Journal of Computational Mathematics, Vol.23, No.4, 2005, 401–407.

SOME ESTIMATIONS FOR DETERMINANT OF THE HADAMARD PRODUCT OF H-MATRICES *1)

Yao-tang Li
(*Department of Mathematics, Yunnan University, Kunming 650091, China*)

Cong-lei Zhong
(*College of Science, Henan University of Science and Technology, Luoyang 417003, China*)

Abstract

In this paper, some new results on the estimations of bounds for determinant of Hadamard Product of two H-matrices are given. Several recent results are improved and generalized.

Mathematics subject classification: 15A06.
Key words: H-matrix, Hadamard Product, Determinant.

1. Introduction

Let $R^{m\times n}$ be the set of all $m\times n$ real matrices and $A=(a_{ij})$ and $B=(b_{ij})\in R^{m\times n}$. The Hadamard product of A and B is defined as an $m\times n$ matrix denoted by $A\circ B:(A\circ B)_{ij}=a_{ij}b_{ij}$. $|A|$ is defined by $(|A|)_{ij}=|a_{ij}|$.

We write $A\geq B$ if $a_{ij}\geq b_{ij}$ for all i,j. A real $n\times n$ matrix A is called a nonsingular M-matrix if $A=sI-B$ satisfies: $s>0$, $B\geq 0$ and $s>\rho(B)$, where $\rho(B)$ is the spectral radius of B. Let M_n denote the set of all $n\times n$ nonsingular M-matrices. Suppose $A=(a_{ij})\in R^{n\times n}$, its comparison matrix $\mu(A)=(m_{ij})$ is defined by

$$m_{ij}=\begin{cases} \mid a_{ij}\mid, & \text{if } \ i=j, \\ -\mid a_{ij}\mid, & \text{if } \ i\neq j. \end{cases}$$

A real $n\times n$ matrix A is called an H-matrix if its comparison matrix $\mu(A)$ is a nonsingular M-matrix. H_n denotes the set of all $n\times n$ H-matrices. Let $A\in R^{n\times n}$. A_k denotes the $k\times k$ successive principal submatrix of A.

In [1], Yao-tang Li and Ji-cheng Li gave an estimation of bounds for determinant of Hadamard product of two H-matrices recently as follows:

Theorem[1,Theorem6]**.** *Let $A=(a_{ij})$ and $B=(b_{ij})\in H_n$, $\prod_{i=1}^{n}a_{ii}b_{ii}>0$. Then*

$$\begin{aligned}\det(A\circ B) &\geq \left(\prod_{i=1}^{n}b_{ii}\right)\det(\mu(A))+\left(\prod_{i=1}^{n}|a_{ii}|\right)\det(\mu(B))\cdot\prod_{k=2}^{n}\sum_{i=1}^{k-1}\left|\frac{a_{ik}a_{ki}}{a_{ii}a_{kk}}\right| \\ &= W_n(A,B).\end{aligned}\tag{1}$$

In this paper, we will improve this result and generalize Jian-zhou Liu's main results on M-matrices in [2] to H-matrices.

* Received March 12, 2003; final revised January 25, 2005.
1) This work is supported by the Science Foundations of the Education Department of Yunnan Province (03Z169A) and the Science Foundations of Yunnan University (2003Z013C).

2. Some Lemmas

In this section, we will give some lemmas that shall be used.

From the definitions and [2, Lemma 3], the following two lemmas are obtained immediately.

Lemma 1. *If $A \in H_n$, A_k is the $k \times k$ successive principal submatrix of A, then $A_k \in H_k$.*

Lemma 2. *If $A = (a_{ij}) \in H_n$, then*

$$\prod_{i=1}^{n} |a_{ii}| \geq |a_{kk}| \det[\mu(A(k))] \geq \det[\mu(A)] \geq 0, \quad k = 1, 2, \cdots, n, \tag{2}$$

where $A(k) \in R^{(n-1)\times(n-1)}$ is the principal submatrix of matrix A obtained by deleting row and column k of A.

