by $h = size(\mathcal{T}) = \sup\{diam(K), K \in \mathcal{T}\}$, where $diam(K)$ is the diameter of $K \in \mathcal{T}$. For any $K \in \mathcal{T}$ and $\sigma \in \varepsilon$, $m(K)$ is the d-dimensional Lebesgue measure of K (i.e. area if $d = 2$, volume if $d = 3$), $m(\sigma)$ is the $(d-1)$-dimensional measure of σ, and $\eta_{K,\sigma}$ denotes the unit normal vector to σ outward to K.

Denote $\varepsilon_{int} = \{\sigma \in \varepsilon; \sigma \not\subset \partial\Omega\}$, $\varepsilon_{ext} = \{\sigma \in \varepsilon; \sigma \subset \partial\Omega\}$. For $K \in \mathcal{T}$, let $V_{K,\sigma} = \{\alpha x_K + (1-\alpha)x, x \in \sigma, \alpha \in [0,1]\}$. For $\sigma \in \varepsilon_{int}$, let $V_\sigma = V_{K,\sigma} \cap V_{L,\sigma}$, where K and L are the control volumes such that $\sigma = K|L$. For $\sigma \in \varepsilon \cap \varepsilon_{ext}$, let $V_\sigma = V_{K,\sigma}$. Denote

$$\zeta = \min_{K\in\mathcal{T}} \min_{\sigma\in\varepsilon_K} \frac{d_{K,\sigma}}{diam(K)}.$$

Denote by $d_{K|L}$ the Euclidean distance between x_K and x_L (which is positive) and by $d_{K,\sigma}$ the distance from x_K to σ. If $\sigma = K|L \in \varepsilon_{int}$, let $d_\sigma = d_{K|L} = d_{K,\sigma} + d_{L,\sigma}$; if $\sigma \in \varepsilon_K \cap \varepsilon_{ext}$, let $d_\sigma = d_{K,\sigma}$.

For any $\sigma \in \varepsilon$, the "transmissibility" through σ is defined by $\tau_\sigma = \frac{m(\sigma)}{d_\sigma}$ if $d_\sigma \neq 0$ and $\tau_\sigma = 0$ if $d_\sigma = 0$. In the results and proofs given below, $\sigma \neq 0$ for all $\sigma \in \varepsilon$ is assumed for simplicity.

Let $\mathcal{T}$ be an restricted admissible mesh in the sense of Definition 1. Let $k \in (0,T)$, be a constant time step, $N_k = max\{n \in \mathcal{N}, nk < T\}$, e.g., divide $[0,T]$ into $N_k + 1$ interval. Denote $t_n = nk$ for $n \in \{0, 1, \cdots, N_k + 1\}$. Denote $\phi^n = \phi(t_n)$, $d_t\phi^n = \frac{\phi^{n+1} - \phi^n}{k}$.

2.2 Finite Volume Approximation

Integrating (1.1) on each cell of the mesh at time $t = t_{n+1}$ yields

$$\int_K u_t^{n+1}(x)dx - \int_{\partial K}[a^{n+1}(x)\nabla u^{n+1}(x) - v^{n+1}(x)u^{n+1}(x)] \cdot \eta_K(x)d\gamma(x)$$
$$+\int_K b^{n+1}(x)u^{n+1}(x)dx = \int_K f^{n+1}(x)dx, \tag{2.1}$$
$$\int_\sigma a^{n+1}(x)\nabla u^{n+1}(x) \cdot \eta_{K,\sigma}(x)d\gamma(x) + \int_\sigma \lambda^{n+1}(x)u^{n+1}(x)d\gamma(x) = \int_\sigma g^{n+1}(x)d\gamma(x). \tag{2.2}$$

Using an "s-points" finite volume scheme for the diffusion terms and an upstream scheme for the convection terms, using an implicit time discretization, one gets, a discretization of (1.1), with unknowns $(U_K^{n+1})_{K\in\mathcal{T}} \cup (U_\sigma^{n+1})_{\sigma\in\varepsilon_{ext}}$, $n = 0, 1, \cdots, N_k$, such that

$$m(K)d_tU_K^n + \sum_{\sigma\in\varepsilon_K}(F_{K,\sigma}^{n+1} + v_{K,\sigma}^{n+1}U_{\sigma,+}^{n+1}) + b_K^{n+1}m(K)U_K^{n+1} = m(K)f_K^{n+1},$$
$$\forall K \in \mathcal{T}, \tag{2.3}$$
$$-F_{K,\sigma}^{n+1} + [m(\sigma)\lambda_\sigma^{n+1} + v_{K,\sigma}^{n+1}]U_\sigma^{n+1} - v_{K,\sigma}^{n+1}U_{\sigma,+}^{n+1} = m(\sigma)g_\sigma^{n+1},$$
$$\forall \sigma \in \varepsilon_K \cap \varepsilon_{ext}, \tag{2.4}$$

where, for $n = 0, 1, \cdots, N_k + 1$,

$$\begin{array}{rcl} F_{K,\sigma}^n &=& -m(K|L)a_\sigma^n\frac{U_L^n - U_K^n}{d_{K,L}}, \quad if\ \sigma = K|L, \\ F_{K,\sigma}^n d_{K,\sigma} &=& -m(\sigma)a_\sigma^n(U_\sigma^n - U_K^n), \quad if\ \sigma \in \varepsilon_K \cap \varepsilon_{ext}, \\ v_{K,\sigma}^n &=& \int_\sigma v(x,t_n) \cdot \eta_{K,\sigma}(x)d\gamma(x), \\ \phi_\sigma^n &=& \frac{1}{m(\sigma)}\int_\sigma \phi(x,t_n)d\gamma(x), \quad for\ \phi = a, \lambda, g; \\ \phi_K^n &=& \frac{1}{m(K)}\int_K \phi(x,t_n)dx, \quad for\ \phi = b, f; \end{array} \tag{2.5}$$

and for $\phi = U$,

$$\phi_{\sigma,+}^n = \begin{cases} \phi_K^n, & if\ v_{K,\sigma}^n \geq 0, \\ \phi_L^n, & otherwise, \end{cases} \qquad if\ \sigma = K|L;$$

$$\phi^n_{\sigma,+} = \begin{cases} \phi^n_K, & if\ v^n_{K,\sigma} \ge 0, \\ \phi^n_\sigma, & otherwise, \end{cases} \qquad if\ \sigma \in \varepsilon_K \cap \varepsilon_{ext}. \tag{2.6}$$

The upstream value $U^{n+1}_{\sigma,+}$ is involved in the discretization (2.4) of the Robin boundary relation in order for the scheme to be well defined with no additional condition on the mesh [1].

Using (2.6) and (2.4), one can eliminate U^{n+1}_σ for all $\sigma \in \varepsilon_K \cap \varepsilon_{ext}$ in (2.4) and obtain

$$U^{n+1}_\sigma = \frac{[\max\{v^{n+1}_{K,\sigma}, 0\} d_{K,\sigma} + m(\sigma) a^{n+1}_\sigma] U^{n+1}_K + d_{K,\sigma} m(\sigma) g^{n+1}_\sigma}{m(\sigma) a^{n+1}_\sigma + [m(\sigma)\lambda^{n+1}_\sigma + v^{n+1}_{K,\sigma} - \min\{v^{n+1}_{K,\sigma}, 0\}] d_{K,\sigma}}, \tag{2.7}$$

thus the numerical unknowns in (2.3) are $(U^{n+1}_K)_{K\in\mathcal{T}}$.