Lemma 3. *If A and $B \in H_n$, then*

$$|a_{kk}| \frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1})]} - \frac{\det[\mu(A_k)]}{\det[\mu(A_{k-1})]} \frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1})]}$$

$$\geq \frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1})]} \sum_{i=1}^{k-1} \left| \frac{a_{ik}a_{ki}}{a_{ii}} \right|, \quad k = 1, 2, \cdots, n. \tag{3}$$

Proof. By Lemma 1,

$$A_k = \begin{pmatrix} A_{k-1} & A_{12}^{(k-1)} \\ A_{21}^{(k-1)} & a_{kk} \end{pmatrix}, \; B_k = \begin{pmatrix} B_{k-1} & B_{12}^{(k-1)} \\ B_{21}^{(k-1)} & b_{kk} \end{pmatrix} \in H_k.$$

Therefore,

$$diag(|a_{11}|, \cdots, |a_{k-1,k-1}|) \geq \mu(A_{k-1})$$

and

$$[\mu(A_{k-1})]^{-1} \geq diag(|a_{11}^{-1}|, \cdots, |a_{k-1,k-1}^{-1}|) > 0.$$

So,

$$|A_{21}^{(k-1)}|[\mu(A_{k-1})]^{-1}|A_{12}^{(k-1)}| \geq |A_{21}^{(k-1)}| diag(|a_{11}^{-1}|, \cdots, |a_{k-1,k-1}^{-1}|)|A_{12}^{(k-1)}|$$

$$= \sum_{i=1}^{k-1} \left| \frac{a_{ik}a_{ki}}{a_{ii}} \right| \geq 0, \tag{4}$$

$$\begin{aligned} \det[\mu(A_k)] &= \det \mu \begin{pmatrix} A_{k-1} & A_{12}^{(k-1)} \\ A_{21}^{(k-1)} & a_{kk} \end{pmatrix} \\ &= \det \begin{pmatrix} \mu(A_{k-1}) & -|A_{12}^{(k-1)}| \\ -|A_{21}^{(k-1)}| & |a_{kk}| \end{pmatrix} \\ &= \det \begin{pmatrix} \mu(A_{k-1}) & 0 \\ 0 & |a_{kk}| - |A_{21}^{(k-1)}|[\mu(A_{k-1})]^{-1}|A_{12}^{(k-1)}| \end{pmatrix} \\ &= \det[\mu(A_{k-1})] \cdot (|a_{kk}| - |A_{21}^{(k-1)}|[\mu(A_{k-1})]^{-1}|A_{12}^{(k-1)}|). \end{aligned} \tag{5}$$

Thus, by (2), (4) and (5), we have:

$$\begin{aligned} 0 &\leq \frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1}]}\sum_{i=1}^{k-1}\left|\frac{a_{ik}a_{ki}}{a_{ii}}\right| \\ &\leq \frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1}]}|A_{21}^{(k-1)}|[\mu(A_{k-1})]^{-1}|A_{12}^{(k-1)}| \\ &= \frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1}]}\left(\frac{|a_{kk}|\det[\mu(A_{k-1})]-\det[\mu(A_k)]}{\det[\mu(A_{k-1})]}\right) \\ &= |a_{kk}|\frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1}]}-\frac{\det[\mu(A_k)]}{\det[\mu(A_{k-1})]}\frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1}]}. \end{aligned}$$

Lemma 4$^{[2,Theorem1]}$**.** *Let* $A=(a_{ij})$ *and* $B=(b_{ij})\in M_n$*. Then*

$$\begin{aligned} \det[\mu(A\circ B)] \quad &\geq \quad a_{11}b_{11}\prod_{i=2}^{n}\left[\frac{\det(A_k)}{\det(A_{k-1})}b_{kk}\right. \\ &\left.+\frac{\det(B_k)}{\det(B_{k-1})}a_{kk}-\frac{\det(A_k)\det(B_k)}{\det(A_{k-1})\det(B_{k-1})}\right]. \end{aligned}$$

Lemma 5$^{[2,Theorem2]}$**.** *Let* $A=(a_{ij})$ *and* $B=(b_{ij})\in M_n$*. Then*

$$\begin{aligned} &\det[\mu(A\circ B)]\geq \det A\prod_{i=1}^{n}b_{ii}+\det B\prod_{i=1}^{n}a_{ii}-\det A\cdot\det B \\ &+\det A\left[\frac{\prod\limits_{i=1,i\neq k}^{n}a_{ii}}{\det[A(k)]}-1\right][b_{kk}\det[B(k)]-\det(B)] \\ &+\det B\left[\frac{\prod\limits_{i=1,i\neq k}^{n}b_{ii}}{\det[B(k)]}-1\right][a_{kk}\det[A(k)]-\det(A)],\quad k=1,2,\cdots,n. \end{aligned} \tag{6}$$

Lemma 6. *Let* $A=(a_{ij})$ *and* $B=(b_{ij})\in H_n$*, then*

$$\begin{aligned} Y_n(A,B) &= |a_{11}b_{11}|\prod_{i=2}^{n}\left\{\frac{\det[\mu(A_k)]}{\det[\mu(A_{k-1})]}|b_{kk}|+\frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1}]}|a_{kk}|\right. \\ &\left.-\frac{\det[\mu(A_k)]\det[\mu(B_k)]}{\det[\mu(A_{k-1})]\det[\mu(B_{k-1}]}\right\} \\ &\geq \det[\mu(A)]\prod_{i=1}^{n}|b_{ii}|+\det[\mu(B)]\prod_{i=1}^{n}|a_{ii}| \\ &\quad-\det[\mu(A)]\det[\mu(B)]+\omega_n(A,B,n) \\ &= \varepsilon_n(A,B)+\omega_n(A,B,n), \end{aligned} \tag{7}$$

where

$$\omega_n(A,B,k)=\det[\mu(A)]\left[\frac{\prod\limits_{i=1,i\neq k}^{n}|a_{ii}|}{\det[\mu(A(k))]}-1\right][|b_{kk}|\det[\mu(B(k))]-\det[\mu(B)]]$$

$$+\det[\mu(B)]\left[\frac{\prod\limits_{i=1,i\neq k}^{n}|b_{ii}|}{\det[\mu(B(k))]}-1\right][|a_{kk}|\det[\mu(A(k))]-\det[\mu(A)]].$$

Proof. By direct verification, $Y_2(A,B)=\varepsilon_2(A,B)$, $\omega_2(A,B,2)=0$ for A and $B\in H_2$, and for A and $B\in H_3$,

$$Y_3(A,B)=Y_2(A_2,B_2)\left[|b_{33}|\frac{\det[\mu(A_3)]}{\det[\mu(A_2)]}\right.$$
$$\left.+|a_{33}|\frac{\det[\mu(B_3)]}{\det[\mu(B_2]}-\frac{\det[\mu(A_3)]\det[\mu(B_3)]}{\det[\mu(A_2)]\det[\mu(B_2)]}\right]. \quad (8)$$

Hence (7) holds for $n=2$. Now, suppose (7) and (8) hold for $n-1$, that is,

$$Y_{n-1}(A,B)\geq\varepsilon_{n-1}(A,B)+\omega_{n-1}(A,B,n-1),$$

$$Y_n(A,B)=Y_{n-1}(A_{n-1},B_{n-1})\left[|b_{nn}|\frac{\det[\mu(A_n)]}{\det[\mu(A_{n-1})]}\right.$$

$$\left.+|a_{nn}|\frac{\det[\mu(B_n)]}{\det[\mu(B_{n-1})]}-\frac{\det[\mu(A_n)]\det[\mu(B_n)]}{\det[\mu(A_{n-1})]\det[\mu(B_{n-1})]}\right].$$

Then, for $A=\begin{pmatrix}A_{n-1} & A_{12}\\ A_{21} & a_{nn}\end{pmatrix}$ and $B=\begin{pmatrix}B_{n-1} & B_{12}\\ B_{21} & b_{nn}\end{pmatrix}\in H_n$, we have