2.3 Discrete Norm Definition

Definition 2 (Discrete Norm). *Let $\phi_\mathcal{T}$ be a function which is a constant on each control volume of $\mathcal{T}$ and on each edge on the boundary with $\phi_\mathcal{T}(x) = \phi_K$ if $x \in K, K \in \mathcal{T}$ and $\phi_\mathcal{T}(x) = \phi_\sigma$ if $x \in \sigma, \sigma \in \varepsilon_{ext}$. One defines the discrete L^2 norm, $L^2(\partial\Omega)$ norm and H^1 semi-norm by*

$$\|\phi_\mathcal{T}\| = \|\phi_\mathcal{T}\|_{L^2(\Omega)} = [\sum_{K\in\mathcal{T}} m(K)(\phi_K)^2]^{\frac{1}{2}}, \ \ \|\phi_\mathcal{T}\|_{L^2(\partial\Omega)} = [\sum_{\sigma\in\varepsilon_{ext}} m(\sigma)(\phi_\sigma)^2]^{\frac{1}{2}},$$
$$|\phi_\mathcal{T}|_{1,\mathcal{T}} = [\sum_{\sigma\in\varepsilon} \tau_\sigma (D_\sigma\phi)^2]^{\frac{1}{2}},$$

where $D_\sigma\phi = |\phi_K - \phi_L|$ if $\sigma = K|L \in \varepsilon_{int}$ and $D_\sigma\phi = |\phi_K - \phi_\sigma|$ if $\sigma \in \varepsilon_K \cap \varepsilon_{ext}, K \in \mathcal{T}$.

Noting this definition, by a similar reasoning procedure as in [1], one has

Lemma 1 (Discrete Norm Inequality). *Let $\phi = \phi_\mathcal{T}$ and its discrete norms be defined as in Definition 2. Let $\Gamma \subset \partial\Omega$ such that its $(d-1)$-dimensional measure $m(\Gamma) \ne 0$ and $\mathcal{O} \subset \Omega$ such that its d-dimensional measure $m(\mathcal{O}) \ne 0$. Then there exists C, only depending on Ω, such that*

$$\|\phi\|^2_{L^2(\Omega)} \le C[|\phi|^2_{1,\mathcal{T}} + \|\phi\|^2_{L^2(\Gamma)}], \quad \|\phi\|^2_{L^2(\partial\Omega)} \le C[|\phi|^2_{1,\mathcal{T}} + \|\phi\|^2_{L^2(\mathcal{O})}].$$

Some useful relations are listed here.

$$a(a-b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2, \tag{2.8}$$

$$k\sum_{n=0}^N \phi^{n+1} d_t\psi^n = \phi^{N+1}\psi^{N+1} - \phi^0\psi^0 - k\sum_{n=0}^N (d_t\phi^n)\psi^n, \tag{2.9}$$

$$k\sum_{n=0}^N \phi^{n+1}\psi^{n+1} d_t\psi^n$$
$$= \frac{1}{2}[\phi^{N+1}(\psi^{N+1})^2 - \phi^0(\psi^0)^2 + k^2\sum_{n=0}^N \phi^{n+1}(d_t\psi^n)^2 - k\sum_{n=0}^N d_t\phi^n(\psi^n)^2], \tag{2.10}$$

$$\|\phi^{N+1}\|^2 \le \|\phi^0\|^2 + \alpha k\sum_{n=0}^N \|d_t\phi^n\|^2 + \frac{1}{8\alpha}k\sum_{n=0}^N (\|\phi^n\|^2 + \|\phi^{n+1}\|^2), \tag{2.11}$$

$$\|\phi^n\|^2 \le 2\|\phi^0\|^2 + 2Tk\sum_{l=0}^{n-1} \|d_t\phi^l\|^2, \tag{2.12}$$

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2, \qquad \text{(Hölder's inequality)}$$
$$|d_t\phi^n(x)| \le C, \qquad \phi = a, b, \lambda, v, div(v);\ x \in \Omega. \tag{2.13}$$

3. Numerical Properties

3.1 Statements of Numerical Properties

Let $e_{\mathcal{T}}^n(x) = e_K^n = u(x_K, t_n) - U_K^n$ for $x \in K, K \in \mathcal{T}$; $e_{\mathcal{T}}^n(x) = e_\sigma^n = u(y_\sigma, t_n) - U_\sigma^n$ for $x \in \sigma, \sigma \in \varepsilon_{ext}$. Under Assumptions 1-3, the L^2 norm estimates are listed below.

Theorem 1 (L^2 Norm Stability). *For finite volume scheme (2.3), (2.4), there is*

$$\begin{aligned} &\|U_{\mathcal{T}}^{N+1}\|^2 + k\sum_{n=0}^{N}|U_{\mathcal{T}}^{n+1}|_{1,\mathcal{T}}^2 + k\sum_{n=0}^{N}\|U_{\mathcal{T}}^{n+1}\|_{L^2(\partial\Omega)}^2 + k^2\sum_{n=0}^{N}\|d_tU_{\mathcal{T}}^n\|^2 \\ \le\ & C\|U_{\mathcal{T}}^0\|^2 + Ck\sum_{n=0}^{N}\|f_{\mathcal{T}}^{n+1}\|^2 + Ck\sum_{n=0}^{N}\|g_{\mathcal{T}}^{n+1}\|_{L^2(\partial\Omega)}^2, \end{aligned}$$

hence the linear algebraic system (2.3) and (2.4) is uniquely solvable.

Theorem 2 (L^2 Norm Error Estimates). *For $\|e_{\mathcal{T}}^0\| = O(h+k)$, there is*

$$\|e_{\mathcal{T}}^{N+1}\| + (k\sum_{n=0}^{N}|e_{\mathcal{T}}^{n+1}|_{1,\mathcal{T}}^2)^{\frac{1}{2}} + [k\sum_{n=0}^{N}\|e_{\mathcal{T}}^{n+1}\|_{L^2(\partial\Omega)}^2]^{\frac{1}{2}} + k(\sum_{n=0}^{N}\|d_te_{\mathcal{T}}^n\|^2)^{\frac{1}{2}} = O(h+k),$$

where $N = 0, 1, \cdots, N_k$.

Let Condition (A) stand for: $v_{K,\sigma}^{n+1} \ge 0$ and $v_{K,\sigma}^n \ge 0$; Condition (B) stand for: $v_{K,\sigma}^{n+1} < 0$ and $v_{K,\sigma}^n < 0$. Under Assumptions 1-4, and Condition (A) or (B), one has the following H^1 semi-norm properties.