$$\begin{aligned}
Y_n(A,B) &= Y_{n-1}(A_{n-1},B_{n-1})\left[|b_{nn}|\frac{\det[\mu(A_n)]}{\det[\mu(A_{n-1})]}\right.\\
&\quad\left.+|a_{nn}|\frac{\det[\mu(B_n)]}{\det[\mu(B_{n-1})]}-\frac{\det[\mu(A_n)]\det[\mu(B_n)]}{\det[\mu(A_{n-1})]\det[\mu(B_{n-1})]}\right]\\
&\geq \varepsilon_{n-1}(A_{n-1},B_{n-1})\left[|b_{nn}|\frac{\det[\mu(A_n)]}{\det[\mu(A_{n-1})]}\right.\\
&\quad\left.+|a_{nn}|\frac{\det[\mu(B_n)]}{\det[\mu(B_{n-1}]}-\frac{\det[\mu(A_n)]\det[\mu(B_n)]}{\det[\mu(A_{n-1})]\det[\mu(B_{n-1})]}\right]\\
&= [\det[\mu(A_{n-1})]\prod_{i=1}^{n-1}|b_{ii}|+\det[\mu(B_{n-1})]\prod_{i=1}^{n-1}|a_{ii}|\\
&\quad-\det[\mu(A_{n-1})]\det[\mu(B_{n-1})]]\times\left[|b_{nn}|\frac{\det[\mu(A_n)]}{\det[\mu(A_{n-1})]}\right.\\
&\quad\left.+|a_{nn}|\frac{\det[\mu(B_n)]}{\det[\mu(B_{n-1})]}-\frac{\det[\mu(A_n)]\det[\mu(B_n)]}{\det[\mu(A_{n-1})]\det[\mu(B_{n-1})]}\right]\\
&= \varepsilon_n(A,B)+\omega_n(A,B,n),
\end{aligned}$$

which completes the proof.

Lemma 7. *If A and $B\in H_n$, then*

$$\varepsilon_n(A,B)\geq W_n(A,B).$$

Proof. Since A and $B\in H_n$, $\mu(A)$ and $\mu(B)\in M_n$. From [5, Corollary 2.2], the result is evident.

3. Main Results

In this section, we give out several estimations of bounds of $\det(A \circ B)$ for A and $B \in H_n$.

Theorem 1. *Let $A = (a_{ij})$ and $B = (b_{ij}) \in H_n$. Then*

$$\det[\mu(A \circ B)] \geq \varepsilon_n(A, B) + \omega_n(A, B, k), \quad k = 1, 2, \cdots, n.$$

Proof. From A and $B \in H_n$, we have $A \circ B \in H_n$, so $\mu(A \circ B) \in M_n$. It is easy to prove that

$$\mu(A \circ B) = \mu[\mu(A) \circ \mu(B)].$$

Hence

$$\det[\mu(A \circ B)] = \det\{\mu[\mu(A) \circ \mu(B)]\}.$$

Obviously,

$$\mu(A), \quad \mu(B) \in M_n.$$

By Lemma 5, we have

$$\begin{aligned} \det[\mu(A \circ B)] &= \det\{\mu[\mu(A) \circ \mu(B)]\} \\ &\geq [\det[\mu(A)] \prod_{i=1}^{n} |b_{ii}| + \det[\mu(B)] \prod_{i=1}^{n} |a_{ii}| \\ &\quad - \det[\mu(A)] \det[\mu(B)]] + \det[\mu(A)] \left[\frac{\prod\limits_{i=1, i \neq k}^{n} |a_{ii}|}{\det[\mu(A(k))]} - 1 \right] \\ &\quad \times [|b_{kk}| \det[\mu(B(k))] - \det[\mu(B)]] + \det[\mu(B)] \\ &\quad \times \left[\frac{\prod\limits_{i=1, i \neq k}^{n} |b_{ii}|}{\det[\mu(B(k))]} - 1 \right] [|a_{kk}| \det[\mu(A(k))] - \det[\mu(A)]] \\ &= \varepsilon_n(A, B) + \omega_n(A, B, n). \end{aligned}$$

Theorem 2. *Let $A = (a_{ij})$ and $B = (b_{ij}) \in H_n$, $\prod\limits_{i=1}^{n} a_{ii} b_{ii} > 0$. Then*

$$\begin{aligned} \det(A \circ B) &\geq \det[\mu(A \circ B)] \\ &\geq \varepsilon_n(A, B) + \omega_n(A, B, k) \\ &\geq W_n(A, B), \quad k = 1, 2, \cdots, n. \end{aligned}$$

Proof. From Lemma 2, $\omega_n(A, B, k) \geq 0$. Hence, we have the following inequality by Lemma 7

$$\varepsilon_n(A, B) + \omega_n(A, B, k) \geq W_n(A, B).$$

By [1, Theorem 3] and Theorem 1, we obtain

$$\begin{aligned} \det(A \circ B) &\geq \det[\mu(A \circ B)] \\ &\geq \varepsilon_n(A, B) + \omega_n(A, B, k) \\ &\geq W_n(A, B), \quad k = 1, 2, \cdots, n. \end{aligned}$$