Theorem 3 (H^1 Semi-Norm Stability). *For finite volume scheme (2.3), (2.4), there is*

$$\begin{aligned} & k\sum_{n=0}^{N}\|d_tU_{\mathcal{T}}^n\|^2 + |U_{\mathcal{T}}^{N+1}|_{1,\mathcal{T}}^2 + \|U_{\mathcal{T}}^{N+1}\|_{L^2(\partial\Omega)}^2 + k^2\sum_{n=0}^{N}|d_tU_{\mathcal{T}}^n|_{1,\mathcal{T}}^2 \\ \le\ & C[\|U_{\mathcal{T}}^0\|^2 + |U_{\mathcal{T}}^0|_{1,\mathcal{T}}^2 + \|U_{\mathcal{T}}^0\|_{L^2(\partial\Omega)}^2] + Ck\sum_{n=0}^{N}\|f_{\mathcal{T}}^{n+1}\|^2 \\ & + Ck\sum_{n=0}^{N}\|d_tg_{\mathcal{T}}^n\|_{L^2(\partial\Omega)}^2 + C\|g_{\mathcal{T}}^{N+1}\|_{L^2(\partial\Omega)}^2 + C\|g_{\mathcal{T}}^0\|_{L^2(\partial\Omega)}^2. \end{aligned}$$

Theorem 4 (H^1 Semi-Norm Error Estimates). *For $\|e_{\mathcal{T}}^0\| + |e_{\mathcal{T}}^0|_{1,\mathcal{T}} + \|e_{\mathcal{T}}^0\|_{L^2(\partial\Omega)} = O(h+k)$, there is*

$$(k\sum_{n=0}^{N}\|d_te_{\mathcal{T}}^n\|^2)^{\frac{1}{2}} + |e_{\mathcal{T}}^{N+1}|_{1,\mathcal{T}} + \|e_{\mathcal{T}}^{N+1}\|_{L^2(\partial\Omega)} + k(\sum_{n=0}^{N}|d_te_{\mathcal{T}}^n|_{1,\mathcal{T}}^2)^{\frac{1}{2}} = O(h+k),$$

where $N = 0, 1, \cdots, N_k$.

Remark 1. It is easy to choose perfect initial evaluation to satisfy the condition $\|e_{\mathcal{T}}^0\| = O(h+k)$ in Theorem 2 and $\|e_{\mathcal{T}}^0\| + |e_{\mathcal{T}}^0|_{1,\mathcal{T}} + \|e_{\mathcal{T}}^0\|_{L^2(\partial\Omega)} = O(h+k)$ in Theorem 4. A natural choice is $U_K^0 = \frac{1}{m(K)}\int_K u_0(x)dx$, $U_\sigma^0 = \frac{1}{m(\sigma)}\int_\sigma u_0(x)dx$.

Remark 2. Theorems 1-4 show that Scheme (2.3), (2.4) has unique solution and 1 order convergence in both temporal norm and spacial L^2 norm and H^1 semi-norm to the original problem (1.1).

Remark 3. Multiplying (2.4) with U_σ^{n+1} (or $U_\sigma^{n+1} - U_\sigma^n$ instead) and summing for all $\sigma \in \varepsilon_K \cap \varepsilon_{ext}$, multiplying (2.3) with U_K^{n+1} (or $U_K^{n+1} - U_K^n$), and summing these two equalities for all $K \in \mathcal{T}$, one can prove Theorem 1 (or Theorem 3). Since the proofs of Theorems 1 and 3 are similar to but easier than those of Theorems 2 and 4 respectively, they are omitted in this paper.

3.2 Error Equation

Replace ϕ with e in (2.6) to define $e^n_{\sigma,+}$, and define $x^n_{\sigma,+}$ as

$$x^n_{\sigma,+} = \begin{cases} x_K, & if\ v^n_{K,\sigma} \geq 0, \\ x_L, & otherwise, \end{cases} \qquad if\ \sigma = K|L;$$

$$x^n_{\sigma,+} = \begin{cases} x_K, & if\ v^n_{K,\sigma} \geq 0, \\ y_\sigma, & otherwise, \end{cases} \qquad if\ \sigma \in \varepsilon_K \cap \varepsilon_{ext}. \tag{3.1}$$

Then $e^n_{\sigma,+} = u(x^n_{\sigma,+}, t_n) - U^n_{\sigma,+}$ for $\sigma \in \varepsilon_K, K \in \mathcal{T}$. Denote

$$\begin{array}{rcl} m(\sigma)R^n_{K,\sigma} & = & \tau_\sigma a^n_\sigma[u(x_L,t_n) - u(x_K,t_n)] - \int_\sigma a(x,t_n)\nabla u(x,t_n)\cdot\eta_{K,\sigma}d\gamma(x), \\ & & \qquad \sigma = K|L \in \varepsilon_{int}, \\ m(\sigma)R^n_{K,\sigma} & = & \tau_\sigma a^n_\sigma[u(y_\sigma,t_n) - u(x_K,t_n)] - \int_\sigma a(x,t_n)\nabla u(x,t_n)\cdot\eta_{K,\sigma}d\gamma(x), \\ & & \qquad \sigma \in \varepsilon_K \cap \varepsilon_{ext}, \\ m(\sigma)r^n_{K,\sigma} & = & \int_\sigma v(x,t_n)\cdot\eta_{K,\sigma}[u(x,t_n) - u(x^n_{\sigma,+},t_n)]d\gamma(x), \\ m(\sigma)\tilde{R}^n_{K,\sigma} & = & \int_\sigma[\lambda(x,t_n) + v(x,t_n)\cdot\eta_{K,\sigma}][u(x,t_n) - u(y_\sigma,t_n)]d\gamma(x), \\ \rho^n_K & = & \frac{1}{m(K)}\int_K b(x,t_n)[u(x,t_n) - u(x_K,t_n)]dx, \\ S^{n+1}_K & = & \frac{1}{m(K)}\int_K[u^{n+1}_t(x) - d_t u^n(x_K)]dx. \end{array} \tag{3.2}$$

For $t \in J$, if $u_t(\cdot,t)$ is a piecewise C^1 function from $\bar{\Omega}$ to R, then one has $|S^{n+1}_K| \leq C_1(h+k)$. Similarly as in [1], one may prove the following estimates are satisfied.

Lemma 2 (Discrete Coefficient Estimates). *Under Assumptions 1, 2 and 3, assume that u is the unique variational solution to (1.1).*

(1) If $u(\cdot,t) \in C^2(\Omega)$, $t \in J$, then there exists a positive constant C only depending on u, a, b and v such that for any $K \in \mathcal{T}$ and $\sigma \in \varepsilon_K$,

$$|R^n_{K,\sigma}| + |\tilde{R}^n_{K,\sigma}| + |r^n_{K,\sigma}| + |\rho^n_K| \leq Ch;$$

moreover, if $u_t(\cdot,t) \in C^2(\Omega)$, $t \in J$, then

$$|d_t R^n_{K,\sigma}| + |d_t \tilde{R}^n_{K,\sigma}| + |d_t r^n_{K,\sigma}| \leq Ch.$$

(2) If $u(\cdot,t) \in H^2(\Omega)$, $t \in J$, then there exist C_1 only depending on d, a and ζ, C_2 only depending on d, v, ζ and p, and C_3 only depending on d, λ, v, ζ and p such that for any $K \in \mathcal{T}$ and $\sigma \in \varepsilon_K$,

$$\begin{array}{rcl} |R^n_{K,\sigma}| & \leq & C_1 h[m(\sigma)d_\sigma]^{-\frac{1}{2}}\|u(\cdot,t_n)\|_{H^2(V_\sigma)}, \\ |r^n_{K,\sigma}| & \leq & C_2 h[m(\sigma)d_\sigma]^{-\frac{1}{p}}\|u(\cdot,t_n)\|_{W^{1,p}(V_\sigma)}, \\ |\tilde{R}^n_{K,\sigma}| & \leq & C_3 h[m(\sigma)d_\sigma]^{-\frac{1}{p}}\|u(\cdot,t_n)\|_{W^{1,p}(V_\sigma)}, \\ |\rho^n_K| & \leq & \|b^n\|_\infty h[m(K)]^{-\frac{1}{p}}\|u(\cdot,t_n)\|_{W^{1,p}(K)}; \end{array}$$

moreover, if $u_t(\cdot,t) \in H^2(\Omega)$, $t \in J$, then

$$\begin{array}{rcl} |d_t R^n_{K,\sigma}| & \leq & C_1 h[m(\sigma)d_\sigma]^{-\frac{1}{2}}\|\int_0^1 u_t(\cdot,\beta t_{n+1} + (1-\beta)t_n)d\beta\|_{H^2(V_\sigma)}, \\ |d_t r^n_{K,\sigma}| & \leq & C_2 h[m(\sigma)d_\sigma]^{-\frac{1}{p}}\|\int_0^1 u_t(\cdot,\beta t_{n+1} + (1-\beta)t_n)d\beta\|_{W^{1,p}(V_\sigma)}, \\ |d_t \tilde{R}^n_{K,\sigma}| & \leq & C_3 h[m(\sigma)d_\sigma]^{-\frac{1}{p}}\|\int_0^1 u_t(\cdot,\beta t_{n+1} + (1-\beta)t_n)d\beta\|_{W^{1,p}(V_\sigma)}, \end{array}$$

for all $p > d$ and such that $p < +\infty$ if $d = 2$ and $p \leq 6$ if $d = 3$.

Denote

$$G^n_{K,\sigma} = -\tau_\sigma a^n_\sigma(e^n_L - e^n_K), \qquad \forall K \in \mathcal{T},\ \sigma = K|L \in \varepsilon_K \cap \varepsilon_{int},$$
$$G^n_{K,\sigma} = -\tau_\sigma a^n_\sigma(e^n_\sigma - e^n_K), \qquad \forall K \in \mathcal{T},\ \sigma \in \varepsilon_K \cap \varepsilon_{ext}.$$

Subtracting (2.3) and (2.4) from (2.1) and (2.2) respectively, one obtains

$$\begin{aligned} & m(K)d_t e_K^n + \sum_{\sigma\in\varepsilon_K} G_{K,\sigma}^{n+1} + \sum_{\sigma\in\varepsilon_K} v_{K,\sigma}^{n+1} e_{\sigma,+}^{n+1} + m(K) b_K^{n+1} e_K^{n+1} \\ = \; & -\sum_{\sigma\in\varepsilon_K} m(\sigma)(R_{K,\sigma}^{n+1} + r_{K,\sigma}^{n+1}) - m(K)(\rho_K^{n+1} - S_K^{n+1}), \qquad \forall K \in \mathcal{T}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} & -G_{K,\sigma}^{n+1} - v_{K,\sigma}^{n+1} e_{\sigma,+}^{n+1} + [m(\sigma)\lambda_\sigma^{n+1} + v_{K,\sigma}^{n+1}] e_\sigma^{n+1} \\ = \; & m(\sigma)(R_{K,\sigma}^{n+1} - \tilde{R}_{K,\sigma}^{n+1} + r_{K,\sigma}^{n+1}), \qquad \forall \sigma \in \varepsilon_K \cap \varepsilon_{ext}. \end{aligned} \tag{3.4}$$

3.3 L^2 Norm Error Estimate - Proof of Theorem 2

Multiplying (3.4) with e_σ^{n+1} and summing for all $\sigma \in \varepsilon_K \cap \varepsilon_{ext}$, multiplying (3.3) with e_K^{n+1}, and summing these two equalities for all $K \in \mathcal{T}$, one knows

$$\begin{aligned} \sum_{i=1}^{5} A_i^n =: \; & \sum_{K\in\mathcal{T}} m(K)(d_t e_K^n) e_K^{n+1} \\ & +[\sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K\cap\varepsilon_{int}} G_{K,\sigma}^{n+1} e_K^{n+1} + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} (G_{K,\sigma}^{n+1} e_K^{n+1} - G_{K,\sigma}^{n+1} e_\sigma^{n+1})] \\ & +[\sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K} v_{K,\sigma}^{n+1} e_{\sigma,+}^{n+1} e_K^{n+1} + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} (\tfrac{1}{2} v_{K,\sigma}^{n+1} e_\sigma^{n+1} - v_{K,\sigma}^{n+1} e_{\sigma,+}^{n+1}) e_\sigma^{n+1}] \\ & + \sum_{K\in\mathcal{T}} m(K) b_K^{n+1} (e_K^{n+1})^2 + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} [m(\sigma)\lambda_\sigma^{n+1} + \tfrac{1}{2} v_{K,\sigma}^{n+1}](e_\sigma^{n+1})^2 \\ = \; & [-\sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K\cap\varepsilon_{int}} m(K)(R_{K,\sigma}^{n+1} + r_{K,\sigma}^{n+1}) e_K^{n+1} \\ & - \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} m(\sigma)(R_{K,\sigma}^{n+1} + r_{K,\sigma}^{n+1})(e_K^{n+1} - e_\sigma^{n+1})] \\ & - \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} m(\sigma) \tilde{R}_{K,\sigma}^{n+1} e_\sigma^{n+1} - \sum_{K\in\mathcal{T}} m(K)(\rho_K^{n+1} - S_K^{n+1}) e_K^{n+1} \\ =: \; & \sum_{i=1}^{3} B_i^n. \end{aligned} \tag{3.5}$$

Now estimate relation (3.5) term by term. From (2.8), one deduces

$$A_1^n = \frac{1}{2k}\|e_\mathcal{T}^{n+1}\|^2 - \frac{1}{2k}\|e_\mathcal{T}^n\|^2 + \frac{k}{2}\|d_t e_\mathcal{T}^n\|^2, \tag{3.6}$$

$$A_2^n = \sum_{\sigma\in\varepsilon} a_\sigma^{n+1} \tau_\sigma (D_\sigma e^{n+1})^2 \geq a_* |e_\mathcal{T}^{n+1}|_{1,\mathcal{T}}^2. \tag{3.7}$$