Theorem 3. *Let* $A = (a_{ij})$ *and* $B = (b_{ij}) \in H_n$, $\prod_{i=1}^{n} a_{ii}b_{ii} > 0$. *Then*

$$\det[\mu(A \circ B)] \geq Y_n(A, B).$$

Proof. Similar to the proof of Theorem 1, by Lemma 4, we have

$$\begin{aligned} \det[\mu(A \circ B)] &= \det\{\mu[\mu(A) \circ \mu(B)]\} \\ &\geq |a_{11}b_{11}| \prod_{i=2}^{n} \left\{ \frac{\det[\mu(A_k)]}{\det[\mu(A_{k-1})]} |b_{kk}| + \frac{\det[\mu(B_k)]}{\det[\mu(B_{k-1}]} |a_{kk}| \right. \\ &\quad \left. - \frac{\det[\mu(A_k)] \det[\mu(B_k)]}{\det[\mu(A_{k-1})] \det[\mu(B_{k-1}]} \right\} \\ &= Y_n(A, B), \end{aligned}$$

where $\mu(A_k)$ and $\mu(B_k), k = 1, 2, \cdots, n$, denote the comparison matrix of the $k \times k$ successive principal submatrix of A and B, respectively.

By [1, Theorem 3], Theorem 3, Lemma 6, and Lemma 7, we can obtain the following theorem, immediately.

Theorem 4. *Let* $A = (a_{ij})$ *and* $B = (b_{ij}) \in H_n$, $\prod_{i=1}^{n} a_{ii}b_{ii} > 0$. *Then*

$$\begin{aligned} \det(A \circ B) &\geq \det[\mu(A \circ B)] \\ &\geq Y_n(A, B) \\ &\geq \varepsilon_n(A, B) + \omega_n(A, B, n) \\ &\geq W_n(A, B). \end{aligned}$$

Now let us consider the following example.

Example 1. Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 4 \end{pmatrix}, \; B = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 2 & 4 \end{pmatrix}.$$

It is easy to know that A, B are H-matrices and $\prod_{i=1}^{3} (a_{ii}b_{ii}) = 34 > 0$,

$$\mu(A) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & 4 \end{pmatrix}, \; \mu(B) = \begin{pmatrix} 4 & -1 & -2 \\ -1 & 3 & 0 \\ -1 & -2 & 4 \end{pmatrix},$$

$$A \circ B = \begin{pmatrix} 12 & 1 & 2 \\ 1 & 6 & 0 \\ 1 & 2 & 16 \end{pmatrix},$$

$$\det(A \circ B) = 1128,$$

$$\begin{aligned} W_3(A, B) &= \left(\prod_{i=1}^{3} b_{ii} \right) \det(\mu(A)) + \left(\prod_{i=1}^{3} |a_{ii}| \right) \\ &\quad \times \det(\mu(B)) \cdot \prod_{k=2}^{3} \sum_{i=1}^{k-1} \left| \frac{a_{ik}a_{ki}}{a_{ii}a_{kk}} \right| \\ &\approx 866, \end{aligned}$$

$$\varepsilon_3(A,B)+\omega_3(A,B,2)\approx 991+41=1032,$$
$$Y_3(A,B)\approx 1062.$$

It is obviously that

$$\det(A\circ B)>Y_3(A,B)>\varepsilon_3(A,B)+\omega_3(A,B,2)\geq W_3(A,B).$$

4. Remark

Theorem 1 and Theorem 3 are the new results on estimations of bounds of $\det[\mu(A\circ B)]$ when A, B are H-matrices. From Lemma 3 and Example 1, we know that Theorem 2 and Theorem 4 strengthen really Theorem 2 and Theorem 6 of [1], respectively. When $A,\ B\in M_k$, A and $B\in H_K$, $a_{ii}\geq 0,\ b_{ii}\geq 0,\quad i=1,2,\cdots,n$, and $\mu(A)=A$, $\mu(B)=B$, $\mu(A_k)=A_k$, $\mu(B_k)=B_k$. So, Theorem 2 and Theorem 4 are the generalizations of Theorem 1 and Theorem 2 of [2], respectively.

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