Note that $v_{K,\sigma}^n = -v_{L,\sigma}^n$, $K \in \mathcal{T}$, $K \cap L = \sigma$, let $v_\sigma^n = |v_{K,\sigma}^n| = |v_{L,\sigma}^n|$,

$$e_{\sigma,-}^n = \begin{cases} e_K^n, & if\ v_{K,\sigma}^n < 0, \\ e_L^n, & otherwise, \end{cases} \qquad if\ \sigma = K|L;$$

$$e_{\sigma,-}^n = \begin{cases} e_K^n, & if\ v_{K,\sigma}^n < 0, \\ e_\sigma^n, & otherwise, \end{cases} \qquad if\ \sigma \in \varepsilon_K \cap \varepsilon_{ext}; \tag{3.8}$$

and notice (2.8), one derives

$$
\begin{aligned}
A_3^n &= \sum_{\sigma\in\varepsilon_K\cap\varepsilon_{int}} v_\sigma^{n+1}e_{\sigma,+}^{n+1}(e_{\sigma,+}^{n+1}-e_{\sigma,-}^{n+1})\\
&\quad+\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}}[v_{K,\sigma}^{n+1}e_{\sigma,+}^{n+1}e_K^{n+1}+(\tfrac{1}{2}v_{K,\sigma}^{n+1}e_\sigma^{n+1}-v_{K,\sigma}^{n+1}e_{\sigma,+}^{n+1})e_\sigma^{n+1}]\\
&= \tfrac{1}{2}\sum_{\sigma\in\varepsilon_{int}} v_\sigma^{n+1}(e_{\sigma,+}^{n+1}-e_{\sigma,-}^{n+1})^2+\tfrac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K} v_{K,\sigma}^{n+1}(e_K^{n+1})^2\\
&\quad+\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{n+1}[e_{\sigma,+}^{n+1}e_K^{n+1}-\tfrac{1}{2}(e_K^{n+1})^2-e_{\sigma,+}^{n+1}e_\sigma^{n+1}+\tfrac{1}{2}(e_\sigma^{n+1})^2]\\
&= \tfrac{1}{2}\sum_{\sigma\in\varepsilon_{int}} v_\sigma^{n+1}(e_{\sigma,+}^{n+1}-e_{\sigma,-}^{n+1})^2+\tfrac{1}{2}\int_\Omega div(v(x,t_{n+1}))[e_{\mathcal{T}}^{n+1}(x)]^2dx\\
&\quad+\tfrac{1}{2}\sum_{\sigma\in\varepsilon_{ext}} v_\sigma^{n+1}(e_{\sigma,+}^{n+1}-e_{\sigma,-}^{n+1})^2\\
&= \tfrac{1}{2}\sum_{\sigma\in\varepsilon} v_\sigma^{n+1}(e_{\sigma,+}^{n+1}-e_{\sigma,-}^{n+1})^2+\tfrac{1}{2}\int_\Omega div(v(x,t_{n+1}))[e_{\mathcal{T}}^{n+1}(x)]^2dx.
\end{aligned}
\tag{3.9}
$$

It is obvious that

$$A_4^n=\int_\Omega b(x,t_{n+1})[e_{\mathcal{T}}^{n+1}(x)]^2dx, \tag{3.10}$$

$$A_5^n=\int_{\partial\Omega}[\lambda(x,t_{n+1})+\tfrac{1}{2}v(x,t_{n+1})\cdot\eta(x)][e_{\mathcal{T}}^{n+1}(x)]^2dx. \tag{3.11}$$

Recall that $R_{K,\sigma}^{n+1}=-R_{L,\sigma}^{n+1}, r_{K,\sigma}^{n+1}=-r_{L,\sigma}^{n+1}$ for $K\in\varepsilon_{int}, K\cap L=\sigma$, hence from Hölder's inequality and Lemma 2,

$$\sum_{i=1}^3 B_i^n\le C(1+\tfrac{1}{\delta}+\tfrac{1}{\epsilon})(h^2+k^2)+\delta|e_{\mathcal{T}}^{n+1}|_{1,\mathcal{T}}^2+\epsilon\|e_{\mathcal{T}}^{n+1}\|_{L^2(\partial\Omega)}^2+C\|e_{\mathcal{T}}^{n+1}\|^2. \tag{3.12}$$

Combining (3.5)-(3.12) leads to

$$
\begin{aligned}
&\tfrac{1}{2k}\|e_{\mathcal{T}}^{n+1}\|^2-\tfrac{1}{2k}\|e_{\mathcal{T}}^n\|^2+\tfrac{k}{2}\|d_te_{\mathcal{T}}^n\|^2+(a_*-\delta)|e_{\mathcal{T}}^{n+1}|_{1,\mathcal{T}}^2\\
&+\int_\Omega[\tfrac{1}{2}div(v(x,t_{n+1}))+b(x,t_{n+1})][e_{\mathcal{T}}^{n+1}(x)]^2dx+\tfrac{1}{2}\sum_{\sigma\in\varepsilon} v_\sigma^{n+1}(e_{\sigma,+}^{n+1}-e_{\sigma,-}^{n+1})^2\\
&+\int_{\partial\Omega}[\lambda(x,t_{n+1})+\tfrac{1}{2}v(x,t_{n+1})\cdot\eta(x)][e_{\mathcal{T}}^{n+1}(x)]^2dx\\
&\le C(1+\tfrac{1}{\delta}+\tfrac{1}{\epsilon})(h^2+k^2)+C\|e_{\mathcal{T}}^{n+1}\|^2+\epsilon\|e_{\mathcal{T}}^{n+1}\|_{L^2(\partial\Omega)}^2.
\end{aligned}
\tag{3.13}
$$

Obviously the last term on the left hand of (3.13) is nonnegative. Multiplying (3.13) with $2k$ and summing for $n=0,1,2,\cdots,N$ leads to

$$
\begin{aligned}
&\|e_{\mathcal{T}}^{N+1}\|^2+2(a_*-\delta)k\sum_{n=0}^N|e_{\mathcal{T}}^{n+1}|_{1,\mathcal{T}}^2+k^2\sum_{n=0}^N\|d_te_{\mathcal{T}}^n\|^2+E_1^N+E_2^N\\
\le\;& C\|e_{\mathcal{T}}^0\|^2+C(1+\tfrac{1}{\delta}+\tfrac{1}{\epsilon})(h^2+k^2)+Ck\sum_{n=0}^N\|e_{\mathcal{T}}^{n+1}\|^2+\epsilon k\sum_{n=0}^N\|e_{\mathcal{T}}^{n+1}\|_{L^2(\partial\Omega)}^2,
\end{aligned}
$$

where

$$
\begin{aligned}
E_1^N &= 2k\sum_{n=0}^N\int_\Omega[\tfrac{1}{2}div(v(x,t_{n+1}))+b(x,t_{n+1})][e_{\mathcal{T}}^{n+1}(x)]^2dx,\\
E_2^N &= 2k\sum_{n=0}^N\int_{\partial\Omega}[\lambda(x,t_{n+1})+\tfrac{1}{2}v(x,t_{n+1})\cdot\eta(x)][e_{\mathcal{T}}^{n+1}(x)]^2dx.
\end{aligned}
$$

For $v\cdot\eta/2+\lambda>0$ a.e. on $\partial\Omega$, one sees $E_1^N\ge0$, $E_2^N\ge\epsilon k\sum_{n=0}^N\|e_{\mathcal{T}}^{n+1}\|_{L^2(\partial\Omega)}^2$; for $v\cdot\eta/2+\lambda=0$ a.e. on $\partial\Omega$, one sees $E_2^N=0$ and $div(v)/2+b>0$ a.e. on $\mathcal{O}$, hence $E_1^N\ge C\epsilon k\sum_{n=0}^N\|e_{\mathcal{T}}^{n+1}\|_{L^2(\mathcal{O})}^2$, with Lemma 1, $\epsilon k\sum_{n=0}^N\|e_{\mathcal{T}}^{n+1}\|_{L^2(\partial\Omega)}^2\le E_1^N+C\epsilon k\sum_{n=0}^N|e_{\mathcal{T}}^{n+1}|_{1,\mathcal{T}}^2$, hence with properly chosen ϵ,δ, under Assumptions 1 and 2, there is

$$\|e_{\mathcal{T}}^{N+1}\|^2 + k\sum_{n=0}^{N}|e_{\mathcal{T}}^{n+1}|_{1,\mathcal{T}}^2 + k^2\sum_{n=0}^{N}\|d_te_{\mathcal{T}}^n\|^2 \le C(\|e_{\mathcal{T}}^0\|^2 + h^2 + k^2) + Ck\sum_{n=0}^{N}\|e_{\mathcal{T}}^{n+1}\|^2,$$

using Gronwall's inequality to the above inequality, one gets for $\|e_{\mathcal{T}}^0\| = O(h+k)$, the left hand of the above relation can be bounded by $C(h^2+k^2)$, $N = 0,1,\cdots,N_k$; another use of Lemma 1 shows Theorem 2 is valid.

3.4 H^1 Semi-norm Error Estimate - Proof of Theorem 4

Multiplying (3.4) with $e_\sigma^{n+1} - e_\sigma^n$ and summing for all $\sigma \in \varepsilon_K \cap \varepsilon_{ext}$, multiplying (3.3) with $e_K^{n+1} - e_K^n$, and summing these two relations for all $K \in \mathcal{T}$ and $n = 0,1,\cdots,N$, rewriting the deduced relation as

$$\begin{aligned}
&\sum_{i=1}^{4} D_i^N =: D_1^N + D_2^N + (I_0^N + Q_0^N + P_0^N) + D_4^N \\
=:\ &\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}[m(K)d_te_K^n(e_K^{n+1} - e_K^n)] \\
&+\sum_{n=0}^{N}[\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K} G_{K,\sigma}^{n+1}(e_K^{n+1} - e_K^n) - \sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} G_{K,\sigma}^{n+1}(e_\sigma^{n+1} - e_\sigma^n)] \\
&+\{\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{int}} v_{K,\sigma}^{n+1}e_{\sigma,+}^{n+1}(e_K^{n+1} - e_K^n) \\
&+\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{n+1}e_{\sigma,+}^{n+1}[(e_K^{n+1} - e_\sigma^{n+1}) - (e_K^n - e_\sigma^n)] \\
&+\tfrac{1}{2}\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{n+1}e_\sigma^{n+1}(e_\sigma^{n+1} - e_\sigma^n)\} \\
&+\{\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K)b_K^{n+1}e_K^{n+1}(e_K^{n+1} - e_K^n) \\
&+\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}}[m(\sigma)\lambda_\sigma^{n+1} + \tfrac{1}{2}v_{K,\sigma}^{n+1}]e_\sigma^{n+1}(e_\sigma^{n+1} - e_\sigma^n)\} \\
=\ &-\sum_{n=0}^{N}\sum_{\sigma\in\varepsilon_K}[m(\sigma)(R_{K,\sigma}^{n+1} + r_{K,\sigma}^{n+1}) + m(K)(\rho_K^{n+1} - S_K^{n+1})](e_K^{n+1} - e_K^n) \\
&+\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} m(\sigma)(R_{K,\sigma}^{n+1} - \tilde{R}_{K,\sigma}^{n+1} + r_{K,\sigma}^{n+1})(e_\sigma^{n+1} - e_\sigma^n) =: \sum_{i=1}^{2} O_i^N.
\end{aligned} \tag{3.14}$$

Estimating its terms one by one, one derives

$$D_1^N = k\sum_{n=0}^{N}\|d_te_{\mathcal{T}}^n\|^2, \tag{3.15}$$

$$\begin{aligned}
D_2^N\ =\ &\sum_{n=0}^{N}\{\sum_{\sigma\in\varepsilon_{int}}\tau_\sigma a_\sigma^{n+1}(e_K^{n+1} - e_L^{n+1})[(e_K^{n+1} - e_L^{n+1}) - (e_K^n - e_L^n)] \\
&+\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}}\tau_\sigma a_\sigma^{n+1}(e_K^{n+1} - e_\sigma^{n+1})[(e_K^{n+1} - e_\sigma^{n+1}) - (e_K^n - e_\sigma^n)]\} \\
=\ &\tfrac{1}{2}\sum_{\sigma\in\varepsilon}\tau_\sigma a_\sigma^{N+1}(D_\sigma e^{N+1})^2 - \tfrac{1}{2}\sum_{\sigma\in\varepsilon}\tau_\sigma a_\sigma^0(D_\sigma e^0)^2 \\
&+\tfrac{1}{2}k^2\sum_{n=0}^{N}\sum_{\sigma\in\varepsilon}\tau_\sigma a_\sigma^{n+1}(D_\sigma d_te^n)^2 - \tfrac{1}{2}k\sum_{n=0}^{N}\sum_{\sigma\in\varepsilon}\tau_\sigma d_ta_\sigma^n(D_\sigma e^n)^2 \\
\ge\ &\tfrac{1}{2}a_*|e_{\mathcal{T}}^{N+1}|_{1,\mathcal{T}}^2 - C|e_{\mathcal{T}}^0|_{1,\mathcal{T}}^2 + \tfrac{1}{2}a_*k^2\sum_{n=0}^{N}|d_te_{\mathcal{T}}^n|_{1,\mathcal{T}}^2 - Ck\sum_{n=0}^{N}|e_{\mathcal{T}}^n|_{1,\mathcal{T}}^2,
\end{aligned} \tag{3.16}$$

where (2.8) has been used for the second equality in (3.16).

Now pay attention to the estimate of D_3^N. First, let $sign(\phi)$ stand for the function $sign(\phi) = 1$ for $\phi > 0$, $sign(\phi) = 0$ for $\phi = 0$, $sign(\phi) = -1$ for $\phi < 0$; for $v_{K,\sigma}^n = 0$, take supplemental definition $sign(\frac{v_{K,\sigma}^{n+1}}{v_{K,\sigma}^n}) = sign(v_{K,\sigma}^{n+1})$. Using the summation-by-parts formula (2.9), one obtains

$$\begin{aligned} I_0^N = & \sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{int}} v_{K,\sigma}^{N+1} e_{\sigma,+}^{N+1} e_K^{N+1} - \sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{int}} v_{K,\sigma}^0 e_{\sigma,+}^0 e_K^0 \\ & -k\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{int}} (d_t v_{K,\sigma}^n) e_{\sigma,+}^n e_K^n \\ & -\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{int}} v_{K,\sigma}^{n+1}(e_{\sigma,+}^{n+1} - e_{\sigma,+}^n) e_K^{n+1} \\ & +\sum_{n=0}^N\sum_{\sigma\in\varepsilon_{int}} v_\sigma^{n+1}(e_{\sigma,+}^{n+1} - e_{\sigma,+}^n)[(e_{\sigma,+}^{n+1} - e_{\sigma,-}^{n+1}) - sign(\tfrac{v_{K,\sigma}^{n+1}}{v_{K,\sigma}^n})(e_{\sigma,+}^n - e_{\sigma,-}^n)] \\ = & \sum_{i=1}^4 I_i^N + \sum_{n=0}^N\sum_{\sigma\in\varepsilon_{int}} v_\sigma^{n+1}(e_{\sigma,+}^{n+1} - e_{\sigma,+}^n)[(e_{\sigma,+}^{n+1} - e_{\sigma,+}^n) - (e_{\sigma,-}^{n+1} - e_{\sigma,-}^n)] \\ = & \sum_{i=1}^4 I_i^N + \tfrac{1}{2}k^2\sum_{n=0}^N\sum_{\sigma\in\varepsilon_{int}} v_\sigma^{n+1}(d_t e_{\sigma,+}^n - d_t e_{\sigma,-}^n)^2 \\ & +\tfrac{1}{2}k^2\sum_{n=0}^N\int_\Omega div(v(x,t_{n+1}))[d_t e_{\mathcal{T}}^n(x)]^2 dx \\ & -\tfrac{1}{2}\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{n+1}(e_K^{n+1} - e_K^n)^2 =: \sum_{i=1}^7 I_i^N, \end{aligned} \tag{3.17}$$

$$\begin{aligned} Q_0^N = & \sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{N+1} e_{\sigma,+}^{N+1}(e_K^{N+1} - e_\sigma^{N+1}) - \sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^0 e_{\sigma,+}^0(e_K^0 - e_\sigma^0) \\ & -k\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} (d_t v_{K,\sigma}^n) e_{\sigma,+}^n(e_K^n - e_\sigma^n) \\ & +\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{n+1}(e_{\sigma,+}^{n+1} - e_{\sigma,+}^n)[(e_K^{n+1} - e_\sigma^{n+1}) - (e_K^n - e_\sigma^n)] \\ & -\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{n+1}(e_{\sigma,+}^{n+1} - e_{\sigma,+}^n)(e_K^{n+1} - e_\sigma^{n+1}) =: \sum_{i=1}^5 Q_i^N, \end{aligned} \tag{3.18}$$

$$\begin{aligned} P_0^N = & \tfrac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{N+1}(e_\sigma^{N+1})^2 - \tfrac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^0(e_\sigma^0)^2 \\ & -\tfrac{1}{2}k\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} (d_t v_{K,\sigma}^n) e_\sigma^n e_\sigma^n \\ & -\tfrac{1}{2}\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{n+1}(e_\sigma^{n+1} - e_\sigma^n) e_\sigma^n =: \sum_{i=1}^4 P_i^N, \end{aligned} \tag{3.19}$$

where the second and the third equalities in (3.17) hold for $sign(\frac{v_{K,\sigma}^{n+1}}{v_{K,\sigma}^n}) = 1$ or 0, e.g. for Condition (A) or (B). Then notice that

$$\begin{aligned} Q_0^N + Q_5^N = & \sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{n+1}(e_K^{n+1} - e_K^n)(e_\sigma^{n+1} + e_\sigma^n) \\ & -\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{N+1} e_K^{N+1} e_\sigma^{N+1} - \sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^0 e_K^0 e_\sigma^0 \\ & +k\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} (d_t v_{K,\sigma}^n) e_K^n e_\sigma^n =: \sum_{i=1}^4 S_i^N, \\ P_0^N + P_4^N = & \tfrac{1}{2}\sum_{n=0}^N\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} v_{K,\sigma}^{n+1}(e_\sigma^{n+1} - e_\sigma^n)^2 =: T_1^N. \end{aligned}$$

Finally, treating the related terms similarly as in the former subsection 3.3 (especially as in

(3.9)), and combining these estimates means

$$
\begin{aligned}
D_3^N = \; & I_0^N + Q_0^N + P_0^N = (\tfrac{1}{2}\sum_{i=0}^{7} I_i^N) + (\tfrac{1}{2}\sum_{i=1}^{4} S_i^N + \tfrac{1}{2}\sum_{i=1}^{4} Q_i^N) + (\tfrac{1}{2}T_1^N + \tfrac{1}{2}\sum_{i=1}^{3} P_i^N) \\
= \; & \tfrac{1}{2}(I_1^N + Q_1^N + P_1^N) + \tfrac{1}{2}(I_5^N + I_7^N + Q_4^N + T_1^N) + \tfrac{1}{2}I_6^N + \tfrac{1}{2}(I_2^N + Q_2^N + P_2^N) \\
& + \tfrac{1}{2}[(I_3^N + Q_3^N + P_3^N) + S_4^N + (I_0^N + I_4^N + S_1^N) + S_2^N + S_3^N] \\
\geq \; & \tfrac{1}{4}\sum_{\sigma\in\varepsilon} v_\sigma^{N+1}(e_{\sigma,+}^{N+1} - e_{\sigma,-}^{N+1})^2 + \tfrac{1}{4}\int_\Omega div(v(x,t_{N+1}))[e_{\mathcal{T}}^{N+1}(x)]^2 dx \\
& + \tfrac{1}{4}k^2 \sum_{n=0}^{N}\sum_{\sigma\in\varepsilon} v_\sigma^{n+1}(d_t e_{\sigma,+}^n - d_t e_{\sigma,-}^n)^2 + \tfrac{1}{4}k^2\sum_{n=0}^{N}\int_\Omega div(v(x,t_{n+1}))[d_t e_{\mathcal{T}}^n(x)]^2 dx \\
& - C[\|e_{\mathcal{T}}^0\|^2 + |e_{\mathcal{T}}^0|_{1,\mathcal{T}}^2 + \|e_{\mathcal{T}}^0\|_{L^2(\partial\Omega)}^2] - \delta k\sum_{n=0}^{N}\|d_t e_{\mathcal{T}}^n\|^2 - \epsilon\|e_{\mathcal{T}}^{N+1}\|_{L^2(\partial\Omega)}^2 \\
& - C(1+\tfrac{1}{\delta})k\sum_{n=0}^{N+1}\|e_{\mathcal{T}}^n\|^2 - C(1+\tfrac{1}{\delta})k\sum_{n=0}^{N+1}|e_{\mathcal{T}}^n|_{1,\mathcal{T}}^2 \\
& - C(1+\tfrac{2}{\delta})k\sum_{n=0}^{N+1}\|e_{\mathcal{T}}^n\|_{L^2(\partial\Omega)}^2 - C\tfrac{1}{\epsilon}\|e_{\mathcal{T}}^{N+1}\|^2.
\end{aligned}
\tag{3.20}
$$

Other terms are easy to deal with. In fact, denote

$$
\begin{aligned}
F_1^N = \; & \tfrac{1}{2}\int_\Omega [b(x,t_{N+1}) + \tfrac{1}{2}div(v(x,t_{N+1}))][e_{\mathcal{T}}^{N+1}(x)]^2 dx \\
& + \tfrac{1}{2}k^2\sum_{n=0}^{N}\int_\Omega [b(x,t_{n+1}) + \tfrac{1}{2}div(v(x,t_{n+1}))][d_t e_{\mathcal{T}}^n(x)]^2 dx, \\
F_2^N = \; & \tfrac{1}{2}\int_{\partial\Omega}[\lambda(x,t_{N+1}) + \tfrac{1}{2}v(x,t_{N+1})\cdot\eta(x)][e_{\mathcal{T}}^{N+1}(x)]^2 dx \\
& + \tfrac{1}{2}k^2\sum_{n=0}^{N}\int_{\partial\Omega}[\lambda(x,t_{n+1}) + \tfrac{1}{2}v(x,t_{n+1})\cdot\eta(x)][d_t e_{\mathcal{T}}^n(x)]^2 dx,
\end{aligned}
$$

using summation-by-parts process (2.10), one shows

$$
\begin{aligned}
D_4^N = \; & F_1^N + F_2^N - \tfrac{1}{2}\int_\Omega b(x,t_0)[e_{\mathcal{T}}^0(x)]^2 dx - \tfrac{1}{2}k\sum_{n=0}^{N}\int_\Omega d_t b^n(x)[e_{\mathcal{T}}^n(x)]^2 dx \\
& - \tfrac{1}{2}\int_{\partial\Omega}[\lambda(x,t_0) + \tfrac{1}{2}v(x,t_0)\cdot\eta(x)][e_{\mathcal{T}}^0(x)]^2 dx \\
& - \tfrac{1}{2}k\sum_{n=0}^{N}\int_{\partial\Omega}[d_t\lambda^n(x) + \tfrac{1}{2}d_t v^n(x)\cdot\eta(x)][e_{\mathcal{T}}^n(x)]^2 dx \\
\geq \; & F_1^N + F_2^N - C[\|e_{\mathcal{T}}^0\|^2 + \|e_{\mathcal{T}}^0\|_{L^2(\partial\Omega)}^2] - Ck\sum_{n=0}^{N}[\|e_{\mathcal{T}}^n\|^2 + \|e_{\mathcal{T}}^n\|_{L^2(\partial\Omega)}^2],
\end{aligned}
\tag{3.21}
$$

using Lemma 2,

$$
O_1^N \leq C\frac{1}{\delta}(h^2 + k^2) + \delta k\sum_{n=0}^{N}\|d_t e_{\mathcal{T}}^n\|^2, \tag{3.22}
$$

$$
\begin{aligned}
O_2^N = \; & \sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} m(\sigma)(R_{K,\sigma}^{N+1} - \tilde{R}_{K,\sigma}^{N+1} + r_{K,\sigma}^{N+1})e_\sigma^{N+1} \\
& - \sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} m(\sigma)(R_{K,\sigma}^0 - \tilde{R}_{K,\sigma}^0 + r_{K,\sigma}^0)e_\sigma^0 \\
& - k\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap\varepsilon_{ext}} m(\sigma)(d_t R_{K,\sigma}^n - d_t\tilde{R}_{K,\sigma}^n + d_t r_{K,\sigma}^n)e_\sigma^n \\
\leq \; & C(1+\tfrac{1}{\epsilon})(h^2+k^2) + C\|e_{\mathcal{T}}^0\|_{L^2(\partial\Omega)}^2 + \epsilon\|e_{\mathcal{T}}^{N+1}\|_{L^2(\partial\Omega)}^2 + Ck\sum_{n=0}^{N}\|e_{\mathcal{T}}^n\|_{L^2(\partial\Omega)}^2,
\end{aligned}
\tag{3.23}
$$

where summation by parts (2.9) has been used to get (3.23).

Combine (3.14)-(3.16) and (3.20)-(3.23), and manipulate the derived relation, one knows

$$\begin{aligned}
&(1-2\delta)k\sum_{n=0}^{N}\|d_te_{\mathcal{T}}^n\|^2+\tfrac{1}{2}a_*|e_{\mathcal{T}}^{N+1}|_{1,\mathcal{T}}^2+\tfrac{1}{2}a_*k^2\sum_{n=0}^{N}|d_te_{\mathcal{T}}^n|_{1,\mathcal{T}}^2+F_1^N+F_2^N\\
&+\tfrac{1}{4}\sum_{\sigma\in\varepsilon}v_\sigma^{N+1}(e_{\sigma,+}^{N+1}-e_{\sigma,-}^{N+1})^2+\tfrac{1}{4}k^2\sum_{n=0}^{N}\sum_{\sigma\in\varepsilon}v_\sigma^{n+1}(d_te_{\sigma,+}^n-d_te_{\sigma,-}^n)^2\\
\le\;& C[\|e_{\mathcal{T}}^0\|^2+|e_{\mathcal{T}}^0|_{1,\mathcal{T}}^2+\|e_{\mathcal{T}}^0\|_{L^2(\partial\Omega)}^2]+C(1+\tfrac{1}{\delta}+\tfrac{1}{\epsilon})(h^2+k^2)\\
&+C(1+\tfrac{1}{\delta})k\sum_{n=0}^{N+1}\|e_{\mathcal{T}}^n\|^2+C(1+\tfrac{1}{\delta})k\sum_{n=0}^{N+1}|e_{\mathcal{T}}^n|_{1,\mathcal{T}}^2+C(1+\tfrac{2}{\delta})k\sum_{n=0}^{N+1}\|e_{\mathcal{T}}^n\|_{L^2(\partial\Omega)}^2\\
&+C\tfrac{1}{\epsilon}\|e_{\mathcal{T}}^{N+1}\|^2+2\epsilon\|e_{\mathcal{T}}^{N+1}\|_{L^2(\partial\Omega)}^2.
\end{aligned}$$

Apply (2.11) and (2.12) for $\phi=e_{\mathcal{T}}$, and notice that for $v\cdot\eta/2+\lambda>0$ a.e. on $\partial\Omega$, $F_1^N\ge 0$, $F_2^N\ge\epsilon_0k^2\sum_{n=0}^{N}\|d_te_{\mathcal{T}}^n\|_{L^2(\partial\Omega)}^2+\epsilon_0\|e_{\mathcal{T}}^{N+1}\|_{L^2(\partial\Omega)}^2$; for $v\cdot\eta/2+\lambda=0$ a.e. on $\partial\Omega\times J$, $F_2^N=0$, $F_1^N\ge\epsilon_0k^2\sum_{n=0}^{N}\|d_te_{\mathcal{T}}^n\|_{L^2(\mathcal{O})}^2+\epsilon_0\|e_{\mathcal{T}}^{N+1}\|_{L^2(\mathcal{O})}^2$; then, for proper positive constants $\epsilon,\delta,\alpha,\epsilon_0$, the above expressions may also be written as

$$\begin{aligned}
&k\sum_{n=0}^{N}\|d_te_{\mathcal{T}}^n\|^2+|e_{\mathcal{T}}^{N+1}|_{1,\mathcal{T}}^2+k^2\sum_{n=0}^{N}|d_te_{\mathcal{T}}^n|_{1,\mathcal{T}}^2\\
\le\;& C[\|e_{\mathcal{T}}^0\|^2+|e_{\mathcal{T}}^0|_{1,\mathcal{T}}^2+\|e_{\mathcal{T}}^0\|_{L^2(\partial\Omega)}^2+h^2+k^2]\\
&+Ck\sum_{n=0}^{N}(k\sum_{l=0}^{n}\|d_te_{\mathcal{T}}^n\|^2)+Ck\sum_{n=0}^{N+1}|e_{\mathcal{T}}^n|_{1,\mathcal{T}}^2,
\end{aligned}$$

using Lemma 1 and Gronwall's inequality, one comes to the conclusion of Theorem 4.

Acknowledgements. The author is grateful to Professors Shen Longjun, Yuan Guangwei and Han Houde for their helpful suggestions and encouragements.

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